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RATIONAL CHEBYSHEV APPROXIMATIONS OF ANALYTIC FUNCTIONS

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ABSTRACT

We present a general procedure for obtaining rational Chebyshev approximations of analytic functions. Surprisingly, the method is a simple generalization of an idea of Newton. The algorithm is developed in detail, and several examples are worked out.

Section 1. SUMMUS

Let us obtain some approximate formulas for the rectification of the arc of the circle. Consider the series

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots ,$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots ,$$

$$\tfrac{1}{2} \sin 2x = \sin x \cos x = x - \frac{2}{3} x^3 + \frac{2}{15} x^5 - \dots .$$

If we consider the first two equations and neglect terms of order five or higher, we obtain

$$3 \sin x - x \cos x \approx 2x,$$

so that

$$x \approx \frac{3 \sin x}{2+\cos x} , \qquad (1\text{-}1)$$

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a formula given by Cardinal Nicolaus Cusanus (1401-1464), and later by the Dutch mathematician and physicist Willebrord Snellius (1580-1626) [3,11], by the use of geometrical considerations.

If we take into account all three of the above series, we obtain, neglecting terms of order seven or more

$$14 \sin x - 6x \cos x + \sin x \cos x \approx 9x,$$

or,

$$x \approx \sin x \frac{14 + \cos x}{9 + 6 \cos x}, \qquad (1\text{-}2)$$

a formula given by Newton [16].

This approach of Newton will become the basis of our method for obtaining rational Chebyshev approximations.

Let us start, then, by extending Newton's idea to obtain better rectifications of the arc of the circle. To this end, we try the expression

$$x \approx A_1 \sin x - A_2 x \cos x + A_3 \sin 2x - A_4 x \cos 2x + \ldots + A_{2s-1} \sin sx - A_{2s} x \cos sx, \qquad (1\text{-}3)$$

where the $\approx$ sign is to be interpreted in the sense that the Maclaurin expansions of both sides agree through the first 2s terms. Note that both sides of the above approximate relation are odd functions of x, as they should be.

Expanding the right-hand side by use of the appropriate Maclaurin expansions, and equating corresponding powers of x, one finds the system of equations

$$A_1 - A_2 + 2\cdot A_3 - A_4 + \ldots + sA_{2s-1} - A_{2s} = 1$$

$$A_1 - 3\cdot A_2 + 2^3 A_3 - 3\cdot 2^2 A_4 + \ldots + s^3 A_{2s-1} - 3s^2 A_{2s} = 0$$

....

$$A_1 - (4s-1)A_2 + 2^{4s-1}A_3 - (4s-1)2^{4s-2}A_4 + \ldots + s^{4s-1}A_{2s-1} - (4s-1)s^{4s-2}A_{2s} = 0$$

of 2s equations with 2s unknowns for the determination of the A's.

Solving this system, x is found as

$$x \approx \frac{A_1 \sin x + A_3 \sin 2x + A_5 \sin 3x + \ldots + A_{2s-1} \sin sx}{1 + A_2 \cos x + A_4 \cos 2x + A_6 \cos 3x + \ldots + A_{2s} \cos sx} . \quad (1\text{-}4)$$

For s = 1, we find $A_1 = 3/2$, $A_2 = 1/2$, which gives Cardinal Cusanus' formula mentioned above.

The value s = 2 gives

$$x \approx \frac{5}{3} \sin x \frac{16 + 5 \cos x}{17 + 16 \cos x + 2 \cos 2x} , \quad (1\text{-}5)$$

s = 3 gives

$$x \approx \frac{7}{5} \sin x \frac{92 + 66 \cos x + 7 \cos^2 x}{82 + 111 \cos x + 36 \cos^2 x + 2 \cos^3 x} , \quad (1\text{-}6)$$

s = 4 gives

$$x \approx \frac{1}{35} \sin x \frac{91648 + 103511 \cos x + 28544 \cos^2 x + 1522 \cos^3 x}{1667 + 2944 \cos x + 1560 \cos^2 x + 256 \cos^3 x + 8 \cos^4 x} , \quad (1\text{-}7)$$

etc.

Setting, for instance, $x = \pi/6$, gives some interesting approximations to π. Equations (1-5), (1-6) and (1-7) give successively

$$5\left(\frac{32 + 5\sqrt{3}}{37 + 16\sqrt{3}}\right) = 3.141592229 \ldots , \qquad (1\text{-}8)$$

$$\frac{21}{5}\left(\frac{389 + 132\sqrt{3}}{436 + 225\sqrt{3}}\right) = 3.14159265346 \ldots , \qquad (1\text{-}9)$$

$$\frac{3}{70}\left(\frac{452224 + 209305\sqrt{3}}{5683 + 3136\sqrt{3}}\right) = 3.141592653589754 \ldots , \qquad (1\text{-}10)$$

the correct value of π, being 3.14159265358979323 8 ... .

It is of interest to note that equations (1-9) and (1-10) improve over Ramanujan's formula [19,5]

$$\frac{63}{25}\left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}\right) = 3.1415926541 \ldots . \qquad (1\text{-}11)$$

If in the right hand side of equation (1-4) one factors out $\sin x$, makes use of the equations $T_n(\cos x) = \cos nx$, $U_n(\cos x) = (\sin(n+1)x)/\sin x$, where T_n and U_n are Chebyshev polynomials of the first and second kinds, respectively, and replaces x by $\cos^{-1}x$, he finds

$$\cos^{-1}x \approx (1-x^2)^{\frac{1}{2}} \frac{A_1\,U_0(x) + A_3\,U_1(x) + \ldots + A_{2s-1}\,U_{s-1}(x)}{T_0(x) + A_2\,T_1(x) + \ldots + A_{2s}\,T_s(x)},$$

which, apart from the square root, gives a rational Chebyshev approximation for the arc cosine.

For instance, Cardinal Cusanus' formula, equation (1-1), becomes

$$\cos^{-1}x \approx (1-x^2)^{\frac{1}{2}} \frac{3\ U_0(x)}{2\ T_0(x) + T_1(x)} = (1-x^2)^{\frac{1}{2}} \frac{3}{2+x} .$$

Section 2. GENERALIZATION OF THE METHOD FOR ARBITRARY ANALYTIC FUNCTIONS.

Let f(z) be analytic at z_0. Then $g(z) = f(\cos z + z_0 - 1)$ is, by composition, analytic at the origin. We can, hence, write

$$g(z) = f(\cos z + z_0 - 1) = \sum_{n=0}^{\infty} g^{(2n)}(0) \frac{z^{2n}}{(2n)!} . \qquad (2\text{-}1)$$

If an explicit expansion of $f(\cos z + z_0 - 1)$ is not available, then successive coefficients in (2-1) are found directly from the formula for Maclaurin expansions, i.e, by simply calculating successive derivatives of (2-1) and setting z = 0. To wit,

$$g(0) = f(z_0), \qquad (2\text{-}2)$$

$$g''(0) = - f'(z_0), \qquad (2\text{-}3)$$

$$g^{(iv)}(0) = 3f''(z_0) + f'(z_0), \qquad (2\text{-}4)$$

$$g^{(vi)}(0) = -15f'''(z_0) - 15f''(z_0) - f'(z_0), \qquad (2\text{-}5)$$

$$g^{(viii)}(0) = 105f^{(iv)}(z_0) + 210f'''(z_0) + 63\ f''(z_0) + f'(z_0), \qquad (2\text{-}6)$$

$$g^{(x)}(0) = -945f^{(v)}(z_0) - 3150f^{(iv)}(z_0) - 2205f'''(z_0) - 255f''(z_0) - f'(z_0), \qquad (2\text{-}7)$$

$$g^{(xii)}(0) = 10395f^{(vi)}(z_0) + 51975f^{(v)}(z_0) + 65835f^{(iv)}(z_0)$$

$+ 21120f'''(z_0) + 1023f''(z_0) + f'(z_0),$ (2-8)

$g^{(xiv)}(0) = -135135f^{(vii)}(z_0) - 945945f^{(vi)}(z_0) - 1891890f^{(v)}(z_0)$

$- 1201200f^{(iv)}(z_0) - 195195f'''(z_0) - 4095f''(z_0) - f'(z_0),$ (2-9)

etc.; the derivatives of odd order at the origin being zero, since g(z) is an even function.

Consider now the expression

$$g(z) \approx A_1\cos z - A_2g(z)\cos z + A_3\cos 2z - A_4g(z)\cos 2z + \dots + A_{2s-1}\cos sz - A_{2s}g(z)\cos sz, \quad (2\text{-}10)$$

where the Ak's are, again, constants to be determined, and the ≈ in (2-10) is to be interpreted in the sense that the Maclaurin expansions of both sides agree through the first 2s terms.

Note that both sides of (2-10) are, of course, even, as they should be.

Observe that the Cauchy product of g(z) and cos mz is

$$g(z)\cos mz = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{g^{(2n-2k)}(0)\,(-1)^k\,m^{2k}\,z^{2n}}{(2n-2k)!\,(2k)!}. \quad (2\text{-}11)$$

Since cos mz is entire, the above Cauchy product will have the same circle of convergence that equation (2-1) has [7].

Using (2-11) to equate powers of z in (2-10) we find, after multiplying through by $(-1)^n\,(2n)!$,

$$(-1)^n g^{(2n)}(0) = A_1 - A_2 \sum_{k=0}^{n} (-1)^{n-k} \binom{2n}{2k} g^{(2n-2k)}(0) + 2^{2n} A_3$$

$$- A_4 \sum_{k=0}^{n} (-1)^{n-k} 2^{2k} \binom{2n}{2k} g^{(2n-2k)}(0) + \ldots + s^{2n} A_{2s-1}$$

$$- A_{2s} \sum_{k=0}^{n} (-1)^{n-k} s^{2k} \binom{2n}{2k} g^{(2n-2k)}(0), \qquad (2\text{-}12)$$

where $\binom{n}{k}$, is the binomial coefficient.

Letting in equation (2-12) $n = 0, 1, 2, \ldots, 2s-1$, we find an algebraic system of 2s equations with 2s unknowns for the determination of the A's. $g(z)$ is then found as

$$g(z) \approx \frac{A_1 \cos z + A_3 \cos 2z + \ldots + A_{2s-1} \cos sz}{1 + A_2 \cos z + A_4 \cos 2z + \ldots + A_{2s} \cos sz}. \qquad (2\text{-}13)$$

Replace now in equation (2-13) above z by $\cos^{-1}(z-z_0+1)$, and make use of the defining equation for Chebyshev polynomials of the first kind $T_n(z) = \cos(n \cos^{-1} z)$, recalling the relation between $f(z)$ and $g(z)$ to obtain

$$f(z) \approx \frac{A_1 T_1(z-z_0+1) + A_3 T_2(z-z_0+1) + \ldots + A_{2s-1} T_s(z-z_0+1)}{T_0(z-z_0+1) + A_2 T_1(z-z_0+1) + \ldots + A_{2s} T_s(z-z_0+1)}, \qquad (2\text{-}14)$$

which gives a rational Chebyshev approximation of $f(z)$ where the only restriction which has been assumed is analyticity of the function at z_0.

Power series of the form given in equation (2-1) are sometimes found Taylor-made in the literature. For instance [9]

$$\exp(\cos z - 1) = 1 - \frac{1}{2} z^2 + \frac{1}{6} z^4 - \frac{31}{720} z^6 + \dots , \qquad (2\text{-}15)$$

where the general coefficient is

$$\frac{(-1)^n \; 2^{1-n}}{n! \; (2n)!} \sum_{k=0}^{n-1} (-1)^k \, 2^k \, (-n)_k \sum_{r=0}^{n-k-1} \frac{(2k-2n)_r}{r!} (n-k-r)_{2n}, \qquad (2\text{-}16)$$

where $(a)_n = a(a+1)(a+2) \dots (a+n-1)$, $(a)_0 = 1$, $a \neq 0$, is Pochhammer's symbol. In series (2-15) $z_0 = 0$.

Also [8],

$$\log \cos z = \sum_{n=1}^{\infty} (-1)^n \, (2^{2n}-1) \, 2^{2n-1} \, B_{2n} \, z^{2n}/[n \, (2n)!], \qquad (2\text{-}17)$$

where the B_{2n} are Bernoulli numbers [1]. In series (2-17) $z_0 = 1$.

It will be noticed that the coefficient of $f^{(j)}(z_0)$ in the sum for $g^{(2i)}(0)$, $(i = 1, 2, \dots , 2s-1, \; j = 1, 2, \dots , 2s-1)$, exemplified in the list given at the beginning of this section, equations (2-2) through (2-9), is also the coefficient of $\cos jz$, evaluated at $z = 0$, in

$$\frac{d^{2j}}{dz^{2j}}(\exp(\cos z - 1)).$$

This provides a simple computer algorithm for generating these coefficients. This observation is due to one of the authors (Rosenthal).

Section 3. ADAPTING THE ALGORITHM FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION.

The method we have developed enables us to find, in simple fashion, a rational Chebyshev approximation for the generalized hypergeometric function ${}_pF_q(z)$:

$$ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n} \frac{z^n}{n!}, \tag{3-1} $$

where none of the b's is zero or a negative integer [18].

The derivative of equation (3-1) is given by [18]

$$ \frac{a_1a_2\cdots a_p}{b_1b_2\cdots b_q} \, {}_pF_q(a_1+1,a_2+1,\ldots,a_p+1;b_1+1,b_2+1,\ldots,b_q+1;z). \tag{3-2} $$

The value of the hypergeometric function at the origin is 1.

Hence, choosing $z_0 = 0$, it is quite simple to determine successive derivatives of ${}_pF_q(z)$ at the origin to find, with the aid of equations (2-2) through (2-9), the values of g(z) and its derivatives at z = 0.

Note, that g(0) and its derivatives at the origin will be given as rational functions of the coefficients of the ${}_pF_q(z)$. In particular, if these coefficients are themselves rational, then the rational Chebyshev approximation will involve only rational coefficients.

As the reader no doubt knows, many known functions are special cases (at most with a multiplicative monomial) of the generalized hypergeometric function. These include the following:

$$\log(1+z) = z\ {}_2F_1(1,1;2;-z) \tag{3-3}$$

$$\sin^{-1}z = z\ {}_2F_1(1/2,1/2;3/2;z^2), \tag{3-4}$$

$$\cos^{-1}z = \tfrac{1}{2}\pi - \sin^{-1}z, \tag{3-5}$$

$$\tan^{-1}z = z\ {}_2F_1(1/2,1;3/2;-z^2), \tag{3-6}$$

$$\sin z = z\ {}_0F_1(-;3/2;-\tfrac{1}{4}z^2), \tag{3-7}$$

$$\cos z = {}_0F_1(-;\tfrac{1}{2};-\tfrac{1}{4}z^2), \tag{3-8}$$

$$e^z = {}_0F_0(-;-;z), \tag{3-9}$$

$$(1-z)^{-a} = {}_1F_0(a;-;z). \tag{3-10}$$

[Note: If a is the reciprocal of a negative integer, then a rational Chebyshev approximation of equation (3-10) will provide a way of estimating roots of positive numbers less than one. The latter restriction is no restriction at all since the reciprocal of a number less than one is greater than one.]

Complete elliptic integrals of the first and second kinds:

$$K = \tfrac{1}{2}\pi\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;k^2) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}\,, \tag{3-11}$$

$$E = \tfrac{1}{2}\pi\,{}_2F_1(\tfrac{1}{2},-\tfrac{1}{2};1;k^2) = \int_0^{\frac{1}{2}\pi} \sqrt{(1-k^2\sin^2\theta)}\ d\theta. \tag{3-12}$$

Bessel functions,

$$J_n(z) = \frac{(z/2)^n}{\Gamma(1+n)}\ {}_0F_1(-;1+n;-\tfrac{1}{4}z^2). \tag{3-13}$$

The error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt = \frac{2x}{\sqrt{\pi}}\; {}_1F_1(1/2;3/2;-x^2). \qquad (3\text{-}14)$$

The incomplete Gamma function,

$$g(\alpha,x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt = \alpha^{-1}x^{\alpha}\,{}_1F_1(\alpha;\alpha+1;-x), \quad \operatorname{Re}(\alpha) > 0. \qquad (3\text{-}15)$$

The incomplete Beta function,

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt = a^{-1}\, x^{a}\,{}_2F_1(a,1-b;a+1;x). \qquad (3\text{-}16)$$

Any computer program which can handle the generalized hypergeometric function can find rational Chebyshev approximations for all of the functions mentioned above, simply by specializing the parameters.

It will be recalled that we mentioned, following equation (3-2), that if the parameters appearing in the hypergeometric function are rational numbers, then the A's, the solutions of the system of equations (2-12), are also rational numbers. This holds in most of the important cases. Note, for instance, that all of the examples given above meet this condition. For this reason we found it desirable to choose a program (we chose REDUCE [20]) which does not execute the operation of division, so that the A's will come out in fractional form.

We will close this section by making a comment which is probably obvious to the reader. If one wishes to go from a given s, the highest order of the Chebyshev polynomials in equation (2-14), to s+1 in the system of equations (2-12), then the matrix of the coefficients for s+1 will be the same as that for s, except that two rows and two columns will be added. Hence, knowing the inverse of the 2sx2s matrix one can find the inverse of the (2s+2)x(2s+2) matrix by using the method of partitioning in the technique known as inversion by bordering.

Section 4. TRANSFORMING THE RATIONAL CHEBYSHEV APPROXIMATIONS INTO FINITE CONTINUED FRACTIONS.

If one has a rational expression

$$f(z) = \frac{p_0\, z^n + p_1\, z^{n-1} + p_2\, z^{n-2} + \dots + p_{n-1}\, z + p_n}{z^n + q_1\, z^{n-1} + q_2\, z^{n-2} + \dots + q_{n-1}\, z + q_n}, \quad (4\text{-}1)$$

then he may write this as [21]

$$f(z) = A_0 + \cfrac{B_1}{z + \cfrac{\alpha_0 z^{n-1} + \alpha_1 z^{n-2} + \dots + \alpha_{n-1}}{z^{n-1} + \beta_1 z^{n-2} + \dots + \beta_{n-1}}}, \quad (4\text{-}2)$$

where $A_0 = p_0$, $B_1 = p_1 - p_0 q_1$,

$$\beta_k = \frac{p_{k+1} - p_0 q_{k+1}}{p_1 - p_0 q_1}, \quad (k = 1, 2, \dots, n-1),$$

$$\alpha_k = q_{k+1} - \beta_{k+1}, \quad (k = 0, 1, \dots, n-2),$$

$$\alpha_{n-1} = q_n.$$

Continuing in this manner one obtains the finite continued fraction

$$f(z) = A_0 + \cfrac{B_1}{z + A_1 + \cfrac{B_2}{z + A_2 + \cfrac{B_3}{z + A_3 + \cdots}}}$$

and evaluates f(z) by performing n divisions. Note, though, that equation (4-1) has twice as many coefficients as an n-degree polynomial, and so, for about the same amount of computer time, n divisions against the n multiplications required to evaluate an n-degree polynomial we obtain roughly twice the accuracy! Put in another form, we get the same accuracy in about half the amount of computer time. This appears to be a good time-saving device for the computer generation of analytic functions.

The reader should note that the algorithm we have discussed may also be used to obtain Chebyshev polynomial approximations. In equation (2-10) set $A_2 = A_4 = \ldots = A_{2s} = 0$,

and in the system of equations (2-12) let $n = 0, 1, 2, \ldots, s-1$.

Section 5. A SUMMARY OF THE PROCEDURE

The algorithm may be summarized as follows:

1. Choose the function f(z) you wish to approximate.

2. Choose z_0, any point at which f(z) is known to be analytic. If dealing with the hypergeometric function it is most wise to choose $z_0 = 0$.

3. Choose $e = \max_{|z-z_0| \leq R} |f(z) - \text{rational Chebyshev approximation}|$,

   where R is to be specified.

4. Construct g(z) in terms of f(z), equation (2-1).

5. Calculate successive derivatives of g(z) at z = 0. The first fourteen derivatives are given in equations (2-2) through (2-9). It is better to generate these derivatives as they are needed.

6. For the exponential and the logarithmic function, these values of g(z) and its derivatives may be picked directly from equations (2-15) and (2-17).

7. Choose s = 1. The parameter s, it will be remembered, is the highest order of the Chebyshev polynomials in equation (2-14).

8. Solve the system of equations (2-12) with s = 1, and construct equation (2-14) with the resulting A's.

9. Test your approximation. If it doesn't meet the requirement stipulated in 3., above, then increase s by one and repeat the procedure. Remember that the matrix of the coefficients will remain identical to that of the previous step, except that two rows and two columns will be added. Only these rows and columns need be generated.

10. Invert by using partitioning and inversion by bordering.

11. Construct equation (2-14) with the new A's.

12. Goto 9.

13. Convert the rational Chebyshev approximation you have obtained into a finite continued fraction, as indicated in Section 4.

Section 6. ILLUSTRATING THE ALGORITHM

We will now give some examples of rational Chebyshev approximations obtained by use of the procedure outlined in the previous section.

a. $$f(z) = \log(1+z)$$

For the interest of the reader we will give the A's, the coefficients of equation (2-14), for the cases s = 1 to s = 5 only. For other values of s, only the rational Chebyshev approximations themselves will be given .

$$s = 1$$

$$A_1 = 0,\ A_2 = -1,$$

$$s = 2$$

$$A_1 = -\frac{8}{5},\ A_2 = \frac{16}{5},\ A_3 = -\frac{8}{5},\ A_4 = \frac{3}{5},$$

$$s = 3$$

$$A_1 = -\frac{105}{94},\ A_2 = \frac{99}{47},\ A_3 = \frac{48}{47},\ A_4 = \frac{33}{47},\ A_5 = \frac{9}{94},\ A_6 = \frac{1}{47},$$

$$s = 4$$

$$A_1 = -\frac{2336}{3609},\ A_2 = \frac{8912}{6015},\ A_3 = \frac{1240}{3609},\ A_4 = \frac{5524}{6015},$$

$$A_5 = \frac{352}{1203},\ A_6 = \frac{816}{6015},\ A_7 = \frac{40}{3609},\ A_8 = \frac{13}{6015},$$

$$s = 5$$

$$A_1 = -\frac{8295}{12668},\ A_2 = \frac{8555}{19002},\ A_3 = \frac{2200}{9501},\ A_4 = \frac{6140}{9501},$$

$$A_5 = \frac{19665}{25336},\ A_6 = -\frac{6855}{12668},\ A_7 = -\frac{9160}{28503},\ A_8 = -\frac{2105}{9501},$$

$$A_9 = -\frac{7195}{228024},\ A_{10} = -\frac{289}{38004}.$$

To list the approximations we will give them in the following format:

$$f(z) \approx \frac{az^k(p_0\ z^n + p_1\ z^{n-1} + p_2\ z^{n-2} + \ldots + p_{n-1}\ z + p_n)}{b(q_0\ z^m + q_1\ z^{m-1} + q_2\ z^{m-2} + \ldots + q_{m-1}\ z + q_m)}, \quad (6\text{-}1)$$

where $k + n \le s$, and $m \le s$. For each s we will simply list the coefficients in (6-1).

Continuing with the logarithmic function we have:

$$s = 1$$

For $s = 1$ we get the approximation $\log(1+z) \approx 0$.

s = 2

$a = 4,\ k = 1,\ n = 1,\ p_0 = 2,\ p_1 = 3;\ b = 1,\ m = 2,\ q_0 = 3,$

$q_1 = 14,\ q_2 = 12.$

s = 3

$a = 3,\ k = 1,\ n = 2,\ p_0 = 3,\ p_1 = 25,\ p_2 = 30;\ b = 1, m = 3,$

$q_0 = 2,\ q_1 = 39,\ q_2 = 120,\ q_3 = 90.$

s = 4

$a = 10,\ k = 1,\ n = 3,\ p_0 = 20,\ p_1 = 344,\ p_2 = 1047,\ p_3 = 798;$

$b = 3,\ m = 4,\ q_0 = 13,\ q_1 = 460,\ q_2 = 2670,\ q_3 = 4820,\ q_4 = 2660.$

s = 5

$a = 5,\ k = 1,\ n = 4,\ p_0 = 1439,\ p_1 = 14523,\ p_2 = 33054,\ p_3 = 17766,$

$p_4 = -\ 3780;\ b = 6,\ m = 5,\ q_0 = 289,\ q_1 = 5655,\ q_2 = 24510,$

$q_3 = 35210,\ q_4 = 13230,\ q_5 = -\ 3150.$

s = 6

$a = 1,\ k = 1,\ n = 5,\ p_0 = 85094,\ p_1 = 16\ 62234,\ p_2 = 87\ 13635,$

$p_3 = 182\ 80080,\ p_4 = 165\ 35610,\ p_5 = 53\ 63820;\ b = 30,\ m = 6,$

$q_0 = 607,\ q_1 = 20532,\ q_2 = 1\ 68105,\ q_3 = 5\ 56640,\ q_4 = 8\ 70030,$

$q_5 = 6\ 40584,\ q_6 = 1\ 78794.$

s = 7

$a = 7,\ k = 1,\ n = 6,\ p_0 = 3\ 78271,\ p_1 = 101\ 49413,\ p_2 = 737\ 90844,$

$p_3 = 2231\ 18260,\ p_4 = 3190\ 00000,\ p_5 = 2118\ 94980,\ p_6 = 517\ 71720;$

$b = 10,\ m = 7,\ q_0 = 53423,\ q_1 = 23\ 91599,\ q_2 = 259\ 85652,$

$q_3 = 1163\ 60580,\ q_4 = 2569\ 82250,\ q_5 = 2944\ 43226,\ q_6 = 1664\ 46588,$

$q_7 = 362\ 40204.$

$s = 8$

$a = 1$, $k = 1$, $n = 7$, $p_0 = 816\ 21520$, $p_1 = 27843\ 07464$,

$p_2 = 2\ 54844\ 59616$, $p_3 = 9\ 66351\ 29360$, $p_4 = 17\ 28618\ 10275$,

$p_5 = 14\ 17447\ 83180$, $p_6 = 3\ 77798\ 72130$, $p_7 = -\ 50423\ 37300$;

$b = 35$, $m = 8$, $q_0 = 4\ 50245$, $q_1 = 250\ 34616$, $q_2 = 3347\ 43444$,

$q_3 = 18346\ 11240$, $q_4 = 49417\ 48350$, $q_5 = 68678\ 79480$,

$q_6 = 46015\ 68972$, $q_7 = 10073\ 91528$, $q_8 = -\ 1440\ 66780$.

The reader should observe that the magnitude of the coefficients increases quite rapidly with increasing s. We shall shortly see that the quality of the approximation also improves very rapidly as s increases.

For the next example, we will simply list the coefficients in equation (6-1). We will begin at $s = 2$.

b. Bessel Functions

$J_0(z)$

$s = 2$

$a = 4$, $k = 0$, $n = 2$, $p_0 = 2$, $p_1 = 0$, $p_2 = -\ 3$; $b = 1$, $m = 2$,

$q_0 = 5$, $q_1 = 0$, $q_2 = -\ 12$.

$s = 3$

$a = -\ 1$, $k = 0$, $n = 3$, $p_0 = 69$, $p_1 = 51$, $p_2 = -\ 368$, $p_3 = -272$;

$b = 1$, $m = 3$, $q_0 = 23$, $q_1 = 17$, $q_2 = 368$, $q_3 = 272$.

$$s = 6$$

$a = 12,\ k = 0,\ n = 6,\ p_0 = 57\ 76742,\ p_1 = 0,\ p_2 = -\ 1838\ 79735,$

$p_3 = 0,\ p_4 = 10070\ 89152,\ p_5 = 0,\ p_6 = -\ 7895\ 61600;\ b = 1,\ m = 6,$

$q_0 = 60\ 35647,\ q_1 = 0,\ q_2 = 3705\ 82236,\ q_3 = 0,\ q_4 = 97163\ 85024,$

$q_5 = 0,\ q_6 = -\ 94747\ 39200.$

$$s = 10$$

$a = 300,\ k = 0,\ n = 10,\ p_0 = 2114\ 63570\ 00545\ 36614,\ p_1 = 0,$

$p_2 = -\ 4\ 28033\ 75450\ 19518\ 86781,\ p_3 = 0,$

$p_4 = 281\ 17868\ 03665\ 81890\ 18624,\ p_5 = 0,$

$p_6 = -\ 6194\ 13498\ 85928\ 62663\ 77984,\ p_7 = 0,$

$p_8 = 31326\ 83622\ 73236\ 69829\ 38624,\ p_9 = 0,$

$p_{10} = -\ 23739\ 05902\ 96182\ 29215\ 88736;\ b = 1,\ m = 10,$

$q_0 = 3272\ 56614\ 14968\ 07057,\ q_1 = 0,$

$q_2 = 9\ 84654\ 95148\ 64179\ 66500,\ q_3 = 0,$

$q_4 = 1597\ 67150\ 04330\ 42594\ 24000,\ q_5 = 0,$

$q_6 = 1\ 57441\ 70286\ 74100\ 89721\ 60000,\ q_7 = 0,$

$q_8 = 76\ 17621\ 44097\ 57337\ 57624\ 32000,\ q_9 = 0,$

$q_{10} = -\ 71\ 21717\ 70888\ 54687\ 64766\ 20800.$

Section 7. COMPLEX VALUES OF THE ARGUMENT.

It is interesting to consider the problem of estimating a rational approximation for complex values of the argument. We will make use of the following simple procedure [22]: Let p_k be real and z complex, $z = x + iy$, in the polynomial expression

$$\sum_{k=0}^{n} p_k z^k = \sum_{k=0}^{n} p_k r^k \exp(iky),$$

where $r = |z|$, and $y = \tan^{-1} y/x = \cos^{-1} x/r$. Then we have

$$\sum_{k=0}^{n} p_k z^k = \sum_{k=0}^{n} p_k r^k \cos ky + i \sum_{k=1}^{n} p_k \sin ky =$$

$$\sum_{k=0}^{n} p_k r^k \cos(k \cos^{-1}(x/r)) + i \sum_{k=1}^{n} p_k r^k \sin(k \cos^{-1}(x/r)) =$$

$$\sum_{k=0}^{n} p_k r^k T_k(x/r) + i (y/r) \sum_{k=1}^{n} p_k r^k U_{k-1}(x/r), \qquad (7\text{-}1)$$

where the T_k's are, of course, Chebyshev polynomials of the first kind and the U_k's are Chebyshev polynomials of the second kind.

By means of equation (7-1) one can split the numerator and denominator of equation (6-1) into real and imaginary parts to obtain an expression of the form

$$\frac{A+iB}{C+iD}, \qquad (7\text{-}2)$$

which is easily broken into real and imaginary parts. Observe, though, that when this separation into real and imaginary components is performed, the numerator and denominator polynomials will be of a degree twice as great as that of polynomial (7-1), and hence the computational labor will be quadrupled, not doubled, as might have been expected at first sight. Of course, if one uses polynomial approximations, rather than rational, then the computational labor is only doubled. But the reader is reminded that polynomial approximations are about half as efficient for purposes of approximation as rational forms are, so that one is back to square one. This increased amount of labor seems to be unavoidable.

Section 8. NUMERICAL VALUES AND GRAPHS OF SOME RATIONAL CHEBYSHEV APPROXIMATIONS.

In this section we present the results of evaluating the rational forms given in Section 6. We will show explicit results for the logarithmic function and the Bessel function of order zero. The runs for different values of the parameter s will be contrasted with the tabulated values given in reference [1]. The latter will be taken, for purposes of comparison, as exact values.

a. $f(z) = \log(1+z)$

Exact Values

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 043249
0.2	0.18232 15567 939546
0.3	0.26236 42644 674911
0.4	0.33647 22366 212129
0.5	0.40546 51081 081644
0.6	0.47000 36292 457356
0.7	0.53062 82510 621704
0.8	0.58778 66649 021190
0.9	0.64185 38861 723948
1.0	0.69314 71805 599453
1.1	0.74193 73447 293773

s = 2

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09530 90096 798213
0.2	0.18230 56300 268097
0.3	0.26229 50819 672131
0.4	0.33628 31858 407080
0.5	0.40506 32911 392405
0.6	0.46927 37430 167598
0.7	0.52943 70434 035239
0.8	0.58598 72611 464968
0.9	0.63928 96781 354051
1.0	0.68965 51724 137931
1.1	0.73735 09506 928779

Error at z = 0.1: 1.17E-06
Error at z = 1.1: 0,005

s = 3

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01804 828502
0.2	0.18232 15892 572853
0.3	0.26236 45457 071409
0.4	0.33647 34567 217112
0.5	0.40546 87500 000000
0.6	0.47001 22399 020808
0.7	0.53064 56273 607298
0.8	0.58781 79485 244169
0.9	0.64190 55780 617015
1.0	0.69322 70916 334661
1.1	0.74205 45002 427792

Error at z = 0.1:-6.79E-10
Error at z = 1.1:-1.17E-04

s = 4

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 047746
0.2	0.18232 15568 731435
0.3	0.26236 42658 952172
0.4	0.33647 22468 540035
0.5	0.40546 51526 425329
0.6	0.47000 37712 544295
0.7	0.53062 86175 919782
0.8	0.58778 74772 161478
0.9	0.64185 54915 476286
1.0	0.69315 00831 529072
1.1	0.74194 22336 783631

Error at z = 0.1:-4.50E-13
Error at z = 1.1:-4.89E-06

s = 5

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 043508
0.2	0.18232 15567 662678
0.3	0.26236 42641 620017
0.4	0.33647 22344 206066
0.5	0.40546 50978 330635
0.6	0.47000 35937 041830
0.7	0.53062 81517 137408
0.8	0.58778 64276 167555
0.9	0.64185 33835 795553
1.0	0.69314 62118 011548
1.1	0.74193 56135 590429

Error at z = 0.1:-2.59E-14
Error at z = 1.1: 1.73E-06

s = 6

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 043249
0.2	0.18232 15567 939536
0.3	0.26236 42644 674133
0.4	0.33647 22366 197337
0.5	0.40546 51080 948204
0.6	0.47000 36291 700854
0.7	0.53062 82507 493159
0.8	0.58778 66638 711976
0.9	0.64185 38833 083111
1.0	0.69314 71735 901349
1.1	0.74193 73294 656556

Error at z = 0.1: 0
Error at z = 1.1: 1.53E-08

s = 7

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 043249
0.2	0.18232 15567 939546
0.3	0.26236 42644 674905
0.4	0.33647 22366 211960
0.5	0.40546 51081 079483
0.6	0.47000 36292 441266
0.7	0.53062 82510 538618
0.8	0.58778 66648 691443
0.9	0.64185 38860 650015
1.0	0.69314 71802 599078
1.1	0.74193 73439 874200

Error at z = 0.1: 0
Error at z = 1.1: 7.42E-10

s = 8

z	log(1+z)
0.0	0.00000 00000 000000
0.1	0.09531 01798 043249
0.2	0.18232 15567 939546
0.3	0.26236 42644 674910
0.4	0.33647 22366 212123
0.5	0.40546 51081 081553
0.6	0.47000 36292 456567
0.7	0.53062 82510 617059
0.8	0.58778 66649 000491
0.9	0.64185 38861 649232
1.0	0.69314 71805 370582
1.1	0.74193 73446 678808

Error at z = 0.1: 0
Error at z = 1.1: 6.15E-11

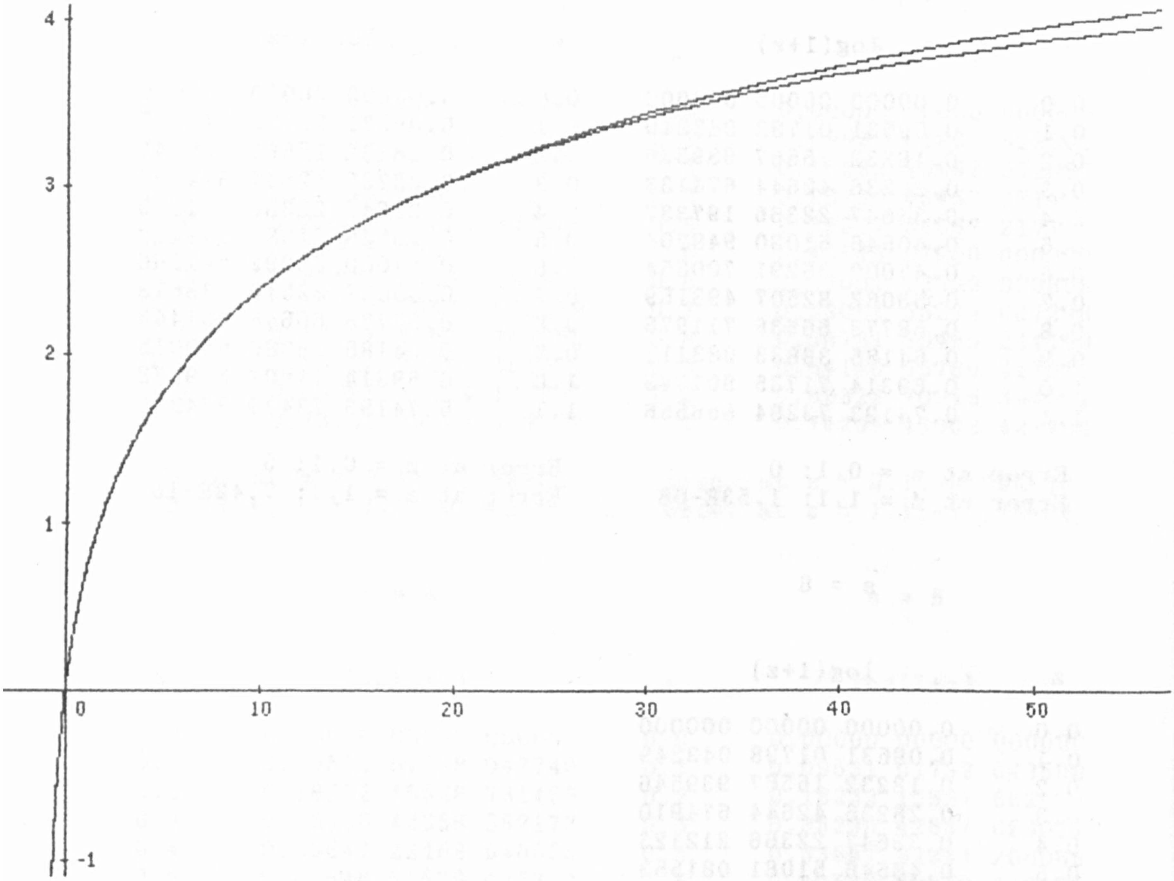

Fig. 1. The curve with the larger ordinate above depicts the graph of the logarithmic function. The curve with the lower ordinate is the algorithm's approximation corresponding to s = 8. Note that the curves do not separate significantly until the magnitude of z is about 30.

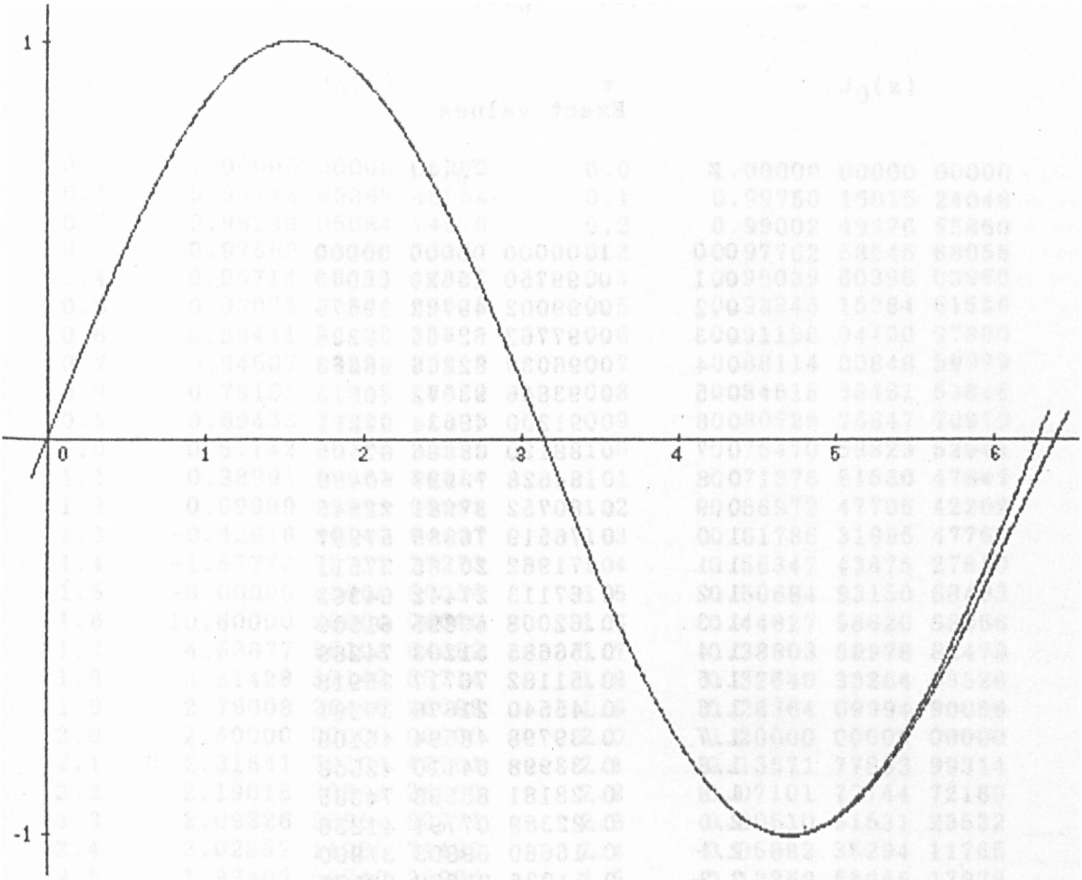

Fig. 2. The curves show the sine function plotted against the algorithmic approximation for s = 8. The curve that intersects the real axis nearer the point z = 6 is the true sine function.

b. $f(z) = J_0(z)$

Exact values

z	$J_0(z)$
0.0	1.00000 00000 00000
0.1	0.99750 15620 66040
0.2	0.99002 49722 39576
0.3	0.97762 62465 38296
0.4	0.96039 82266 59563
0.5	0.93846 98072 40813
0.6	0.91200 48634 97211
0.7	0.88120 08886 07405
0.8	0.84628 73527 50480
0.9	0.80752 37981 22545
1.0	0.76519 76865 57967
1.1	0.71962 20185 27511
1.2	0.67113 27442 64363
1.3	0.62008 59895 61509
1.4	0.56685 51203 74289
1.5	0.51182 76717 35918
1.6	0.45540 21676 39381
1.7	0.39798 48594 46109
1.8	0.33998 64110 42558
1.9	0.28181 85593 74385
2.0	0.22389 07791 41236
2.1	0.16660 69803 31990
2.2	0.11036 22669 22174
2.3	0.05553 97844 45602
2.4	0.00250 76832 97244
2.5	-0.04838 37764 68198

s = 2

z	$J_0(z)$
0.0	1.00000 00000 00000
0.1	0.99748 95397 48954
0.2	0.98983 05084 74576
0.3	0.97662 33766 23377
0.4	0.95714 28571 42857
0.5	0.93023 25581 39535
0.6	0.89411 76470 58824
0.7	0.84607 32984 29319
0.8	0.78181 81818 18182
0.9	0.69433 96226 41509
1.0	0.57142 85714 28571
1.1	0.38991 59663 86554
1.2	0.09999 99999 99999
1.3	-0.42816 90140 84507
1.4	-1.67272 72727 27273
1.5	-8.00000 00000 00000
1.6	10.60000 00000 00000
1.7	4.53877 55102 04082
1.8	3.31428 57142 85714
1.9	2.79008 26446 28099
2.0	2.50000 00000 00000
2.1	2.31641 79104 47761
2.2	2.19016 39344 26230
2.3	2.09826 98961 93771
2.4	2.02857 14285 71429
2.5	1.97402 59740 25974

Error at z = 0.1: 1.20E-05
Error at z = 1.5: 8.512
Error at z = 2.5:-2.022

s = 3

z	$J_0(z)$
0.0	1.00000 00000 00000
0.1	0.99750 15615 24048
0.2	0.99002 49376 55860
0.3	0.97762 58545 68055
0.4	0.96039 60396 03960
0.5	0.93846 15384 61539
0.6	0.91198 04400 97800
0.7	0.88114 00848 99939
0.8	0.84615 38461 53846
0.9	0.80725 75847 70970
1.0	0.76470 58823 52941
1.1	0.71876 81580 47647
1.2	0.66972 47706 42202
1.3	0.61786 31995 47767
1.4	0.56347 43875 27840
1.5	0.50684 93150 68493
1.6	0.44827 58620 68966
1.7	0.38803 59978 82478
1.8	0.32640 33264 03326
1.9	0.26364 09994 90056
2.0	0.20000 00000 00000
2.1	0.13571 77853 99314
2.2	0.07101 72744 72169
2.3	0.00610 61531 23532
2.4	-0.05882 35294 11765
2.5	-0.12359 55056 17978

Error at z = 0.1: 5.42E-10
Error at z = 1.5: 0.005
Error at z = 2.5: 0.075

s = 6		s = 10	
z	$J_0(z)$	z	$J_0(z)$
0.0	1.00000 00000 00000	0.0	1.00000 00000 00000
0.1	0.99750 15620 66040	0.1	0.99750 15620 66040
0.2	0.99002 49722 39576	0.2	0.99002 49722 39576
0.3	0.97762 62465 38249	0.3	0.97762 62465 38296
0.4	0.96039 82266 57938	0.4	0.96039 82266 59564
0.5	0.93846 98072 14225	0.5	0.93846 98072 40813
0.6	0.91200 48632 17224	0.6	0.91200 48634 97211
0.7	0.88120 08863 32675	0.7	0.88120 08886 07405
0.8	0.84628 73359 87120	0.8	0.84628 73527 50480
0.9	0.80752 36414 25774	0.9	0.80752 37981 22545
1.0	0.76519 88991 81058	1.0	0.76519 76865 57967
1.1	0.71962 28437 77746	1.1	0.71962 20185 27512
1.2	0.67113 39900 87712	1.2	0.67113 27442 64364
1.3	0.62008 81243 85951	1.3	0.62008 59895 61514
1.4	0.56685 88740 92599	1.4	0.56685 51203 74305
1.5	0.51183 42373 87263	1.5	0.51182 76717 35967
1.6	0.45541 34601 77972	1.6	0.45540 21676 39523
1.7	0.39800 38749 36571	1.7	0.39798 48594 46502
1.8	0.34001 77192 20127	1.8	0.33998 64110 43589
1.9	0.28186 89650 63377	1.9	0.28181 85593 76972
2.0	0.22397 01919 55021	2.0	0.22389 07791 47447
2.1	0.16672 95358 25093	2.1	0.16660 69803 46316
2.2	0.11054 77454 23837	2.2	0.11036 22669 54003
2.3	0.05581 53758 52507	2.3	0.05553 97845 13916
2.4	0.00291 01468 75270	2.4	0.00250 76834 39234
2.5	-0.04780 55089 54713	2.5	-0.04838 37761 81732

Error at z = 0.1: 0
Error at z = 1.5:-6.57E-06
Error at z = 2.5:-5.78E-04

Error at z = 0.1: 0
Error at z = 1.5:-4.90E-14
Error at z = 2.5:-2.86E-10

The algorithm is seen to be very stable. As the value of s increases, the quality of the approximations improves notably. The last example above, $J_0(z)$ for s = 10, gives remarkable agreement throughout the range $0 \le |z| \le 2.5$.

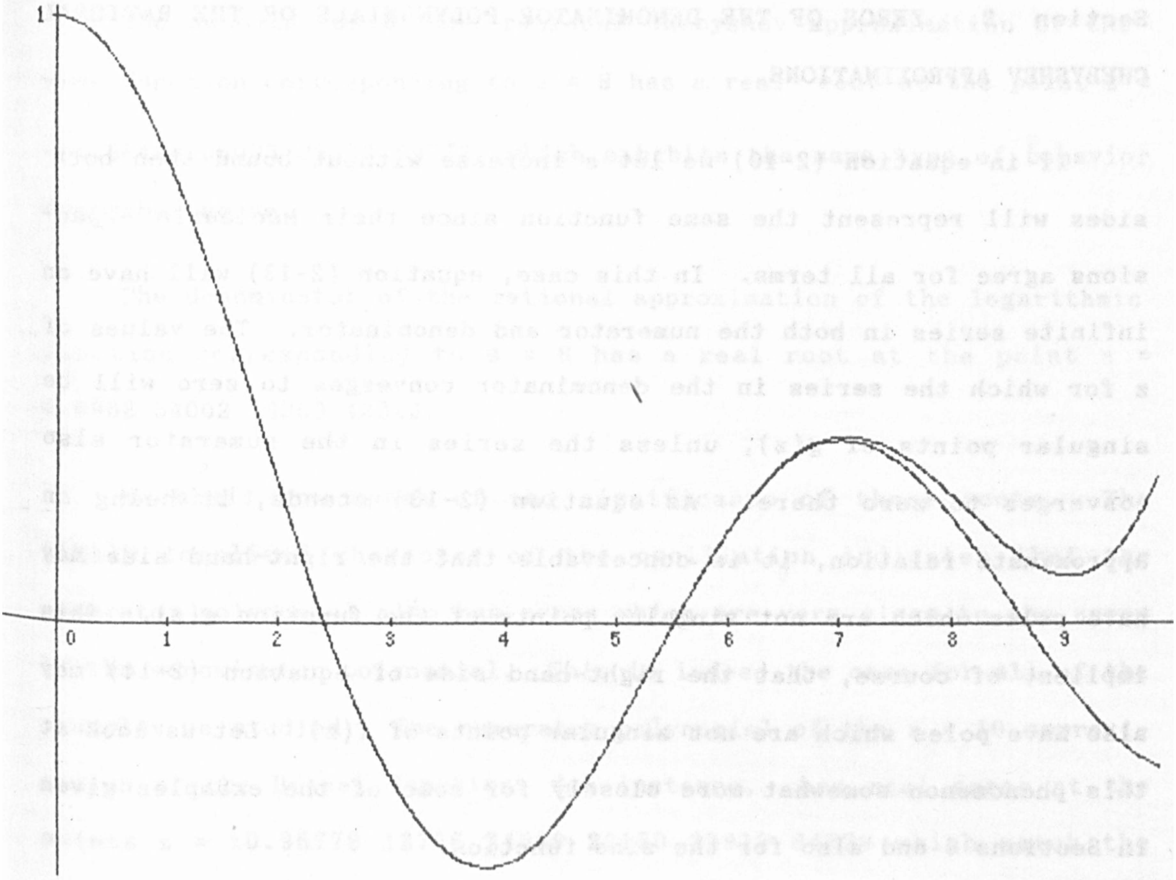

Fig. 3. This figure shows the Bessel function of the first kind of order zero, $J_0(z)$, plotted against the rational Chebyshev approximation corresponding to $s = 10$. After $z = 9$, the Bessel function continues to oscillate, while the approximation separates from this behavior. The two functions move apart after $z = 7$. The algorithm approximates the first zero of the Bessel function to be 2.40482 55580, and the second zero to be 5.51960 87207. These results compare favorably with the correct values 2.40482 55577 and 5.52007 81103 given in reference [1].

Section 9. ZEROS OF THE DENOMINATOR POLYNOMIALS OF THE RATIONAL CHEBYSHEV APPROXIMATIONS.

If in equation (2-10) we let s increase without bound then both sides will represent the same function since their Maclaurin expansions agree for all terms. In this case, equation (2-13) will have an infinite series in both the numerator and denominator. The values of z for which the series in the denominator converges to zero will be singular points of g(z), unless the series in the numerator also converges to zero there. As equation (2-13) stands, it being an approximate relation, it is conceivable that the right-hand side may have poles which are not singular points of the function g(z). This implies, of course, that the right-hand side of equation (2-14) may also have poles which are not singular points of f(z). Let us look at this phenomenon somewhat more closely for some of the examples given in Sections 6 and also for the sine function.

The denominator polynomial of the rational Chebyshev approximation for the Bessel function $J_0(z)$ corresponding to s = 10 has real zeros at the points z = ±0.95778 12766 24968 22726 05909 45945. Yet, the graph given in Fig. 3, and the table of values of this function do not seem to indicate any abnormal behavior in the neighborhood of this point. However, if we analyze the rational approximation within ±E-18 of this point, then the rational form is seen to undergo marked oscillations with nearly infinite slope. Nevertheless, as soon as we are within ±E-17 of the point in question, the erratic behavior disappears and the algorithm again represents the correct values of the Bessel function $J_0(z)$.

The denominator of the rational Chebyshev approximation of the sine function corresponding to s = 8 has a real root at the point z = - 1.38499 06093 05933 39272, which exhibits the same type of behavior described above.

The denominator of the rational approximation of the logarithmic function corresponding to s = 8 has a real root at the point z = 0.0952 54002 16069 42323.

We shall now speak of the significance of these roots. The highly localized character of the oscillation indicates that the numerator polynomial also has zeros which are very close to the zeros of the denominator polynomial. This is indeed the case for all of the examples we studied. The numerator polynomial of the s = 10 approximation of the Bessel function, for instance, has real zeros at the points z = ±0.95778 12766 24968 22150 32913 84229 which match the zeros of the denominator polynomial through seventeen decimal places. The s = 8 approximation of the logarithmic function also has a zero of the numerator polynomial which coincides with the zero of the denominator mentioned above within the limits of the program error. The oscillatory behavior is then simply a reflection of the computer's arithmetic inability to handle 0/0. The algorithm, we see, is a self-correcting one that introduces zeros in the numerator and denominator polynomials in a way that ensures the correct approximation to the function for a given value of s.

In essence, our method provides a rational approximation $P_s(z)/Q_s(z)$ such that its Taylor expansion about the point z_0 agrees with the Taylor expansion of f(z) through the first 2s terms. This requirement may be written as

$$Q_s(z)f(z) - P_s(z) = (z-z_0)^{2s+1} \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

and it is equivalent to the criterion for choosing the sth diagonal entry in the Padé table for $z_0 = 0$.

Because of the proximity of the real zeros of the numerator and denominator polynomials of the Bessel function approximation corresponding to s = 10, we chose to divide out the zeros and try out the outcome against the tabulated values given before. The resulting expression is:

a = 300, k = 0, n = 8, p_0 = 2114 63570 00545 36614, p_1 = 0,
p_2 = - 4 26093 90407 09759 89175.46176 795488, p_3 = 0,
p_4 = 277 26992 93536 91260 65928.27801 8860, p_5 = 0,
p_6 = - 5939 78281 24995 79471 89316.44693 046, p_7 = 0,
p_8 = 25878 00631 84966 42221 73861.61878 68
b = 1, m = 8,
q_0 = 3272 56614 14968 07057, q_1 = 0,
q_2 = 9 87657 02358 79227 26257.68351 76325 9, q_3 = 0,
q_4 = 1606 73172 24978 36037 41999.24516 095, q_5 = 0,
q_6 = 1 58915 63013 73742 21409 56470.02415 1, q_7 = 0,
q_8 = 77 63401 89554 89925 73188 73022.34814

The tabulated values resulting from this approximation are

z	$J_0(z)$
0.0	1.00000 00000 00000
0.1	0.99750 15620 66040
0.2	0.99002 49722 39576
0.3	0.97762 62465 38296
0.4	0.96039 82266 59564
0.5	0.93846 98072 40813
0.6	0.91200 48634 97211
0.7	0.88120 08886 07405
0.8	0.84628 73527 50480
0.9	0.80752 37981 22545
1.0	0.76519 76865 57967
1.1	0.71962 20185 27512
1.2	0.67113 27442 64364
1.3	0.62008 59895 61514
1.4	0.56685 51203 74305
1.5	0.51182 76717 35967
1.6	0.45540 21676 39523
1.7	0.39798 48594 46502
1.8	0.33998 64110 43589
1.9	0.28181 85593 76972
2.0	0.22389 07791 47447
2.1	0.16660 69803 46316
2.2	0.11036 22669 54003
2.3	0.05553 97845 13916
2.4	0.00250 76834 39234
2.5	-0.04838 37761 81732

Error at z = 0.1: 0
Error at z = 1.5:-4.90E-14
Error at z = 2.5:-2.86E-10

These are exactly the same values, to fifteen decimal accuracy, obtained with the s = 10 approximation of the Bessel function $J_0(z)$ before the roots are divided out! - These results imply a substantial saving in computer time since the number of divisions required for a given approximation is reduced by two.

A comment is in order, though it is probably obvious to the reader. The results shown in the above table were obtained by dividing the numerator polynomial by its real roots, and the denominator polynomial by its corresponding real roots. Slightly better accuracy is obtained (though the above table does not indicate it) if we divide both numerator and denominator polynomials by either the real roots of the numerator *or* the real roots of the denominator, since in this case all we are doing is dividing numerator and denominator of the s = 10 approximation by a common factor.

It is worth emphasizing that the rational Chebyshev approximations our algorithm provides are not optimal, in the sense that error does not remain constant within the range of approximation. Rather, error is least when one is sufficiently near the point z_0 and the quality of the approximation deteriorates as we move away from the point in question. The importance of the method lies, we believe, in the extreme simplicity with which it can provide rational Chebyshev approximations of any accuracy for a wide variety of functions. These non-optimal approximations may easily be used to obtain optimal Chebyshev approximations. Several algorithms have been developed to this effect.

Let us now speak of the origin of the problem that has occupied us in the last nine sections.

Section 10. SOME HISTORY.

About one hundred and twenty five years ago, the Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894) set himself the problem of finding the best rational approximation of a continuous function specified on an interval [a,b].

Specifically, he wanted to determine parameters p_0, p_1, ... ,p_n; q_0, q_1, ... , q_m in the expression

$$Q(x) = s(x)\,\frac{p_0\, x^n + p_1\, x^{n-1} + \ldots + p_n}{q_0\, x^m + q_1\, x^{m-1} + \ldots + q_m}\,, \qquad (10\text{-}1)$$

where m and n are given, and s(x) is a function continuous on [a,b], so that the deviation of Q(x) from a chosen continuous function f(x)

$$H_Q = \max_{a \le x \le b} |f(x) - Q(x)| \qquad (10\text{-}2)$$

shall be a minimum.

Chebyshev established the beautiful existence theorem [6,2]:

The function P(x), which deviates least from the function f(x) than does any other function of the type exemplified by equation (10-1) is completely characterized by the following property: If the function can be expressed in the form

$$P(x) = s(x)\,\frac{a_0\, x^{n-\sigma} + a_1\, x^{n-\sigma-1} + \ldots + a_{n-\sigma}}{b_0\, x^{m-\tau} + b_1\, x^{m-\tau-1} + \ldots + b_{m-\tau}} = s(x)\,\frac{A(x)}{B(x)}$$

where $0 \le \sigma \le n$, $0 \le \tau \le m$, $b_0 \ne 0$ and the fraction $\frac{A(x)}{B(x)}$ is irreducible, then the number N of consecutive points of the interval [a,b] at which the difference f(x) - P(x), with alternate change of sign, takes on the value H_P, is not less than $m + n + 2 - d$, where $d = \min(\sigma,\tau)$; in case $P(x) \equiv 0$, then $N \ge n + 2$.

Chebyshev did not provide a constructive approach to the problem of finding the rational approximations whose existence is guaranteed by the above theorem. He, and E. Solotarev did work out one example, based on the theory of Jacobian elliptic functions, which meets the requirements of the theorem [21]. Since that time, though, many people have sought to obtain an explicit method of attack for determining these rational approximations [12,13,14]. The problem is especially complicated by the fact that the class of continuous functions is a very broad one. Most of the methods of attack which have been developed deal with a more restrictive class of functions: bounded variation, analytic, or the like.

A substantial advance was made by H. Padé in his now classic thesis of 1892 [17]. Padé's method, mentioned briefly at the end of the last section, yields excellent rational approximations of analytic functions by means of solutions of a system of linear algebraic equations [23]. The method is an extension of some earlier work of Frobenius [9]. It, however, does not provide rational Chebyshev approximations. It is known that rational forms in Chebyshev polynomials yield better accuracy than ordinary rational forms [22].

Maehly gave a method for obtaining rational Chebyshev approximations of functions of bounded variation on the unit interval [15,22]. It has the substantial disadvantage that it requires that the given function be first expanded in a series of Chebyshev polynomials. If the function is anywhere complicated, these expansions may be devilishly hard to obtain.

To the best of our knowledge, no method is known for obtaining rational Chebyshev approximations which is better, more direct, or more powerful than the one we have presented in this paper. The method was discovered by one of the authors (Castellanos) as a result of his work on formulas to approximate π while in preparation of "The Ubiquitous π" [5]. The delicate and time-consuming task of carrying the algorithm into a working computer program was done by the other author (Rosenthal).

As explained in Section 1, the algorithm is a simple and immediate extension of an idea of Newton. The reader no doubt knows than when Abel was asked how he had managed to escalate so rapidly to the highest echelons of mathematical fame, he is said to have replied [4]: "By studying the masters, not their pupils." Laplace in turn gave the counsel "Lisez Euler, lisez Euler, c'est notre maître à tous." "Read Euler, read Euler, he is our master in everything." Gauss used to refer to Euler, Laplace, Lagrange, and Legendre with the complimentary "clarissimus," the clearest ones, while for Newton he reserved the term "summus," the supreme one [4]. If what little we have contributed with this paper may help to entice others to study the great masters, then that, and not the algorithm itself, will be our most rewarding contribution.

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