ON A CLASS OF APPROXIMATION OPERATORS OVER LOCAL FIELDS

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1. Introduction. Let K be a local field, p its prime element, $\mathcal{P}$ its prime ideal and $\mathcal{O}$ the ring of integers in K. Then $\mathcal{O}/\mathcal{P} \simeq GF(q)$ where $q=p^c$ for some prime p and $c \in N$. Denote by $|\cdot|$ the nonarchimedean norm in K, then the Haar measure of the sphere $\mathcal{P}^{-k} = \{x: |x| \leq q^k\}$ is q^k, $k \in Z$. Let $\{e_0=0, e_1, \ldots, e_{q-1}\}$ be the full set of coset representatives of $\mathcal{P}$ in $\mathcal{O}$, and meanwhile let us consider an additive character $\chi \in K_+^\wedge$ that is trivial on $\mathcal{O}$ but is not on $\mathcal{P}^{-1}$ such that the relations

$$\chi(e_l x) = \chi(x)^l, \quad for \quad x \in \mathcal{P}^{-1}, \ l \in \{0, 1, \cdots, q-1\}$$

hold.

As is in [5], by using the norm $|\cdot|_\gamma$ we define the mapping λ: $K \to R^+$ as $\lambda(x) = |x|_\gamma$ and the order '$\leq$' in K. Thus, the interval $[\alpha, \beta)$ means the set $\{x: \alpha \leq x < \beta\}$, $[0, p^{-1}) = \mathcal{O}$, $[0, p^{-k-1}) = \mathcal{P}^{-k}$ etc. Let $\{z^{(s)} + \mathcal{O}: s \in P\}$ with $\lambda(z^{(s)}) \in P$ be the full set of coset representatives of $\mathcal{O}$ in K, where $z^{(0)} < z^{(1)} < \ldots$. Denote by $\chi(t, s) = \chi(z^{(s)} t)$, $s \in P$, the complete system of characters over $\mathcal{O}$. Note that we can write $\chi(t, s) = \chi(\lambda^{-1}(s)t)$ for $s \in P$. For simplicity in what follows we will write $\{0, 1, \ldots, q-1\}$ in place of $\{e_0, \ldots, e_{q-1}\}$ if no ambiguity occurs. For $t \in K$ we often write $t = T + t'$ in which $\lambda(T) \in P$ and say that T is the main part of t.

Let $\omega \in K$ be a nonzero parameter, $\psi_\omega(x)$ a function in $L(K)$ which vanishes outside $[0, \omega)$. Denote by $\psi_\omega^{(k)}(x) = \psi_\omega(x + \lambda^{-1}(k))$, $k = 0, 1, \ldots, \lambda(\Omega)$. For $f \in L(\mathcal{O})$, let $a_j = a_j(f)$ be the Fourier coefficients of f with respect to the system $\chi(\cdot, j)$, $j \in P$. Consider the kernel $K_\omega(t) \equiv \int_{[0,\omega)} \psi_\omega(x) \bar{\chi}_{\lambda(x)}(T) \chi(xt) dx$. We have proved in [5],

$$K_\omega(t) = \begin{cases} \sum_{k=0}^{\lambda(\Omega)} a_0^{(k)} \chi(t, k), & t \in \mathcal{O}, \\ \sum_{k=0}^{\lambda(\Omega)} a_{\lambda([-T])}^{(k)} \chi(t', k), & t \bar{\in} \mathcal{O}, \end{cases} \tag{1}$$

where $a_j^{(k)} = a_j(\psi_\omega^{(k)})$.

Taking $q' > q$ and setting

$$\begin{cases} a_0^{(k)} = 1 - |\omega|_r^{-1}(k + c^{-1}), \quad k \in \mathbb{P}, \\ a_{lq^s}^{(\nu)} = |\omega|_r^{-1} A_l q'^{-s-1}, \\ a_j^{(\nu)} = 0, \text{ for } \quad j \neq 0, lq^s; \; l = 1, \cdots, q-1, \; \nu \in \{0, 1, \cdots, \lambda(\Omega)\}, \; s \in \mathbb{P}, \end{cases} \tag{2}$$

where $c = 2(q'-1)(q-1)^{-1}$ and $A = (e^{-2\pi i l} - 1)^{-1}$, we will study the type ofthe maximal operator associated $f*K$.

2. <u>The case $\mathcal{O}$, the ring of integers in K</u>.

Let $D(t,k) = \sum_{\nu=0}^{k-1} \chi(t,\nu)$, and $K(t,k) = k^{-1} \sum_{\nu=1}^{k} D(t,\nu)$, $t \in \mathcal{O}$, $k \in \mathbb{N}$. By induction we can prove

<u>Lemma 2.1</u> For any $\nu \in \mathbb{P}$, $x \in \mathcal{O}$, we have the following formula

$$K(t, q^{\nu+1})$$

$$= q^{-\nu-1} D(t, q^{\nu+1}) + \sum_{j=0}^{\nu} q^{j-\nu-1} D(t, q^j) D(\mathfrak{p}^{-j-1} t, q^{\nu-j})(q-1) K(\mathfrak{p}^{-j} t, q-1). \tag{3}$$

<u>Lemma 2.2</u> Let

$$\mathcal{K}_1 f(x) = \sup_{\nu \in \mathbb{P}} \left| \int_{\mathcal{O}} f(x-t) K(t, q^\nu) \, dt \right|, \tag{4}$$

then $\mathcal{K}_1 f$ is of type (r,r) for $1 < r \leq \infty$ and is of weak type $(1,1)$ (See Lemma 4 of [6], p.28).

<u>Proof</u> (cf. [2]) Apply the Calderón-Zygmund decomposition([3], p.148).

The Half-Hardy-Littlewood maximal function of f is defined by

$$\mathcal{M}_1 f(x) = \sup_{k \in \mathbb{P}} \int_{\mathfrak{p}^k} q^k |f(x-t)| \, dt. \tag{5}$$

<u>Lemma 2.3</u> There is a constant $c = c_q$ such that $\mathcal{M}_1 f(x) \leq c \mathcal{K}_1 f(x)$. And consequently, if $\mathcal{K}_1 f \in L^r(\mathcal{O})$ for some r, $1 \leq r < \infty$, then so does $\mathcal{M}_1 f$.

<u>Proof</u> It is easy to check the recurrence formula

$$q^k D(t, q^k)[qK(\mathfrak{p}^{-k} t, q) - D(\mathfrak{p}^{-k} t, q)] = q^{k+1} K(t, q^{k+1}) - q^k K(t, q^k) D(\mathfrak{p}^{-k} t, q), \tag{6}$$

$k \in \mathbb{P}$, $t \in \mathcal{O}$, hence

$$\sup_{k\in \mathbb{P}} \int_{\rho^k} q^k |f(x-t)| \cdot |(qK(p^{-k}t ,q)-D(p^{-k}t,q))| dt \leq 2q\mathcal{K}_1 f(x). \tag{7}$$

We conclude that

$$\min_{z\in\mathcal{O}} | -D(z,q) + qK(z,q)| = \min\{q(\max_{z\in[p,p^{-1})} | 1-\chi(p^{-1}z)|)^{-1} ,2^{-1}q(q-1)\}$$
$$= c > 0. \tag{8}$$

Applying (8) t o (7) we see that

$$\mathcal{M}_1 f(x) \leq 2qc^{-1}\mathcal{K}_1 f(x). \tag{9}$$

<u>Theorem 2.4</u> Let

$$\mathcal{K}_2 f(x) = \sup_{n\in\mathbb{N}} \int_{\mathcal{O}} |f(x-t)K(t,n)| dt , \tag{10}$$

t hen $\mathcal{K}_2 f$ is of weak type (1,1) and of type (r,r) for $1<r\leq\infty$.

<u>Proof</u> Let $n=sq^{\nu}+n'$, where s is in $\{1,\dots,q-1\}$, $\nu\in\mathbb{P}$ and $0\leq n'<q^{\nu}$, then

$$\int_{\mathcal{O}} |nK(t ,n)f(x-t)| dt$$
$$\leq (2^{-1}s(s-1)q^{\nu} + sn')\mathcal{M}_1 f(x) + sq^{\nu}\int_{\mathcal{O}} | K(t ,q^{\nu})f(x-t)| dt$$
$$+ n' \int_{\mathcal{O}} | K(t,n')f(x-t)| dt. \tag{11}$$

Applying the inequality (11) successively we obtain

$$\int_{\mathcal{O}} |nK(t ,n)f(x-t)| dt$$
$$\leq nq^2\mathcal{M}_1 f(x)+(q-1)\{\int_{\mathcal{O}} |q^{\nu} K(t ,q^{\nu})f(x-t)|dt +\dots+\int_{\mathcal{O}} |qK(t,q)f(x-t)| dt\},$$

hence

$$\mathcal{K}_2 f(x)\leq q^2\mathcal{M}_1 f(x) + q\mathcal{K}_1 f(x). \tag{12}$$

this shows $\mathcal{K}_2 f$ is of weak type (1,1) and of type (r,r) for $1<r\leq\infty$

together with $\mathcal{M}_1 f$ and $\mathcal{K}_1 f$.

3. The general case.

In this section we consider the maximal operator associated $f * K_\omega(t)$:

$$\mathcal{K} f(x) = \sup_{\omega \geq \omega_0} \left| \int_K f(x-t) K_\omega(t)\,dt \right|, \tag{13}$$

where ω_0 is any fixed nonzero element in K.

The following lemma will be applied several times.

Lemma 3.1 Let

$$\mathcal{A} f(x) = \sum_{s=0}^{\infty} q'^{-s-1} \Big\{ \sum_{l=1}^{q-1} \mathcal{M}_1 f(x-h_{l0}) + q^{-1} \sum_{l=1}^{q-1} \mathcal{M}_1 f(x-h_{l1}^{(s)})$$
$$+ q^{-2} \sum_{j=0}^{q(q-1)} \mathcal{M}_1 f(x-h_{j2}^{(s)}) + \ldots + q^{-s} \sum_{j=1}^{q^{s-1}(q-1)} \mathcal{M}_1 f(x-h_{js}^{(s)}),$$

then $\mathcal{A}f$ is of weak type (1,1).

Theorem 3.2 The operator $\mathcal{K} f(x)$ is of weak type (1,1) and of type (r,r) for $1 < r \leq \infty$.

Proof We break the proof into three steps.

Step 1 Let $t \in \mathcal{O}$. By (1) we have for $\omega \geq p^{-1}$,

$$\left| \int_{\mathcal{O}} K_\omega(t) f(x-t)\,dt \right| \leq c_1 \mathcal{K}_1 f(x) + c_2 \mathcal{M}_2 f(x). \tag{14}$$

Step 2 For $t \bar{\in} \mathcal{O}$, we have

$$K_\omega(t) = \sum_{\nu=0}^{\lambda(\Omega)-1} a^{(\nu)}_{\lambda([-T])} \chi(t', \nu) + \chi(t', \lambda(\Omega)) \int_{[0,\omega')} \psi_\omega(\Omega + x) \chi(x, \lambda(T))\,dx.$$

Applying Abel's transform twice we deduce

$$K_\omega(t) = D(t, \lambda(\Omega)) a^{(\lambda(\Omega)-1)}_{\lambda([-T])} + \chi(t', \lambda(\Omega)) \int_{[0,\omega')} \psi_\omega(\Omega + x) \chi(x, \lambda(T))\,dx$$
$$= K_{\omega 1}(t) + K_{\omega 2}(t), \text{ say.} \tag{15}$$

Step 2.1 For $K_{\omega 1}(t)$, $t \in \complement\mathcal{O}$, we have

$$\left| \int_{\complement\mathcal{O}} K_{\omega 1}(t) f(x-t)\,dt \right| \leq A \frac{\lambda(\Omega)}{|\omega|_r} \sum_{s=0}^{\infty} q'^{-s-1} \sum_{l=1}^{q-1} \mathcal{M} f(x-h_{sl}), \tag{16}$$

where A is a constant and $h_{sl} \in K$.

Step 2.2 For $K_{\omega 2}(t)$, $t \in \mathcal{GO}$, we have

$$K_{\omega 2}(t) = \chi(t', \lambda(\Omega))\left\{\sum_{j=0}^{q^s-1} a_j^{(\lambda(\Omega))}\int_{[0,\omega')}\chi((T+\lambda^{-1}(j))x)dx + \sum_{j=q^s}^{q^{s+1}-1} + \sum_{j=q^{s+1}}^{\infty}\right\},$$

hence

$$\left|\int_{\mathcal{GO}} K_{\omega 2}(t)f(x-t)dt\right| \le AM(|\omega|_r q(q-1))^{-1}\sum_{s=0}^{\infty} q'^{-s-1}\{\mathcal{M}_1 f(x-p^{-s-1})+\ldots+\mathcal{M}_1 f(x-e_{q-1}p^{-s-1})$$
$$+AMq^{-1}|\omega|_r^{-1}\sum_{s=1}^{\infty} q'^{-s-1}\{\mathcal{M}_1 f(x-p^{-s-1})+\ldots+\mathcal{M}_1 f(x-e_{q-1}p^{-s-1})\}$$
$$+AM|\omega|_r^{-1}\sum_{s=0}^{\infty} q'^{-s-1}\{\mathcal{M}_1 f(x-h_{q-1,0})+\ldots+\mathcal{M}_1 f(x-h_{1,0}) + \ldots$$
$$+q^{-s}[\mathcal{M}_1 f(x-h^{(s)}_{q^s(q-1),1})+\ldots+\mathcal{M}_1 f(x-h^{(s)}_{q^{s-1}(q-1),s})]\}. \tag{17}$$

Step 3 Let c be a constant depending only on ω_0, q, q', from (14), (16) and (17) we get

$$\mathcal{K}f(x) \le c_1\mathcal{K}f(x)+c_2\mathcal{M}_1 f(x)+c_3\sum_{s=0}^{\infty} q'^{-s-1}\sum_{l=1}^{q-1}\mathcal{M}_1 f(x-h_{sl})+$$
$$c_3\sum_{s=0}^{\infty} q'^{-s-1}\{\mathcal{M}_1 f(x-h_{10})+\ldots+\mathcal{M}_1 f(x-h_{q-1,0})+$$
$$q^{-1}[\mathcal{M}_1 f(x-h^{(s)}_{11})+\ldots+\mathcal{M}_1 f(x-h^{(s)}_{q-1,1})+\ldots+$$
$$q^{-s}[\mathcal{M}_1 f(x-h^{(s)}_{1s})+\ldots+\mathcal{M}_1 f(x-h^{(s)}_{q^{s-1}(q-1)s})]\}. \tag{18}$$

By Lemma 3.2 we see readily from (18) that $\mathcal{K}f$ is of weak type (1,1).

4. A comparison between $\mathcal{K}f$ and $\mathcal{M}f$.

We will provide a comparison between $\mathcal{K}f$ an d $\mathcal{M}f$.

Lemma 4.1 The operator

$$\mathcal{K}_3 f(x) = \sup_{k\in P}\left|\int_K K_{p^{k-1}}(t)f(x-t)dt\right| \tag{19}$$

is of weak type (1,1).

Proof For $k \in \mathbf{P}$, we have

$$K_{p^{k-1}}(t) = \int_{[0,p^{k-1})}\psi_{p^{k-1}}(y)\bar{\chi}(yT)\chi(yt)dy$$
$$=\sum_{j=0}^{q^k-1} a_j^{(0)} q^{-k}\xi_{p^{-k}}(t)+\sum_{\substack{j=q^k\\ \lambda^{-1}(j)+T\in p^{-k}}}^{\infty} a_j^{(0)}\int_{p^{-k}}\chi((\lambda^{-1}(j)+T)y)dy(1-\xi_{p^{-k}}(t)); \tag{20}$$

hence

$$\left|\int_{\mathbb{K}} K_{p^{k+1}}(t)f(x-t)\,dt\right| \le \sum_{j=0}^{q^k-1}|a_j^{(0)}|\mathcal{M}_2 f(x) + \sum_{j=q^k}^{\infty}|a_j^{(0)}|\mathcal{M}_2 f(x-h_{jk}), \tag{21}$$

where

$$\mathcal{M}_2 f(x) = \sup_{k\in P}\left| q^{-k}\int_{\rho^{-k}} f(x-t)\,dt\right|. \tag{22}$$

Thus,

$$\mathcal{K}_3 f(x) \le \mathcal{M}_2 f(x) + \sum_{k=0}^{\infty}\sum_{s=k}^{\infty}\sum_{l=1}^{q-1}|A_l|q^{k-1}q'^{-s-1}\mathcal{M}_2 f(x-h_{kl}^{(s)}). \tag{23}$$

Applying the same idea as the proof in Theorem 3.2 we conclude from (23) that $\mathcal{K}_3 f(x)$ is of weak type (1,1).

Lemma 4.2 There is a constant $A_0>0$ such that the inequality

$$A_0\mathcal{M}_2 f(x) \le \mathcal{K}_3 f(x) + \sum_{k=0}^{\infty}\sum_{s=k}^{\infty}\sum_{l=1}^{q-1}|A_l|q^{k-1}q'^{-s-1}\mathcal{M}_2 f(x-h_{kl}^{(s)}) \tag{24}$$

holds.

Lemma 4.3 Let S be the class of functions being in $L(\mathbb{K})$ with compact support. Then to every $\lambda>0$ and every $f\in S$, the measure $|\{\mathcal{M}_2 f(x)>\lambda\}|$ is always finite.

Proof In fact, if $f\in S$ with $\operatorname{supp}(f)\subset\rho^{-N}$ for some $N\in\mathbb{N}$, we can prove

$$|\{\mathcal{M}_2 f(x)>\lambda\}| \le q^N + \|f\|_1\lambda^{-1}.$$

Lemma 4.4 There exists a $k_0\in\mathbb{N}$ such that the operator

$$\mathcal{M}_3 f(x) = \sup_{j\ge k_0} q^{-j}\int_{\rho^j}|f(x-t)|\,dt,\quad f\in S \tag{25}$$

is of weak type (1,1).

Proof We can choose $k_0\in\mathbb{N}$ such that

$$\frac{Aq'}{q'-q} + \frac{q-1}{2}\left(\frac{q}{q'}\right)^{k_0} < 1 \tag{26}$$

then from (20)

$$(1-\mathcal{E}(k_0))\mathcal{M}_3 f(x) \le \mathcal{K}_3 f(x) + A\sum_{k=k_0}^{\infty}\sum_{j=k}^{\infty} q^{k-1}q'^{-j-1}\mathcal{M}_3 f(x-h_{jk}^{(\omega)}), \tag{27}$$

where $\mathcal{E}(k_0)=(q-1)(2q(q'-1))^{-1}q^{k_0}q'^{-k_0}$, $A=(q-1)\max|A_l|$.

Assume now that $\mathcal{M}_3 f$ were not of weak type (1,1) for such k_0, we would have by virtue of the fact that $|\{\mathcal{M}_3 f(x)>\lambda\}|=0$ provided $\lambda \geq \|f\|_1 q^{-1}$,

$$\sup_{\substack{0<\lambda\leq\|f\|_1 q^{-1}\\ f\in S,\ \|f\|_1\neq 0}} |\{\mathcal{M}_3 f(x)>\lambda\}|\lambda\|f\|_1^{-1}=\infty . \tag{28}$$

It f ollows from (27) that for any given $N>0$, there is an $f=f_N\in S$ with $\|f\|_1\neq 0$, such that

$$\sup_{0<\lambda\leq\|f\|_1 q^{-1}} |\{\mathcal{M}_3 f(x)>\lambda\}|\cdot\lambda\|f\|_1^{-1}>N. \tag{29}$$

Now (27) should read as

$$\mathcal{M}_3 f(x)(1-\varepsilon(k_0))\leq \mathcal{K}_3 f(x) + \frac{A}{qq'}\left(\frac{q}{q'}\right)^{k_0}\sum_{l=1}^{q-1}\sum_{k,j=0}^{\infty}\left(\frac{q}{q'}\right)^{k} q'^{-j}\mathcal{M}_3 f(x-h'_{jkl}) \tag{30}$$

Taking a positive number sequence $\{c_{jk}\}$ such that

$$\begin{cases} c_0 + \dfrac{A(q-1)}{qq'}\displaystyle\sum_{k,j=0}^{\infty}\left(\frac{q}{q'}\right)^{k+k_0} q'^{-j} c_{jk}\leq 1-\varepsilon(k_0), \\ c_{jk}>0,\ j,k=0,1,\ldots \\ \sum_{j,k} c_{jk}^{-1}\ (q-1)^{-1}, \end{cases} \tag{31}$$

we get from (30)

$$N(1-(q-1)\sum_{k,j} c_{jk}^{-1})\leq \frac{M}{c}. \tag{32}$$

Since N can be made arbitrary large, (32) leads to a contradiction. Therefore (28) is false.

<u>Th eorem 4.1</u> That the weak type (1,1) of $\mathcal{K}f$ implies that of $\mathcal{M}f$.

<u>Proof</u> Choose $k_0\in\mathbb{N}$ as in Lemma 4.4, and let

$$\mathcal{M}_4 f(x) = \sup_{1\leq k<k_0} q^{-k}\int_{\mathcal{P}^k}|f(x-t)|\,dt. \tag{33}$$

It is easy to see th at $\mathcal{M}_4 f$ is of weak type (1,1).

Obviously, we have

$$\mathcal{M}f(x) = \sup_{k\in\mathbb{Z}} q^{-k}\int_{\mathcal{P}^k}|f(x-t)|\,dt \quad .$$

$$\leq \sup_{k\geq k_0} q^{-k}\int_{\mathfrak{p}^{-k}}|f(x-t)|\,dt + \sup_{1\leq k<k_0} q^{-k}\int_{\mathfrak{p}^{k}}|f(x-t)|\,dt$$

$$+ \sup_{k<1} q^{-k}\int_{\mathfrak{p}^{k}}|f(x-t)|\,dt \tag{34}$$

By Lemma 4.4, th e first term of the right side of (34) is of weak type (1.1), so is the third term, since $\mathcal{M}_1 f(x)\leq c_1\mathcal{K}_1 f(x)$ by Lemma 2.3 and $\mathcal{K}_1 f(x)$ is of weak type (1,1), by Lemma 2.2.

Thus from (34) we see that the weak type (1,1) of $\mathcal{K}f$ implies that of $\mathcal{M}f$.

References

1. N. Fuju, A maximal inequality for H functions on a generalized Walsh-Paley group, Proc. Amer. Math. Soc., 77:1(1979),111-116.
2. F. Shipp, Uber Gewissen Maximaloperatoren, Ann. Univ. Sci. Budapest Eotvos Sect. Math., 18(1974),189-195.
3. M. H. Taibleson, Fourier analysis on local fields, Princeton University Press, 1975.
4. S. Y ano, Cesaro summability of Walsh-Fourier series, Tohoku Math. Jour., (2)9(1957),267-272.
5. W. X. Zheng , A clss of approximation identity kernels on local fields, Jour. Approx. Theory & Appl., 1:1(1984), 65-76.
6. W. X. Zheng , Further on a class of approximation iden tity operators over local fields, Scientia Sinica, A 1(1987),22-33.

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