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LACUNARY INTERPOLATION BY SPLINES

(0,1,2,4) CASE

JOHN BURKETT AND A.K. VARMA

Lacunary interpolation at special knots has received considerable attention. In 1955, J. Suranyi and P. Turan [7] initiated the study of what they called $(0,2)$ interpolation. By $(0,2)$ interpolation, we mean the problem of finding the algbraic polynomial of degree $\leq 2n-1$, if it exists, whose values and second derivatives are prescribed on n given nodes. Several authors have studied interpolation of types $(0,1,3)$ and $(0,1,2,4)$ at special knots [see [3], [6], [8], [2]].

If the interpolating functions are not polynomials, lacunary interpolation may assume a very different character. In 1973 A. Meir and A. Sharma [4] obtained error bounds for some classes of quintic splines which interpolate $(0,2)$ data based on equidistant knots. In order to describe related later developments in the theory of lacunary interpolation by splines, let us denote by $S_{n,q}^{(r)}$ the class of splines $S(x)$ such that

$$\text{(1.1)}\qquad \begin{cases} i)\ S(x) \in c^r[0,1], \\ ii)\ S(x) \text{ is a polynomial of degree } q \text{ in} \\ \qquad [x_i, x_{i+1}],\ i = 0,1,\dots,n-1 \end{cases}$$

where

$$\text{(1.2)}\qquad 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$$

In 1979, Prasad and Varma [5] obtained the following results.

THEOREM A. *Given arbitrary numbers* $f(x_i)$, $i = 0,1,\dots,n$, $f^{(p)}(z_i)$, $i = 0,1,\dots,n-1$, $p = 0,3$, $2z_i = x_i + x_{i+1}$; $f'(x_0)$, $f'(x_n)$; *there exists a unique* $S_n(x) \in S^{(2)}_{n,5}$ *such that*

$$(1.3)\qquad \begin{cases} S_n(x_i) = f(x_i), i = 0,1,\dots,n;\ S'_n(x_0) = f'(x_0);\ S'_n(x_n) = f'(x_n) \\ S_n^{(p)}(z_i) = f^{(p)}(z_i),\ i = 0,1,\dots,n-1,\ p = 0,3, \end{cases}$$

From formula (3.1) and (3.6) in [5] pages 1076, we can explicitly obtain the polynomials in each of the pieces. The main advantage in the above scheme is that, contrary to the known results in spline interpolation theory, we do not have to solve a system of equations. After these results appeared, Prof. Schoenberg, in a personal conversation to the second author, asked the natural question: can you construct other lacunary polynomial splines from $S^{(r)}_{n,q}$ of character similar to that given in Theorem A but with r greater than 2. First, we looked at various lacunary splines of the class $C^{(3)}[0,1]$, involving different function data at nodes and midpoints, but none turned out to be local in character. Next, we turned to the case $r = 4$. Here, we obtained results analogous to those in Theorem A. At this point, we would like to remark that when the second author visited the University of Alberta a few years ago, Prof. A. Meir showed a keen interest in the above problem. The authors are grateful to Prof. A. Meir. Our main results are as follows:

THEOREM 1. *Given arbitrary numbers* $f^{(j)}(x_i)$, $i = 0,1,\dots,n$; $j = 0,1,2$; $f^{(iv)}(z_i)$, $i = 0,1,\dots,n-1$ *where* $2z_i = x_i + x_{i+1}$, $f'''(x_0)$, $f'''(x_n)$ *there exists a unique* $S_n(x) \in S^{(4)}_{n,8}$ *such that*

$$(1.4)\qquad \begin{cases} S_n^{(j)}(x_i) = f^{(j)}(x_i),\ i = 0,1,\dots,n;\ j = 0,1,2 \\ S_n^{(iv)}(z_i) = f^{(iv)}(z_i),\ i = 0,1,\dots,n-1 \\ S'''_n(x_0) = f'''(x_0),\ S'''_n(x_n) = f'''(x_n). \end{cases}$$

Proof of Theorem 1 is given in section 3. We refer to this case as $(0,1,2,4)$ interpolation by splines. From the construction of these splines as given by (3.1) and (3.4), the local character of these splines becomes evident. Next, we state the following convergence results.

THEOREM 2. *Let $f \in c^l[0,1], \ell \geq 4$. Then, for the unique spline $S_n(x)$ associated with f and satisfying (1.4) we have*

$$\left|S_n^{(r)}(x) - f^{(r)}(x)\right| \leq c_{r,l}\delta^{l-r}w(f^{(l)},\delta),\ r = 0,1,2,3,4,\ l = 4,5,6,7,8. \tag{1.5}$$

Also, for $f \in c^9[0,1]$

$$\left|S_n^{(r)}(x) - f^{(r)}(x)\right| \leq c_{r,9}\delta^{9-r} \max_{0\leq x\leq 1} |f^{(9)}(x)|,\ \delta = \max_{i=0,1,\ldots,n-1} h_i \tag{1.6}$$

Other interesting contributions to the subject of lacunary polynomial spline interpolation are due to S. Demko [1] T. Fawzy and L. L. Schamaker [2]. See also the references mentioned in [3].

2. Preliminaries. If $Q(z)$ is any polynomial of degree eight on $[0,1]$, then we have

$$\begin{aligned} Q(z) &= Q(0)A_o(z) + Q(1)A_0(1-z) + Q'(0)B_0(z) \\ &- Q'(1)B_0(1-z) + Q''(0)C_0(z) + Q''(1)C_2(1-z) \\ &+ Q'''(0)D_0(z) - Q'''(1)D_0(1-z) + Q^{(iv)}(\frac{1}{2})E_1(z), \end{aligned} \tag{2.1}$$

where

$$A_0(z) = 1 - 35z^4 + 84z^5 - 70z^6 + 20z^7 \tag{2.2}$$

$$B_0(z) = z - \frac{70}{3}z^4 + \frac{175}{3}z^5 - 56z^6 + \frac{70}{3}z^7 - \frac{10}{3}z^8, \tag{2.3}$$

$$C_0(z) = \frac{z^2}{2} - \frac{20}{3}z^4 + \frac{50}{3}z^5 - \frac{35}{2}z^6 + \frac{26}{3}z^7 - \frac{5}{3}z^8, \tag{2.4}$$

$$D_0(z) = \frac{z^3}{6} - \frac{5}{6}z^4 + \frac{5}{3}z^5 - \frac{5}{3}z^6 + \frac{5}{6}z^7 - \frac{z^8}{6}, \tag{2.5}$$

$$E_1(z) = \frac{z^4}{9} - \frac{4}{9}z^5 + \frac{2}{3}z^6 - \frac{4}{9}z^7 + \frac{z^8}{9}. \tag{2.6}$$

Next, from Taylors formula we have

$$f^{(l)}(x_{i+1}) = \sum_{j=l}^{p-1} \frac{f^{(j)}(x_i)}{(j-l)!}h_i^{j-l} + \frac{f^{(p)}(\eta_{1,l})h_i^{p-l}}{(p-l)!} \tag{2.7}$$

$$f^{(l)}(x_{i-1}) = \sum_{j=l}^{p-1} \frac{f^{(j)}(x_i)}{(j-l)!}(-h_{i-1})^{j-l} + \frac{f^{(p)}(\eta_{2,l})(-h_{i-l})^{p-l}}{(p-l)!} \tag{2.8}$$

$$f^{(iv)}(z_i) = \sum_{j=4}^{p-1} \frac{f^{(j)}(x_i)}{(j-4)!}\left(\frac{h_i}{2}\right)^{j-4} + \frac{f^{(p)}(\eta_3)(\frac{h_i}{2})^{p-4}}{(p-4)!} \tag{2.9}$$

$$f^{(iv)}(z_{i-1}) = \sum_{j=4}^{p-1} \frac{f^{(j)}(x_i)\left(-\frac{h_{i-1}}{2}\right)^{j-4}}{(j-4)!} + \frac{f^{(p)}(\eta_4)\left(-\frac{h_{i-1}}{2}\right)^{p-4}}{(p-4)!} \tag{2.10}$$

where $x_i < \eta_{1,l} < x_{i+1}$, $x_{i-1} < \eta_{2,l} < x_i$, $x_i < \eta_3 < z_i$, and $z_{i-1} < \eta_4 < x_i$.

Let $t = \dfrac{x - x_i}{h_i}$

$$\begin{aligned} f(x) &= f_i + (x - x_i)f_i' + \frac{(x-x_i)^2}{2!}f_i'' + \frac{(x-x_i)^3}{3!}f_i''' + \frac{(x-x_i)^4}{4!}f^{(iv)}(\eta_0) \\ &= f_i + th_if_i' + \frac{(th_i)^2}{2!}f_i'' + \frac{(th_i)^3}{3!}f_i''' + \frac{(th_i)^4}{4!}f^{(iv)}(\eta_0) \end{aligned} \tag{2.11}$$

LEMMA 2.1:' The following identities are valid:

$$
(2.12)\qquad \begin{cases} A_0(t)+A_0(1-t)=1,\ A_0(1-t)+B_0(t)-B_0(1-t)=t, \\ A_0(1-t)-2B_0(1-t)+2C_0(t)+C_0(1-t)=t^2, \\ A_0(1-t)-3B_0(1-t)+6C_0(1-t)+6D_0(t)-6D_0(1-t)=t^3 \\ A_0(1-t)-4B_0(1-t)+12C_0(1-t)-24D_0(1-t)+24E_1(t)=t^4 \\ A_0(1-t)-5B_0(1-t)+20C_0(1-t)-60D_0(1-t)+60E_1(t)=t^5 \\ A_0(1-t)-6B_0(1-t)+30C_0(1-t)-120D_0(1-t)+90E_1(t)=t^6 \\ A_0(1-t)-7B_0(1-t)+42C_0(1-t)-210D_0(1-t)+105E_1(t)=t^7 \\ A_0(1-t)-8B_0(1-t)+56C_0(1-t)-336D_0(1-t)+105E_1(t)=t^8 \end{cases}
$$

The identities can be derived from (2.1) and the uniqueness of this interpolation formula.

3. Proof of Theorem 1. Let $x=x_i+th_i$, $h_i=x_{i+1}-x_i$, $0\le t\le 1$. Here, our object is to prove that there exists a unique $S_n(x)\in S^{(4)}_{n,8}$ satisfying the conditions of Theorem 1. We define $S_n(x)$ by the following representation.

$$
(3.1)\qquad \begin{aligned} S_n(x) &= f_iA_0(t)+f_{i+1}A_0(1-t)+h_if_i'B_0(t)-h_if_{i+1}'B_0(1-t) \\ &+h_i^2f_i''C_0(t)+h_i^2\,f_{i+1}''C_0(1-t)+h_i^3S_n'''(x_i)D_0(t) \\ &-h_i^3S_n'''(x_{i+1})D_0(1-t)+h_i^4\,f^{(iv)}_{i+\frac12}E_1(t) \end{aligned}
$$

where $f_i^{(p)}=f^{(p)}(x_i)$, $f_{i+1}^{(p)}=f^{(p)}(x_{i+1})$, $f^{(iv)}_{i+\frac12}=f^{(iv)}(z_i)$. Next, we set

$$
(3.2)\qquad S_n'''(0)=f'''(0),\ S_n'''(1)=f'''(1).
$$

Clearly, $S_n(x)$ as defined by (3.1) belongs to $C^3[0,1]$, no matter how we choose $S_n'''(x_i)$, $i=1,2,\ldots,n-1$. They are uniquely determined by the condiditons

$$
(3.3)\qquad S_n^{(iv)}(x_i+)=S_n^{(iv)}(x_i-),\ i=1,2,\ldots,n-1.
$$

A simple computation shows that

$$
\begin{aligned}
5h_i^3h_{i-1}^3(h_i+h_{i-1})S_n'''(x_i) &= 210\left(h_{i-1}^4 f_{i+1} - h_i^4 f_{i-1}\right.\\
&\left.+(h_i^4 - h_{i-1}^4)f_i\right) - 70h_ih_{i-1}\left(h_{i-1}^3 f_{i+1}' + h_i^3 f_{i-1}'\right.\\
&\left.+2(h_i^3 + h_{i-1}^3)f_i'\right) + 5h_i^2h_{i-1}^2\left(h_{i-1}^2 f_{i+1}'' - h_i^2 f_{i-1}''\right.\\
&\left.+8(h_i^2 - h_{i-1}^2)f_i''\right) + \frac{2}{3}h_i^4h_{i-1}^4\left(f_{i+\frac{1}{2}}^{(iv)} - f_{i-\frac{1}{2}}^{(iv)}\right).
\end{aligned}
\tag{3.4}
$$

Thus, $S_n(x)$ defined by (3.1), (3.2) and (3.4) satisfy all the conditions of the theorem. This completes the proof of Theorem 1.

4. Auxilary Lemma. In order to prove Theorem 2 we need the following:

LEMMA 4.1: Let $f \in C^{(p)}[0,1]$, $4 \le p \le 8$, then $|f'''(x_i) - S_n'''(x_i)| \le \dfrac{c_p h_i h_{i-1}(h_i^{p-4} + h_{i-1}^{p-4})}{h_i + h_{i-1}} w(f^{(p)}, \delta)$ and for $f \in c^9[0,1]$

$$|f'''(x_i) - S_n'''(x_i)| \le \frac{c_p h_i h_{i-1}(h_i^5 + h_{i-1}^5)}{h_i + h_{i-1}} \max_{0 \le x \le 1} |f^{(9)}(x)|,$$

where $\delta = \max\limits_{i=0,1,\ldots,n-1}(h_i)$ and $w(f,\delta)$ denotes the modulus of continuity of f.

Proof of this lemma is a simple consequence of (2.7)-(2.10), and (3.4).

5. Proof of Theorem 2. Let $x = x_i + th_i, h_i = x_{i+1} - x_i$ and $f \in c^4[0,1]$. Then using (3.1) we obtain

$$S_n(x) - f(x) = \lambda_i(t) + \mu_i(t) \tag{5.1}$$

where

$$\lambda_i(t) = h_i^3\left(S_n'''(x_i) - f'''(x_i)\right)D_0(t) - h_i^3\left(S_n'''(x_{i+1}) - f'''(x_{i+1})\right)D_0(1-t) \tag{5.2}$$

and

$$\mu_i(t) = f_i A_0(t) + f_{i+1} A_0(1-t) + h_i\left(f_i' B_0(t) - f_{i+1}' B_0(1-t)\right) +$$

$$(5.3) \qquad + h_i^2\left(f_i'' C_0(t) + f_{i+1}'' C_0(1-t)\right) + h_i^3\left(f_i''' D_0(t) - f_{i+1}''' D_0(1-t)\right)$$

$$+ h_i^4 f_{i+\frac{1}{2}}^{(iv)} E_1(t) - f(x)$$

Next, on using (5.2) and Lemma 4.1 we obtain

$$(5.4) \qquad |\lambda_i(t)| \le C_1 h_i^4 \max_{0\le t\le 1} |D_0(t)| w(f^{(4)}, \delta).$$

Also, from (2.7), (2.11), (2.12) and (5.3) we have

$$\mu_i(t) = f_i A_0(t) + A_0(1-t)\left[\sum_{j=0}^{3} \frac{f_i^{(j)} h_i^j}{j!} + \frac{f^{(iv)}(\eta_{1,0})}{4!} h_i^4\right]$$

$$+ h_i f_i' B_0(t) - h_i\left[\sum_{j=1}^{3} \frac{f_i^{(j)} h_i^{j-1}}{(j-1)!} + \frac{f^{(iv)}(\eta_{1,1})}{3!} h_i^3\right] B_0(1-t)$$

$$+ h_i^2 f_i'' C_0(t) + h_i^3 f_i''' D_0(t) + h_i^4 f_{i+\frac{1}{2}}^{(iv)} E_1(t)$$

$$+ h_i^2 C_0(1-t)\left[\sum_{j=2}^{3} \frac{f_i^{(j)}}{(j-2)!} h_i^{j-2} + \frac{f^{(iv)}(\eta_{1,2})}{2!} h_i^2\right]$$

$$- h_i^3\left[f_i''' + h_i f^{(iv)}(\eta_{1,3})\right] D_0(1-t)$$

$$- \left(f_i + t h_i f_i' + \frac{t^2 h_i^2}{2!} f_i'' + \frac{t^3 h_i^3}{3!} f_i''' + \frac{t^4 h_i^4 f^{(iv)}(\eta_0)}{4!}\right)$$

$$= h_i^4\left[\frac{f^{(iv)}(\eta_{1,0})}{24} A_0(1-t) - \frac{f^{(iv)}(\eta_{1,1})}{6} B_0(1-t)\right.$$

$$+ \frac{f^{(iv)}(\eta_{1,2})}{2} C_0(1-t) - f^{(iv)}(\eta_{1,3}) D_0(1-t)$$

$$\left. + f^{(iv)}(z_i) E_1(t) - \frac{t^4}{24} f^{(iv)}(\eta_0)\right]$$

on using the identity given in (2.12) corresponding to t^4 we have

$$\begin{aligned}\mu_i(t) = h_i^4 \Bigg[& \left(\frac{f^{(iv)}(\eta_{1,0}) - f^{(iv)}(\eta_0)}{24}\right) A_0(1-t) + \\ & + \left(\frac{f^{(iv)}(\eta_0) - f^{(iv)}(\eta_{1,1})}{6}\right) B_0(1-t) \\ & + \left(\frac{f^{(iv)}(\eta_{1,2}) - f^{(iv)}(\eta_0)}{2}\right) C_0(1-t) \\ & + \left(f^{(iv)}(\eta_0) - f^{(iv)}(\eta_{1,3})\right) D_0(1-t) \\ & + \left(f^{(iv)}(z_i) - f^{(iv)}(\eta_0)\right) E_1(t)\Bigg].\end{aligned}$$

From this we obtain

$$|\mu_i(t)| \le C_6 h_i^4 \, w(f^{(iv)}, \delta) \tag{5.5}$$

on combining (5.1), (5.4) and (5.5) we obtain (1.5) for $r = 0$, $l = 4$. Proof in the other cases are similar. We omit the details.

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