CONSTRUCTIVE THEORY
OF FUNCTIONS, Varna '91
Sofia, 1992, pp. 19-29

ERROR BOUNDS FOR COMPOUND QUADRATURE RULES IN THE SPACE OF RIEMANN INTEGRABLE FUNCTIONS

B. Büttgenbach, P. Lünebach, G. Lüttgens*, R. J. Nessel

This paper is devoted to some comments and extensions of results, previously obtained by the Bulgarian school of approximation, and thus heavily relies to work of Vasil Popov. Since the remainders of some Newton–Cotes formulas can explicitly be represented in terms of differences of that order which is characteristic for the exactness of the rule, error bounds for the associated compound quadrature processes via τ–moduli of corresponding orders may now be established in a completely elementary way, including good constants. In fact, some of the constants resulting from this approach are best possible. In the general case a quantitative extension of a familiar Pólya criterion, concerned with the convergence of quadrature procedures in the space of Riemann integrable functions, is suggested which admits a systematic treatment of error bounds for compound quadrature rules, due to K. Ivanov.

Let $C = C[a,b]$, R or B be the spaces of real-valued functions, defined on the compact interval $[a,b]$ of the real axis $\mathbb{R}$ which are continuous, Riemann integrable or bounded, respectively. Obviously, $C \subset R \subset B$, and under the familiar *sup*–norm $\|f\|_B := \sup\{|f(u)| : a \le u \le b\}$ each of these spaces is a Banach space. Given $f \in R[0,1]$, consider the elementary midpoint, trapezoidal, Simpson, 3/8–, and Milne rule, i.e.,

$$Q^{Mi} f := f(1/2), \qquad Q^{Tr} f := [f(0) + f(1)]/2, \tag{1}$$

$$Q^{Si} f := [f(0) + 4\,f(1/2) + f(1)]/6,$$

$$Q^{3/8} f := [f(0) + 3\,f(1/3) + 3\,f(2/3) + f(1)]/8,$$

$$Q^{Mil} f := [7\,f(0) + 32\,f(1/4) + 12\,f(1/2) + 32\,f(3/4) + 7\,f(1)]/90,$$

for the approximate calculation of the integral $\int_0^1 f(u)\,du$. Then for the remainders $Rf := Qf - \int f$ there hold true the representations

$$R^{Mi} f = \int_0^{1/2} [-\Delta_h^2 f(\tfrac{1}{2} - h)]\,dh, \qquad R^{Tr} f = \int_0^{1/2} [\Delta_h^2 f(0) + \Delta_{-h}^2 f(1)]\,dh, \tag{2}$$

$$R^{Si} f = \frac{2}{3} \int_0^{1/4} [\Delta_h^4 f(0) + \frac{2}{3} \Delta_h^4 f(\tfrac{1}{2} - 2h) + \Delta_{-h}^4 f(1)]\,dh,$$

*Supported by Deutsche Forschungsgemeinschaft Grant No. Ne 171/11-1

$$R^{3/8}f = \frac{3}{4}\int_0^{1/6} [\Delta_h^4 f(0) + 3\,\Delta_h^4 f(\frac{1}{3}) + 3\,\Delta_{-h}^4 f(\frac{2}{3}) + \Delta_{-h}^4 f(1)]\,dh,$$

$$R^{Mil}f = \int_0^{1/8} \Big(\frac{28}{45}[\Delta_h^6 f(0) + \Delta_{-h}^6 f(1)] + \frac{232}{75}[\Delta_h^6 f(\frac{1}{4}) + \Delta_{-h}^6 f(\frac{3}{4})]$$

$$+\frac{28}{675}[\Delta_h^6 f(\frac{1}{4}-h) + \Delta_{-h}^6 f(\frac{3}{4}+h)] + \frac{8}{225}[\Delta_h^6 f(\frac{1}{2}-2h) + \Delta_{-h}^6 f(\frac{1}{2}+2h)]\Big)\,dh,$$

where $\Delta_h^r f(x) := \Delta_h(\Delta_h^{r-1} f(x))$, $\Delta_h f(x) := f(x+h) - f(x)$, $r \in \mathbf{N}$ (set of natural numbers) denotes the r-th difference with increment $h \in \mathbf{R}$. Note that the differences occurring precisely correspond to the order of exactness of the relevant rule. Representations for R^{Mi}, R^{Tr} are well–known and used by several authors (cf. [12], [14], [16, pp. 42, 44, 54]), concerning R^{Si}, $R^{3/8}$ and R^{Mil} see [4]. Of course it would be interesting to have a general procedure to establish representations like those of (2) for a wide class of elementary quadrature formulas

$$R^{el}f := Q^{el}f - \int_0^1 f(u)\,du := \sum_{i=1}^{j} a_i f(x_i) - \int_0^1 f(u)\,du, \tag{3}$$

where $0 \le x_1 < \ldots < x_j \le 1$ and the formula is assumed to be exact on $\mathcal{P}_{r-1}$ for some $r \in \mathbf{N}$, i.e., $R^{el}p = 0$ for all $p \in \mathcal{P}_{r-1}$, the set of algebraic polynomials of degree $(r-1)$. But so far we have to leave this as an open problem.

Nevertheless, the rules (1), though particular, generate those compound quadrature processes, most commonly used in the applications. Indeed, if for the remainders (3) of an elementary rule Q^{el} there holds true the representation

$$R^{el}f = \sum_{l=1}^{m} \int_0^s b_l\,\Delta_h^r f(y_l + c_l\,h)\,dh \tag{4}$$

for some $m, r \in \mathbf{N}$, $0 < s < 1$, $b_l, c_l \in \mathbf{R}$ and $y_l \in \{x_i : 1 \le i \le j\}$, then it follows for the compound quadrature process

$$R_{(n)}f := Q_{(n)}f - \int_a^b f(u)\,du := \frac{b-a}{n}\sum_{k=1}^{n}\sum_{i=1}^{j} a_i\,f(a + (k-1+x_i)\,\frac{b-a}{n}) - \int_a^b f(u)\,du \tag{5}$$

that for $f \in R[a,b]$ (see [4])

$$R_{(n)}f = \sum_{k=1}^{n}\sum_{l=1}^{m}\int_0^{s(b-a)/n} b_l\,\Delta_h^r f(a + (k-1+y_l)\,\frac{b-a}{n} + c_l\,h)\,dh.$$

Observing that any backward difference Δ_{-h} may also be rewritten as a forward one Δ_h, it is obvious that the representations (2) are all of the form (4) (note that $y_{l_1} = y_{l_2}$ for $l_1 \ne l_2$ is possible).

Once representations of type (2) are available, estimates of the remainders versus the r-th τ–modulus

$$\tau_r(\delta, f) := \tau_r(\delta, f; [a,b]) := \bar{\int_a^b} \omega_r(\delta, f, x)\,dx,$$

$$\omega_r(\delta, f, x) := \omega_r(\delta, f, x; [a,b]) := \sup\{|\Delta_h^r f(u)| : u, u+rh \in U_\delta(x)\},$$

$$U_\delta(x) := U_\delta(x; [a,b]) := [x-\delta, x+\delta] \cap [a,b],$$

are immediate. This notion, independently introduced by P. P. Korovkin and Bl. Sendov around 1968, has turned out to be very useful in approximation theory. Here it is given for an arbitrary $f \in B[a,b]$ as the upper Riemann integral $\bar{\int}$ over the local modulus of continuity $\omega_r(\delta, f, x)$. It follows that (see [16, p. 11])

$$f \in R[a,b] \iff \tau_1(\delta, f) = o(1) \qquad (\delta \to 0+),$$

quite parallel to the classical assertion

$$f \in C[a,b] \iff \omega_1(\delta, f) = o(1) \qquad (\delta \to 0+)$$

in terms of the ordinary modulus of continuity

$$\omega_r(\delta, f) := \sup\{|\Delta_h^r f(u)| \, : \, u, u + rh \in [a,b], |h| \le \delta\}.$$

For further elementary properties of the τ–modulus one may consult the treatment in [16] (and the literature cited there). Thus it is an immediate consequence of the definitions that

$$\tau_r(\delta, f) \le (b-a)\, \|\omega_r(\delta, f, x)\|_B = (b-a)\, \omega_r(2\delta/r, f) \qquad (f \in B, \delta > 0). \tag{6}$$

Let us also recall the familiar fact that in the special case of affine transformations the usual integration by substitution remains valid for upper Riemann integrals. Indeed, if $A(u) := a + (u-c)(b-a)/(d-c)$ for $u \in [c,d]$ is the affine transformation of the interval $[c,d]$ onto $[a,b]$, then for every $g \in B[a,b]$

$$\bar{\int_a^b} g(x)\, dx = \bar{\int_c^d} g(A(u))\, A'(u)\, du. \tag{7}$$

Corollary 1: *For $f \in R[0,1]$ there hold true the estimates*

$$|R^{Mi} f| \le \frac{1}{2} \tau_2 (\frac{1}{2}, f), \qquad |R^{Tr} f| \le \tau_2 (\frac{1}{2}, f),$$

$$|R^{Si} f| \le \frac{4}{9} \tau_4 (\frac{1}{2}, f), \qquad |R^{3/8} f| \le \frac{9}{8} \tau_4 (\frac{1}{3}, f),$$

$$|R^{Mil} f| \le \frac{286}{135} \tau_6 (\frac{3}{8}, f).$$

Proof : Concerning the estimates for R^{Mi}, R^{Tr} see [16, pp. 42, 54]. Obviously,

$$|\Delta_{\pm h}^r f(c)| \le \omega_r (\frac{\alpha r}{2}, f, c \pm \frac{rh}{2})$$

for $h \in [0, \alpha]$ such that $c, c + rh \in [a,b]$, and

$$|\Delta_{\pm h}^r f(c \mp nh)| \le \omega_r (\frac{\alpha r}{2}, f, c \mp nh \pm \frac{\alpha r}{2})$$

for $h \in [0, \alpha]$, $n \leq r$ such that $c - nh, c - nh + rh \in [a, b]$. Therefore in view of the representations (2)

$$|R^{Si} f| \leq \frac{2}{9} \int_0^{1/4} \left[3\,|\Delta_h^4 f(0)| + |\Delta_h^4 f(\frac{1}{2} - 2h)| + |\Delta_{-h}^4 f(\frac{1}{2} + 2h)| + 3\,|\Delta_{-h}^4 f(1)|\right] dh$$

$$\leq \frac{2}{9} \bar{\int_0^{1/4}} \left[3\,\omega_4(\frac{1}{2}, f, 2h) + \omega_4(\frac{1}{2}, f, 1 - 2h) + \omega_4(\frac{1}{2}, f, 2h) + 3\,\omega_4(\frac{1}{2}, f, 1 - 2h)\right] dh$$

$$= \frac{4}{9} \left[\bar{\int_0^{1/2}} \omega_4(\frac{1}{2}, f, h)\, dh + \bar{\int_{1/2}^{1}} \omega_4(\frac{1}{2}, f, h)\, dh\right] = \frac{4}{9} \tau_4(\frac{1}{2}, f),$$

$$|R^{3/8} f| \leq \frac{3}{4} \bar{\int_0^{1/6}} \left[\omega_4(\frac{1}{3}, f, 2h) + \omega_4(\frac{1}{3}, f, 1 - 2h)\right] dh$$

$$+ \frac{9}{4} \int_0^{1/6} \left[|\Delta_{-h}^4 f(\frac{1}{3} + 4h)| + |\Delta_h^4 f(\frac{2}{3} - 4h)|\right] dh$$

$$\leq \frac{3}{8} \left[\bar{\int_0^{1/3}} \omega_4(\frac{1}{3}, f, h)\, dh + \bar{\int_{2/3}^{1}} \omega_4(\frac{1}{3}, f, h)\, dh\right]$$

$$+ \frac{9}{4} \bar{\int_0^{1/6}} \left[\omega_4(\frac{1}{3}, f, 4h) + \omega_4(\frac{1}{3}, f, 1 - 4h)\right] dh$$

$$= \left[\frac{3}{8} \bar{\int_0^{1/3}} + \frac{3}{8} \bar{\int_{2/3}^{1}} + \frac{9}{16} \bar{\int_0^{2/3}} + \frac{9}{16} \bar{\int_{1/3}^{1}}\right] \omega_4(\frac{1}{3}, f, h)\, dh \leq \frac{9}{8} \tau_4(\frac{1}{3}, f).$$

The estimate for R^{Mil} may be deduced analogously. ∎

The constants occurring in the assertions of Corollary 1 may be compared with those obtained in [4] in connection with corresponding estimates versus $\omega_r(\delta, f)$ (cf. (6)). From a good estimate for the elementary rule (3) one immediately obtains a similar one for the compound process (5). Indeed,

Theorem 1: *Given the elementary rule* (3), *suppose that there holds true the estimate*

$$|R^{el} g| \leq c\, \tau_r(\delta, g; [0, 1]) \qquad (g \in R[0, 1]).$$

Then one has for $f \in R[a, b]$

$$|R_{(n)} f| \leq c\, \tau_r(\delta(b - a)/n, f; [a, b]).$$

Proof: Setting $g(x) := f(a + x(b-a))$ for $f \in R[a,b]$, one has

$$\omega_r(\delta, g, x; [0,1]) = \sup\{|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(a + y(b-a) + kh(b-a))| : \qquad y, y+rh \in U_\delta(x;[0,1])\}$$

$$= \sup\{|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} f(z+kh)| : \qquad z, z+rh \in [a+(b-a)(x-\delta), a+(b-a)(x+\delta)] \cap [a,b]\}$$

$$= \omega_r(\delta(b-a), f, a+(b-a)x; [a,b]).$$

With $g_k(x) := f(a + (k-1+x)(b-a)/n)$ and $A_k := a + k(b-a)/n$ it therefore follows that

$$\omega_r(\delta, g_k, x; [0,1]) = \omega_r\Big(\delta\frac{b-a}{n}, f, a+(k-1+x)\frac{b-a}{n}; [A_{k-1}, A_k]\Big).$$

This as well as the assumption and (7) then imply for the compound process (5) that

$$|R_{(n)}f| \le \frac{b-a}{n}\sum_{k=1}^{n}|R^{el}g_k| \le c\,\frac{b-a}{n}\sum_{k=1}^{n}\tau_r(\delta, g_k; [0,1])$$

$$= c\,\frac{b-a}{n}\sum_{k=1}^{n}\int_0^1 \omega_r\Big(\delta\,\frac{b-a}{n}, f, a+(k-1+x)\,\frac{b-a}{n}; [A_{k-1}, A_k]\Big)\,dx$$

$$\le c\,\frac{b-a}{n}\sum_{k=1}^{n}\int_0^1 \omega_r\Big(\delta\,\frac{b-a}{n}, f, a+(k-1+x)\,\frac{b-a}{n}; [a,b]\Big)\,dx$$

$$= c\,\tau_r\Big(\delta\,\frac{b-a}{n}, f; [a,b]\Big). \quad \blacksquare$$

Corollary 2: *For $f \in R[a,b]$ there hold true the estimates*

$$|R^{Mi}_{(n)}f| \le \frac{1}{2}\tau_2\Big(\frac{b-a}{2n}, f\Big), \qquad |R^{Tr}_{(n)}f| \le \tau_2\Big(\frac{b-a}{2n}, f\Big),$$

$$|R^{Si}_{(n)}f| \le \frac{4}{9}\tau_4\Big(\frac{b-a}{2n}, f\Big), \qquad |R^{3/8}_{(n)}f| \le \frac{9}{8}\tau_4\Big(\frac{b-a}{3n}, f\Big),$$

$$|R^{Mil}_{(n)}f| \le \frac{286}{135}\tau_6\Big(\frac{3(b-a)}{8n}, f\Big).$$

Concerning the estimates for $R^{Mi}_{(n)}$, $R^{Tr}_{(n)}$ see [16, pp. 42, 54, 65], for corresponding ones versus $\omega_r(\delta, f)$ (and continuous functions) see [4]. It may be mentioned that the previous elementary approach to the estimates of Corollary 2 delivers rather good constants which in some cases are even best possible. Indeed, it was shown in [8] (cf. [16, p. 52]) that

$$A^{Mi}_n\binom{C[0,1]}{R[0,1]} := \sup\Big\{\frac{|R^{Mi}_{(n)}f|}{\tau_2(1/2n, f)}; f \in \binom{C}{R}\setminus \mathcal{P}_1\Big\} = \frac{1}{2},$$

which, by the way, also holds true for those apparently smaller constants (cf. (6)), determined via the analogous estimates versus $\omega_2(1/2n, f)$ (see [5]). Moreover, for the corresponding constants with regard to the trapezoidal rule one also has that (see [5])

$$A_n^{Tr}\binom{C[0,1]}{R[0,1]} := \sup\left\{\frac{|R_{(n)}^{Tr} f|}{\tau_2(1/2n, f)} : f \in \binom{C}{R} \setminus \mathcal{P}_1\right\} = 1.$$

The preceding approach to error bounds is elementary, in fact leads to good constants, but on the other hand is rather specific. It may therefore be worthwhile to conclude with the observation that even in the present frame of Riemann integrable functions which nevertheless is quite natural in connection with quadrature, one may proceed via standard K-functional techniques to deduce error bounds for arbitrary quadrature processes, this time, however, without reasonable candidates for constants. Let us develop the matter in the context of the familiar Pólya criterion on the convergence for Riemann integrable functions.

To this end, for any $\delta > 0$ consider the local supremum

$$M(\delta, f, x) := \sup\{|f(y)| : y \in U_\delta(x; [a,b])\} \tag{8}$$

of $f \in B[a,b]$ and introduce the following set of norms

$$\|f\|_\delta := \int_a^b M(\delta, f, x)\,dx \qquad (\delta > 0) \tag{9}$$

on the space B. Among the elementary properties note that (cf. [10])

$$\|f\|_{2\delta} \le 2\,\|f\|_\delta, \qquad \|f\|_{\lambda\delta} \le 2(1+\lambda)\,\|f\|_\delta \qquad (\lambda > 0), \tag{10}$$

$$M(\delta, f, x) \in R[a,b] \text{ for } f \in R[a,b].$$

Inspired by work of V. Popov (see [1], [15]) one may then introduce a K-functional via

$$K_r(\delta, f) := \inf_{g \in ACR^{(r)}} [\|f-g\|_\delta + \delta^r \int_a^b |g^{(r)}(u)|\,du], \tag{11}$$

where $ACR^{(r)} := ACR^{(r)}[a,b]$ is the set of r-times absolutely continuous functions with r-th derivative being Riemann integrable. Let us mention that instead of the family of norms (9) the notion of a K-functional in [15] was based upon

$$\|f\|_{l_Z} := \sum_{i=1}^m |f(x_{i-1})|(x_i - x_{i-1}) \qquad (f \in B[a,b]),$$

where $Z = \{x_0, \dots, x_m\}$, $a = x_0 < \dots < x_m = b$ denotes a partition of $[a,b]$. This leads to the candidate (see [15])

$$\tilde{K}_r(\delta, f) := \inf_{g \in ACR^{(r)}} \Big[\sup_{\|Z\| \le \delta} \Big(\|f-g\|_{l_Z} + \delta^r \int_a^b |g^{(r)}(u)|\,du\Big)\Big]$$

which, however, can be shown to be equivalent to (11) (see [10]). The approach via (9,11) seems to have some technical advantages. Thus it follows by the routine argument (cf. [1], [3], [9], [15]) that for $f \in R[a,b]$, $\delta > 0$, $r \in \mathbf{N}$ there holds true the equivalence

$$2^{-r}\tau_r(\delta, f) \le K_r(\delta, f) \le C_r \tau_r(\delta, f). \tag{12}$$

Based upon their Banach–Steinhaus theorem with rates, Butzer–Scherer–Westphal [6] suggested the following quantitative extension of the familiar Pólya criterion (see [13], [2, p. 32]), concerned with the convergence of quadrature processes on the Banach space $C[a,b]$: Given a sequence of (arbitrary) quadrature rules

$$Q_n f := \sum_{k=1}^{n} a_{kn} f(x_{kn}) \qquad (n \in \mathbf{N}) \tag{13}$$

with knots $a \le x_{1n} < \ldots < x_{nn} \le b$ and weights $0 \ne a_{kn} \in \mathbf{R}$, the conditions

$$|Q_n f| \le M_1 \|f\|_B \qquad (f \in C[a,b]),$$

$$|Q_n g - \int_a^b g(u)\,du| \le M_2 \varphi_n^r \|g^{(r)}\|_B \qquad (g \in C^{(r)}[a,b])$$

are necessary and sufficient for the error estimate on C

$$|Q_n f - \int_a^b f(u)\,du| \le M_3 \omega_r(\varphi_n, f) \qquad (f \in C[a,b]).$$

Here $\{\varphi_n\}$ is a decreasing nullsequence and $C^{(r)}[a,b]$ denotes the set of r-times continuously differentiable functions. The following result (cf. [1]) may therefore be considered as a quantitative extension of the corresponding Pólya criterion (see [13], also [7] for a functionalanalytical approach), concerned with the convergence of quadrature procedures for Riemann integrable functions.

Theorem 2: *Given a sequence of (arbitrary) quadrature rules* (13), *the conditions*

(i) $\qquad |Q_n f| \le M_1 \|f\|_{\varphi_n} \qquad\qquad (f \in R[a,b]),$ (14)

(ii) $\qquad |Q_n g - \int_a^b g(u)\,du| \le M_2 \varphi_n^r \int_a^b |g^{(r)}(u)|\,du \qquad (g \in ACR^{(r)}[a,b])$

for some decreasing nullsequence $\{\varphi_n\}$ are necessary and sufficient for the error estimate on R

$$|Q_n f - \int_a^b f(u)\,du| \le M_3 \tau_r(\varphi_n f) \qquad (f \in R[a,b]). \tag{15}$$

Proof : <u>(14)→(15):</u> In view of the definition (8) for any $f \in R$, $\delta > 0$ (cf. (10))

$$\int_a^b |f(x)|\,dx \le \int_a^b M(\delta, f, x)\,dx =: \|f\|_\delta.$$

Therefore the assumption (14)(i) implies (with $R_n := Q_n - \int$)

$$|R_n f| \le |Q_n f| + \int_a^b |f(x)|\,dx \le M_1 \|f\|_{\varphi_n} + \|f\|_\delta = (1+M_1)\|f\|_{\varphi_n} \qquad (16)$$

by setting $\delta = \varphi_n$. On the other hand, for arbitrary $f \in R$ and $g \in ACR^{(r)}$ one has by (14, 16) that

$$\begin{aligned}|R_n f| &\le |R_n(f-g)| + |R_n g| \le (1+M_1)\|f-g\|_{\varphi_n} + M_2 \varphi_n^r \int_a^b |g^{(r)}(u)|\,du \\ &\le \max\{1+M_1, M_2\}\Big[\|f-g\|_{\varphi_n} + \varphi_n^r \int_a^b |g^{(r)}(u)|\,du\Big].\end{aligned}$$

Therefore, taking the infimum over $g \in ACR^{(r)}$ and using the right hand estimate of (12) immediately deliver that for every $f \in R$

$$|R_n f| \le \max\{1+M_1, M_2\}\, K_r(\varphi_n, f) \le M_3\, \tau_r(\varphi_n, f).$$

<u>(15)→(14):</u> The assumption and (12) imply (cf. (11))

$$|R_n f| \le M_3\, \tau_r(\varphi_n, f) \le M_3\, 2^r\, K_r(\varphi_n, f) \le 2^r M_3 \begin{cases} \|f\|_{\varphi_n} & (f \in R) \\ \varphi_n^r \int_a^b |f^{(r)}(x)|\,dx & (f \in ACR^{(r)}), \end{cases}$$

thus the assertions of (14). ∎

To verify the conditions of (14), the validity of the Jackson-type inequality (14)(ii) is controlled by the Peano kernel theorem: If the quadrature rule (13) is exact on $\mathcal{P}_{r-1}$ for some $r \in \mathbf{N}$, then for the remainder there holds true the representation (cf. [2, p. 39])

$$R_n g = \int_a^b K_{nr}(u)\, g^{(r)}(u)\,du \qquad (g \in ACR^{(r)}) \qquad (17)$$

with Peano kernel $K_{nr}(u) := R_n((\cdot - u)_+^{r-1})/(r-1)!$. Therefore, in view of the estimate

$$|R_n g| \le \|K_{nr}\|_B \int_a^b |g^{(r)}(u)|\,du \qquad (g \in ACR^{(r)}),$$

the Peano theorem leads to the candidate $\varphi_n^r = \|K_{nr}\|_B$, provided condition (14)(i) can be established for this sequence $\{\varphi_n\}$, too. This coupling, a consequence of the definition (11) of a K-functional, causes some additional difficulties if compared with the situation in C. It follows, however, that for compound quadrature rules this can be handled.

Lemma 1: *For the compound quadrature process* (5) *there holds true*

$$|Q_{(n)} f| \le M_1 \|f\|_{1/n} \qquad (f \in R[a,b]).$$

Proof: Recalling $A_k := a + k(b-a)/n$ one has that for every $f \in R[a,b]$ (cf. (8, 10))

$$|Q_{(n)}f| \le \sum_{k=1}^{n} \int_{A_{k-1}}^{A_k} \Big[\sum_{i=1}^{j} |a_i|\, |f(a + (k-1+x_i)\frac{b-a}{n})|\Big]\, dx$$

$$\le \sum_{i=1}^{j} |a_i| \sum_{k=1}^{n} \int_{A_{k-1}}^{A_k} M(\frac{b-a}{n}, f, x)\, dx = \sum_{i=1}^{j} |a_i|\, \|f\|_{(b-a)/n}$$

$$\le \sum_{i=1}^{j} |a_i|\, 2(1+b-a)\, \|f\|_{1/n} =: M_1\, \|f\|_{1/n}. \quad \blacksquare$$

Lemma 2: *Let the elementary rule Q^{el} (cf. (3)) be exact on $\mathcal{P}_{r-1}$ for some $r \in \mathbf{N}$. Then for the compound quadrature process* (5) *there holds true the Jackson-type inequality*

$$|Q_{(n)}g - \int_a^b g(u)\, du| \le M_2\, n^{-r} \int_a^b |g^{(r)}(u)|\, du \qquad (g \in ACR^{(r)}[a,b]).$$

Proof: Since Q^{el} is exact on $\mathcal{P}_{r-1}$, the same holds true for $Q_{(n)}$. Moreover, if K_r^{el} is the r-th Peano kernel of the elementary rule Q^{el}, then the r-th Peano kernel $K_{(n)r}$ of the compound rule $Q_{(n)}$ may be represented via

$$K_{(n)r}(x) = (\frac{b-a}{n})^r \sum_{k=1}^{n} K_r^{el}\Big(\frac{n}{b-a}(x-a) - (k-1)\Big)\, \kappa_{[A_{k-1},A_k]}(x),$$

where κ_D is the characteristic function of the set $D \subset \mathbf{R}$. Therefore by (17) for every $g \in ACR^{(r)}$

$$|R_{(n)}g| = (\frac{b-a}{n})^r \Big| \int_a^b \sum_{k=1}^{n} K_r^{el}(\frac{n}{b-a}(u-a) - (k-1))\, \kappa_{[A_{k-1},A_k]}(u)\, g^{(r)}(u)\, du \Big|$$

$$\le (\frac{b-a}{n})^r \sum_{k=1}^{n} \int_{A_{k-1}}^{A_k} |K_r^{el}(\frac{n}{b-a}(u-a) - (k-1))|\, |g^{(r)}(u)|\, du$$

$$\le (b-a)^r\, \|K_r^{el}\|_{B[0,1]}\, n^{-r} \sum_{k=1}^{n} \int_{A_{k-1}}^{A_k} |g^{(r)}(u)|\, du =: M_2\, n^{-r} \int_a^b |g^{(r)}(u)|\, du. \quad \blacksquare$$

As a consequence of the preceding Lemma 1, 2 and Theorem 2 one therefore has

Corollary 3: *Let the elementary rule* (3) *be exact on $\mathcal{P}_{r-1}$ for some $r \in \mathbf{N}$. Then for the compound quadrature process* (5) *there holds true the error estimate*

$$|Q_{(n)}f - \int_a^b f(u)\, du| \le M\, \tau_r(1/n, f) \qquad (f \in R[a,b]). \tag{18}$$

This result is due to Ivanov [8] (cf. [16, p. 49]) where it is shown by some interpolation technique, similar to that usually employed in the course of the proof of (12) (cf. [3]). Let us conclude with the remark that the estimate (18) is indeed sharp in the following sense: For every abstract modulus of continuity ω (cf. [17, p. 96]) there exists a counterexample $f_\omega \in C[a,b]$ such that (see [4], [11])

$$\tau_r(\delta, f_\omega) = \mathcal{O}(\omega(\delta^r)), \qquad |Q_{(n)} f_\omega - \int_a^b f_\omega(u)\,du| \neq o(\omega(n^{-r})).$$

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