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Euler Splines from 3-directional Box Splines

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Dedicated to the memory of Professor Vasil A. Popov

Abstract: *Euler spline curves based on 3-directional box spline are introduced. Some of their properties are derived. A short and soft proof of "correctness" of cardinal interpolation with 3-directional box spline shifted along one of the directions is found.*

1. Introduction

The univariate exponential Euler spline is defined for any $z \in \mathbb{C}$, $z \neq 0$, by

$$\Phi_n(t;z) := \sum_{j=-\infty}^{\infty} z^j M_n(t-j), \quad t \in \mathbb{R}.$$

Here M_n is a univariate B-spline defined inductively by

$$M_1(x) := \begin{cases} 1 & \text{for } |x| < 1/2 \\ 1/2 & \text{for } |x| = 1/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad M_n(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} M_{n-1}(x+\tau)d\tau.$$

A definition by its Fourier transform $M_n\hat{}$ is also possible. We have $M_n\hat{}(\xi) = (\mathrm{sinc}\frac{\xi}{2})^n$, where $\mathrm{sinc}\,\xi = \frac{\sin\xi}{\xi}$.

The case when $|z| = 1$ is of main interest. The function $\varphi_n(t;u) = \Phi_n(t;e^{iu})$ was introduced by Schoenberg [4] and was studied in detail in [3], [5], [6]. Let us mention some of its properties:

(1.1) $\varphi_n(t+j;u) = e^{iju}\varphi_n(t;u)$ for $j \in \mathbb{Z}$;
(1.2) $\varphi_n(t;-u) = \varphi_n(-t;u) = \overline{\varphi_n(t;u)}$;
(1.3) $\varphi_n(t;0) \equiv 1$;
(1.4) $\varphi_1(t;u) = 1$ for $|t| < \frac{1}{2}$.

According to (1.1) and (1.2) we may consider only the case $0 \le t \le \frac{1}{2}$, $0 \le u \le \pi$. Then

(1.5) $$\varphi_n(t;u) = 0 \quad \text{if and only if} \quad t = \frac{1}{2} \text{ and } u = \pi.$$

Moreover, for fixed $u \in (0,\pi)$ and $n \ge 2$ we have

(1.6) $$\arg \varphi_n(t;u) \text{ is strictly increasing} \quad \text{and} \quad 0 < \arg \varphi_n(t;u) < \frac{u}{2} \quad \text{for} \quad 0 < t < \frac{1}{2},$$

and the curve $\Gamma_n(t) = \varphi_n(t;u)$ is strictly convex and left turning for $n \ge 3$.

In this paper we investigate the properties of Euler spline curves which corresponds to 3-directional box splines. As usual (cf. [2]) we define inductively for $k, \ell, m \in \mathbb{N}$:

(1.7) $$M_{k,\ell,0}(x,y) := M_k(x) \ \ M_\ell(y)$$

and

(1.8) $$M_{k,\ell,m}(x,y) := \int_{-\frac{1}{2}}^{\frac{1}{2}} M_{k,\ell,m-1}(x+\tau, y+\tau)d\tau.$$

An equivalent definition by Fourier transform $M_{k,\ell,m}\hat{\ }(\xi,\eta) = (\text{sinc}\frac{\xi}{2})^k(\text{sinc}\frac{\eta}{2})^\ell(\text{sinc}\frac{\xi+\eta}{2})^m$ is also possible. We set

(1.9) $$\psi_{k,\ell,m}(x,y;u,v) = \sum_{j_1,j_2=-\infty}^{\infty} e^{i(j_1u+j_2v)}M_{k,\ell,m}(x-j_1, y-j_2).$$

This function is known as the symbol or the characteristic polynomial of the cardinal interpolation problem (CIP) for the shifted box spline $\phi(\xi,\eta) = M_{k,\ell,m}(x+\xi, y+\eta)$. Solvability of CIP with ϕ refers to solvability of the discrete convolution equations

(1.10) $$\sum_{j_1,j_2=-\infty}^{\infty} c_{j_1,j_2}\phi(\nu_1-j_1,\nu_2-j_2) = d_{\nu_1,\nu_2}, \quad (\nu_1,\nu_2) \in \mathbf{Z}^2,$$

for a given data $d = (d_{\nu_1,\nu_2})$. This problem is called correct [2] if for any bounded sequence d of data there is a unique bounded sequence c satisfying (1.10), or, in other words, if $\psi_{k,\ell,m}(x,y;u,v) \ne 0$ for all u and v.

Our goal is to find some properties (like convexity, monotonicity of the argument, etc.) of curves generated by taking $\psi_{k,\ell,m}$ on the lines $x = \frac{1}{2}j$, $y = \frac{1}{2}j$ and $x - y = \frac{1}{2}j$ for $j \in \mathbf{Z}$. These properties are stated in Lemma 2 and Lemma 3. We define

(1.11a) $$\Gamma_m(t) = \Gamma_m(k,\ell,u,v;t) := \psi_{k,\ell,m}(t,t;u,v).$$

A change of the coordinates in (1.9) gives the equalities

(1.12) $$\psi_{k,\ell,m}(x,y;u,v) = \psi_{\ell,m,k}(y-x,-x;v,-u-v) = \psi_{m,k,\ell}(-y,x-y;-u-v,u).$$

Thus we receive the following equations which can be used as a definition of $\Gamma_n(t)$ instead of (1.11a)

(1.11b) $$\Gamma_k(t) = \Gamma_k(\ell,m,v,-u-v;t) = \psi_{\ell,m,k}(0,-t;u,v),$$

(1.11c) $$\Gamma_\ell(t) = \Gamma_\ell(m,k,-u-v,u;t) = \psi_{m,k,\ell}(-t,0;u,v).$$

In sections 3 and 4 some of the properties of $\Gamma_n(t)$ are derived. The main result is

Theorem 1. *Let $k, \ell, m \in \mathbb{N}$. Then $\Gamma_m(t) \neq 0$ for $|t| < \frac{1}{2}$ and $\Gamma_m(0)$ is a positive real number.*

According to (1.11) and Theorem 1 we have that the symbol $\psi_{k,\ell,m}(x, y; u, v)$ does not vanish for (x, y) satisfying one of the conditions

$$(*) \qquad \begin{aligned} x &= 0, \quad |y| < \frac{1}{2}; \\ |x| &< \frac{1}{2}, \quad y = 0; \\ |x| &< \frac{1}{2}, \quad y = x. \end{aligned}$$

So Theorem 1, in particular, is a short and soft proof of the theorems of de Boor, Höllig and Riemenschneider [2] (case $x = y = 0$) and of Sivakumar [7] (case (*)).

Theorem 2. *If (x, y) is from the "star" region (*), then CIP for $\phi = M_{k,\ell,m}(x + \cdot, y + \cdot)$ is correct.*

An extension of the methods in the proof of Theorem 1 is applied by authors in [1] for proving Theorem 2 for the region $\Omega = \{(x, y) \in \mathbb{R}^2; |x| < \frac{1}{2}, |y| < \frac{1}{2}, |x - y| < \frac{1}{2}\}$.

2. Two simple cases

Let first mention that (1.7), (1.8) and (1.9) give the formulas

$$\psi_{k,\ell,0}(x, y; u, v) = \varphi_k(x; u)\varphi_\ell(y; v), \tag{2.1}$$

$$\psi_{k,\ell,m}(x, y; u, v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_{k,\ell,m-1}(x + \tau, y + \tau; u, v). \tag{2.2}$$

Now we examine two cases in which the symbol coincides with the univariate Euler spline.

Proposition 1. *Given $k, \ell, m \in \mathbb{N}$ we have*

$$\begin{aligned} \psi_{k,\ell,m}(x, y; 0, v) &= \varphi_{\ell+m}(y; v), \\ \psi_{k,\ell,m}(x, y; u, 0) &= \varphi_{k+m}(x; u), \\ \psi_{k,\ell,m}(x, y; u, -u) &= \varphi_{k+\ell}(x - y; u). \end{aligned}$$

In particular, $\Gamma_n(t) \neq 0$ for $|t| < \frac{1}{2}$ in the cases $u = 0$, $v = 0$ and $u + v = 0$.

Proof: According to (1.12) and (1.2) it is enough to consider only the case $u = 0$. Here, using (2.1) and (1.3) we receive $\psi_{k,\ell,0}(x, y; 0, v) = \varphi_\ell(y, v)$. For $m > 0$ we use (2.2) and induction to yield

$$\psi_{k,\ell,m}(x, y; 0, v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_{\ell+m-1}(y + \tau; v)d\tau = \varphi_{\ell+m}(y; v).$$

Now $\Gamma_n(t) \neq 0$ for $|t| < \frac{1}{2}$ follows from (1.11a) and (1.5). ■

Proposition 2. *The following equality holds true for $n \in \mathbb{N}$*

$$\Gamma_n(1,1,u,v;t) = \varphi_{n+1}(t;u+v).$$

Proof: We use (2.1), (1.4) and (1.1) to receive

$$\psi_{1,1,0}(t,t;u,v) = \varphi_1(t;u)\varphi_1(t;v) = \varphi_1(t;u+v).$$

The induction by n yields

$$\psi_{1,1,n}(t,t;u,v) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_{1,1,n-1}(t+\tau,t+\tau;u,v)d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_n(t+\tau;u+v)d\tau = \varphi_{n+1}(t;u+v).$$

Now the proposition follows from (1.11a). ■

3. Preliminaries

Let the curve $\xi_0 : \mathbb{R} \to \mathbb{C}$ be given. The sequence of curves $\{\xi_n\}$ is defined inductively by

$$\xi_n(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi_{n-1}(t+\tau)d\tau, \quad n = 1,2,\ldots \tag{3.1}$$

Given $0 < \alpha < \pi$ we suppose that the following properties hold true

$$\xi_n(t+j) = e^{i2j\alpha}\xi_n(t) \quad \text{for each} \quad j \in \mathbb{Z}; \tag{3.2a}$$

$$\xi_n(t) = \overline{\xi_n(-t)}, \tag{3.2b}$$

and then

$$e^{-i\alpha}\xi_n(\frac{1}{2}+t) = e^{i\alpha}\xi_n(-\frac{1}{2}+t) = \overline{e^{-i\alpha}\xi_n(\frac{1}{2}-t)}. \tag{3.2c}$$

It is necessary to have the properties (3.2) only for $n = 0$. Then they can be inductively verified for each $n \in \mathbb{N}$ using (3.1).

Taking the derivatives of both sides in (3.1) we receive

$$\xi_n'(t) = \xi_{n-1}(t+\frac{1}{2}) - \xi_{n-1}(t-\frac{1}{2}) = (e^{i2\alpha}-1)\ \xi_{n-1}(t-\frac{1}{2})$$

and therefore from (3.2b)

$$\arg \xi_n'(t) = \frac{\pi}{2} + \alpha + \arg \xi_{n-1}(t-\frac{1}{2}) = \frac{\pi}{2} + \alpha - \arg \xi_{n-1}(\frac{1}{2}-t). \tag{3.3}$$

On the other hand (3.2) gives that $\xi_n(0)$ and $e^{-i\alpha}\xi_n(\frac{1}{2})$ are real numbers and according to (3.1), (3.2b) and (3.2c) we have

$$\xi_n(0) = 2\int_0^{\frac{1}{2}} \operatorname{Re} \xi_{n-1}(\tau)d\tau, \quad e^{-i\alpha}\xi_n(\frac{1}{2}) = 2\int_0^{\frac{1}{2}} \operatorname{Re}\,(e^{-i\alpha}\xi_{n-1}(\tau))d\tau. \tag{3.4}$$

First we prove one technical result.

Lemma 1. *Let $0 < \beta < \pi$ and the curve $\xi : [0, \frac{1}{2}] \to \mathbf{C}$ have the properties:*

(i) $\xi(0) \neq 0$, $\quad \xi(\frac{1}{2}) \neq 0$ *and* $\quad \arg \xi(0) = 0$, $\arg \xi(\frac{1}{2}) = \beta$;

(ii) $\xi'(t) \neq 0$ *and* $\beta < \arg \xi'(t) < \pi$ *for* $0 < t < \frac{1}{2}$.

Then for $0 < t < \frac{1}{2}$:

(a) $\xi(t) \neq 0$ *and* $\quad 0 < \arg \xi(t) < \beta$;

(b) $\arg \xi(t)$ *is strictly increasing.*

Proof: From (i) we have Im $\xi(0) = 0$ and Im $(e^{-i\beta}\xi(\frac{1}{2})) = 0$. On the other hand (ii) gives that Im $\xi(t)$ and Im $(e^{-i\beta}\xi(t))$ are strictly increasing. Hence

$$\operatorname{Im} \xi(t) > 0 \quad \text{and} \quad \operatorname{Im} (e^{-i\beta}\xi(t)) < 0 \quad \text{for} \quad 0 < t < \frac{1}{2},$$

which proves (a). We compare (ii) and (a) to receive

$$0 < \arg \xi'(t) - \arg \xi(t) < \pi.$$

Thus (b) holds true, too. ■

The following two lemmas consider some properties of the curves ξ_n.

Lemma 2. *Let $0 < \alpha \leq \frac{\pi}{2}$, $\xi_0(t) \neq 0$ and $\alpha - \frac{\pi}{2} < \arg \xi_0(t) < \frac{\pi}{2}$ for $0 < t < \frac{1}{2}$. Then for each $n \in \mathbb{N}$ we have*

(a) $\xi_n(0) > 0$ *and* $\quad e^{-i\alpha}\xi_n(\frac{1}{2}) > 0$;

(b) $\xi_n(t) \neq 0$ *and* $\quad 0 < \arg \xi_n(t) < \alpha$ *for* $\quad 0 < t < \frac{1}{2}$;

(c) $\arg \xi_n(t)$ *is strictly increasing;*

(d) ξ_n *is a strictly convex left turning curve for* $n \geq 2$.

Proof: The conditions on $\arg \xi_0(t)$ give that Re $\xi_0(t)$ and Re $(e^{-i\alpha}\xi_0(t))$ are positive. Thus from (3.4) we get (a). From (3.3) we have that $\alpha < \arg \xi_1'(t) < \pi$ for $0 < t < \frac{1}{2}$. Applying Lemma 1 for $\xi = \xi_1$ and $\beta = \alpha$ we receive (b) and (c) for $n = 1$.

In the induction step we receive (a) from (3.4) and (b) for $n - 1$. Then condition (ii) of Lemma 1 is verified using (3.3), (b) for $n - 1$ and $\alpha \leq \frac{\pi}{2}$. The application of Lemma 1 for $\xi = \xi_n$ gives (b) and (c) for n, while (d) is an immediate consequence of (c) for $n - 1$ and (3.3). ■

Lemma 3. *Let $\frac{\pi}{2} < \alpha < \pi$. Assume that $\xi_0(t) \neq 0$, $\arg \xi_0(t)$ is strictly increasing and $0 < \arg \xi_0(t) < \alpha$ for $0 < t < \frac{1}{2}$. Denote by μ the maximal integer such that $\xi_n(0)$ and $e^{-i\alpha}\xi_n(\frac{1}{2})$ are positive for all $n < \mu$. Then the following properties of the curves $\xi_n(t)$ hold true for $0 < t < \frac{1}{2}$:*

(a) if $\quad n = 1, 2, \ldots, \mu - 1$, $\quad$ *then* $\xi_n(t) \neq 0$, $\quad \arg \xi_n(t)$ *is strictly increasing and* $0 < \arg \xi_n(t) < \alpha$;

(b) if $\xi_\mu(0) \leq 0$, *then* $e^{-i\alpha}\xi_\mu(\frac{1}{2}) > 0$, $\xi_\mu(t) \neq 0$ *and* $\frac{\pi}{2} < \arg \xi_\mu(t) < \pi$. *Moreover for* $n > \mu$ *we have* $\xi_n(0) < 0$, $e^{-i\alpha}\xi_n(\frac{1}{2}) > 0$, $\arg \xi_n(t)$ *is strictly decreasing and* $\alpha < \arg \xi_n(t) < \pi$;

(c) if $e^{-i\alpha}\xi_\mu(\frac{1}{2}) \leq 0$, *then* $\xi_\mu(0) > 0$, $\xi_\mu(t) \neq 0$ *and* $\alpha - \pi < \arg \xi_\mu(t) < \alpha - \frac{\pi}{2}$. *Moreover for* $n > \mu$ *we have* $\xi_n(0) > 0$, $e^{-i\alpha}\xi_n(\frac{1}{2}) < 0$, $\arg \xi_n(t)$ *is strictly decreasing and* $\alpha - \pi < \arg \xi_n(t) < 0$;

(d) *for $1 \le n \le \mu$ the curve ξ_n is strictly convex and left turning, while for $n \ge \mu + 2$ it is strictly convex and right turning.*

Proof: We obtain (a) using an induction argument. The statement for $n = 0$ is contained in the assumption of the lemma. Let now suppose that it holds for $n-1$. Then using (3.3) we receive that $\arg \xi_n'(t)$ is strictly increasing and

$$\frac{\pi}{2} < \arg \xi_n'(t) < \frac{\pi}{2} + \alpha \quad \text{for} \quad 0 < t < \frac{1}{2}. \tag{3.5}$$

Therefore for some $\tau \in [0, \frac{1}{2}]$ we have that Im $\xi_n(t)$ increases for $t \in (0, \tau)$ and decreases for $t \in (\tau, \frac{1}{2})$. Hence

$$\text{Im } \xi_n(t) > \min\{\text{Im } \xi_n(0), \text{Im } \xi_n(\tfrac{1}{2})\}. \tag{3.6}$$

On the other hand Im $\xi_n(0) = 0$ and Im $\xi_n(\frac{1}{2}) > 0$ subject to $n < \mu$. Thus Im $\xi_n(t) > 0$ for $0 < t < \frac{1}{2}$. By the same reasoning Im $(e^{-i\alpha}\xi_n(t)) < 0$ and therefore $0 < \arg \xi_n(t) < \alpha$ for $0 < t < \frac{1}{2}$. In order to prove the increasing of $\arg \xi_n(t)$ we fix $t \in (0, \frac{1}{2})$ and set $\gamma = \arg \xi_n(t)$. Then $0 < \gamma < \alpha$ and therefore Im $(e^{-i\gamma}\xi_n(0)) < 0 <$ Im $(e^{-i\gamma}\xi_n(\frac{1}{2}))$. Hence there exist $\tau_0 \in (0, t)$ and $\tau_1 \in (t, \frac{1}{2})$ such that $\arg \xi_n'(\tau_j) \in (\gamma, \gamma + \pi)$ for $j = 0, 1$. Taking in account the increasing of $\arg \xi_n'$, we receive $\arg \xi_n'(t) \in (\gamma, \gamma + \pi)$, i.e. $0 < \arg \xi_n'(t) - \arg \xi_n(t) < \pi$. This shows that $\arg \xi_n(t)$ is strictly increasing and completes the proof of (a).

Let now $n = \mu$. Then (3.5) and (3.6) hold true. If $\xi_\mu(0) \le 0$, then Im $(e^{-i\frac{\pi}{2}}\xi_\mu(0)) \ge 0$. But it follows from (3.5) that Im $(e^{-i\frac{\pi}{2}}\xi_\mu(t))$ is increasing and therefore

$$\text{Im } (e^{-i\frac{\pi}{2}}\xi_\mu(t)) > 0 \quad \text{for} \quad 0 < t \le \frac{1}{2}. \tag{3.7}$$

Hence $\xi_\mu(t) \ne 0$ and $e^{-i\alpha}\xi_\mu(\frac{1}{2}) > 0$. The last inequality shows that Im $\xi_\mu(\frac{1}{2}) > 0$ and according to (3.6) and Im $\xi_\mu(0) = 0$ we have

$$\text{Im } (e^{-i\pi}\xi_\mu(t)) = -\text{ Im } \xi_\mu(t) < 0 \quad \text{for} \quad 0 < t < \frac{1}{2}.$$

Comparing this with (3.7) we receive

$$\frac{\pi}{2} < \arg \xi_\mu(t) < \pi.$$

Finally applying Lemma 2 to the curves $-\overline{\xi_{\mu+j}(t)}$, $j = 0, 1, \dots$ with $\beta = \pi - \alpha$, we receive (b). The proof of (c) is analogous. The convexity property (d) follows from increasing *or* decreasing of $\arg \xi_n'(t)$, which is a consequence of (3.3) and the monotonicity of $\arg \xi_{n-1}(t)$ in (a) *or* (b) and (c), respectively. ■

In the proof of Theorem 1 we shall use the following corollary of Lemma 3.

Lemma 4. *Let ξ_n be the curves from Lemma 3. Then $\xi_n(t) \ne 0$ for $0 < t < \frac{1}{2}$ and at least one of the sequences $\{\xi_n(0)\}$ and $\{e^{-i\alpha}\xi_n(\frac{1}{2})\}$ has only positive terms.*

It is easy to see that Lemma 4 holds true in the case when $\arg \xi_0(t)$ is nondecreasing.

4. Proof of Theorem 1

We shall examine the symbol $\psi_{k,\ell,m}(x,y;u,v)$ on the lines $x=\frac{1}{2}j$, $y=\frac{1}{2}j$ and $x-y=\frac{1}{2}j$ for $j\in\mathbf{Z}$. According to (1.12) it is enough to consider only one of these lines. In order to apply Lemma 2 and Lemma 3, we now check the properties (3.1) and (3.2) for the curves

$$\xi_m(t)=\xi_m(k,\ell,u,v,j,s;t):=e^{-i(\frac{j+s}{2}u+\frac{s}{2}v)}\psi_{k,\ell,m}(\frac{j+s}{2}+t,\frac{s}{2}+t;u,v). \tag{4.1}$$

We receive (3.1) as an immediate consequence of (2.2). According to (1.2) and (1.1), for any integer s we have

$$e^{-i\frac{s}{2}u}\varphi_n(\frac{s}{2}-t;u)=\overline{e^{-i\frac{s}{2}u}\varphi_n(t-\frac{s}{2};u)}=\overline{e^{-i\frac{s}{2}u}\varphi_n(\frac{s}{2}+t;u)}.$$

Using (2.1) we receive

$$\xi_0(-t)=e^{-i\frac{j+s}{2}u}\varphi_k(\frac{j+s}{2}-t;u)\ e^{-i\frac{s}{2}v}\varphi_\ell(\frac{s}{2}-t;v)$$
$$=\overline{e^{-i\frac{j+s}{2}u}\varphi_k(\frac{j+s}{2}+t;u)}\ \overline{e^{-i\frac{s}{2}v}\varphi_\ell(\frac{s}{2}+t;v)}=\overline{\xi_0(t)}.$$

On the other hand (2.1) and (1.1) give

$$\xi_0(t+j)=\varphi_k(t+j;u)\ \varphi_\ell(t+j;v)=e^{iju}\varphi_k(t;u)\ e^{ijv}\varphi_\ell(t;v)=e^{ij(u+v)}\xi_0(t).$$

Hence (3.2) holds true for the curves (4.1) with $\alpha=\frac{u+v}{2}$.

Let $-\pi<u\le\pi$ and $-\pi\le v<\pi$. We suppose that $u\ne0$, $v\ne0$ and $u+v\ne0$ (cases $u=0$, $v=0$ and $u+v=0$ have been already considered by Proposition 1). We divide the proof into two parts: Case 1 ($0<|u+v|\le\pi$) and Case 2 ($\pi<|u+v|<2\pi$). It is important that the change of the directions given by (1.12) does not change the case. Really, if the pair (u,v) is in Case 1, then $(v,-u-v)$ and $(-u-v,u)$ are also in Case 1. The same is true for Case 2, but here we take $2\pi-u-v$ or $-2\pi-u-v$ instead of $-u-v$, if $\pi<u+v<2\pi$ or $-2\pi<u+v<-\pi$, respectively.

On the other hand the formula $M_{k,\ell,m}(-x,-y)=M_{k,\ell,m}(x,y)$ gives that

$$\psi_{k,\ell,m}(-x,-y;-u,-v)=\overline{\psi_{k,\ell,m}(x,y;u,v)}$$

and therefore only $u+v>0$ is essential. We also suppose that $k+\ell>2$, because the case $k=\ell=1$ was considered by Proposition 2 and then Theorem 1 follows from (1.5).

Case 1. $0<u+v\le\pi$

From the definition (4.1) and (2.1) we have

$$\xi_0(t)=e^{-i\frac{j+s}{2}u}\varphi_k(\frac{j+s}{2}+t;u)\ e^{-i\frac{s}{2}v}\varphi_\ell(\frac{s}{2}+t;u). \tag{4.2}$$

Let $0<t<\frac{1}{2}$. Then using (1.6), (1.1) and (1.2) we receive that $\arg(e^{-i\frac{j+s}{2}u}\varphi_k(\frac{j+s}{2}+t;u))$ is between 0 and $\frac{u}{2}$, while $\arg(e^{-i\frac{s}{2}v}\varphi_\ell(\frac{s}{2}+t;v))$ is between 0 and $\frac{v}{2}$. Taking in account that $\frac{u+v}{2}>0$ we receive

$$\min\{0,\frac{u}{2},\frac{v}{2}\}<\arg\xi_0(t)<\max\{\frac{u+v}{2},\frac{u}{2},\frac{v}{2}\}. \tag{4.3}$$

Hence $\frac{u+v}{2}-\frac{\pi}{2}<\arg\xi_0(t)<\frac{\pi}{2}$, subject to $\frac{u+v}{2}\le\frac{\pi}{2}$. Now the application of Lemma 2 yields $\xi_m(t)\ne0$ for $0\le t\le\frac{1}{2}$ and $\xi_m(0)>0$. In particular (for $j=0$ and $s=0,1$), we receive the proposition of Theorem 1 in this case.

Case 2. $\pi < u + v < 2\pi$.

Here $0 < |u|, |v| \le \pi$ yields $u, v > 0$ and according to (1.6), (4.2) and (4.3) we have that $\arg \xi_0(t)$ increases and $0 < \arg \xi_0(t) < \frac{u+v}{2}$ for $0 < t < \frac{1}{2}$. The application of Lemma 3 gives that $\xi_m(t) \neq 0$ for $0 < t < \frac{1}{2}$ and therefore $\Gamma_m(t) \neq 0$ for $0 < |t| < \frac{1}{2}$.

In order to prove $\Gamma_m(0) > 0$ we set $\eta_k(t) = \xi_k(\ell, m, v, 2\pi - u - v, 0, -1; t)$ and $\zeta_\ell(t) = \xi_\ell(m, k, 2\pi - u - v, u, -1, 0; t)$ and define

$$r_0(k,\ell,m) = \psi_{k,\ell,m}(0,0;u,v) = \xi_m(0) = e^{-i(\pi - \frac{u}{2})}\eta_k(\frac{1}{2}),$$
$$r_1(k,\ell,m) = e^{-i\frac{u+v}{2}}\psi_{k,\ell,m}(\frac{1}{2},\frac{1}{2};u,v) = -\zeta_\ell(0) = e^{-i\frac{u+v}{2}}\xi_m(\frac{1}{2}),$$
$$r_2(k,\ell,m) = e^{-i(\frac{u}{2} - \pi)}\psi_{k,\ell,m}(\frac{1}{2},0;u,v) = \eta_k(0) = -e^{-i(\pi - \frac{v}{2})}\zeta_\ell(\frac{1}{2}),$$

where the last two equalities in the definitions of r_j are derived from (1.12) and (4.1). We apply Lemma 4 to the curves ξ_m, η_k and ζ_ℓ to receive that there is at least one positive number in each of the following pairs:

$$r_0(k,\ell,m) \quad \text{and} \quad r_1(k,\ell,m),$$
$$r_0(k,\ell,m) \quad \text{and} \quad r_2(k,\ell,m),$$
$$-r_1(k,\ell,m) \quad \text{and} \quad -r_2(k,\ell,m).$$

Comparing these results we conclude that $r_0(k,\ell,m) > 0$ and therefore $\Gamma_m(0) > 0$. This completes the proof of the theorem. ■

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