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Modulus of Smoothness of the Brownian Paths in the L^p Norm

BY ZBIGNIEW CIESIELSKI (SOPOT)

Abstract: *It shown that that for each p, $1 \leq p < \infty$, almost every path of the Brownian Motion satisfies Lipschitz condition with exponent $\frac{1}{2}$ in the L^p norm on I.*

1. Introduction

In this note by Brownian Motion we understand a Gaussian real valued stochastic process $(X(t),\ 0 \leq t \leq 1)$ over a probability space $(P, \Omega, \mathcal{F})$ with continuous trajectories on I and with covariance

$$E(X(t)X(s)) = \min(t, s),$$

where E is the mean.

The modulus of smoothness of $f \in L^p(I)$ is defined by formula

$$\omega_p(f;\delta) = \sup_{|h|<\delta} \left(\int_{I(h)} |f(t+h) - f(t)|^p \, dt \right)^{1/p},$$

where $I(h) = \{t \in I : t + h \in I\}$, and in case $p = \infty$ the L^p norm is being replaced by the L^∞ norm. It is classical result going back to N. Wiener [7] and P. Lévy [5] that for each α, $0 < \alpha < \frac{1}{2}$,

$$P\{\omega_\infty(X(\cdot);\delta) = O(\delta^\alpha) \ \text{ as } \ \delta \to 0_+\} = 1,$$

and moreover

$$P\{\omega_\infty(X(\cdot);\delta) = O(\delta^{\frac{1}{2}}) \ \text{ as } \ \delta \to 0_+\} = 0.$$

Our aim is to proof the following

THEOREM. *For each p, $1 \leq p < \infty$, we have*

$$P\{\omega_p(X(\cdot);\delta) = O(\delta^{\frac{1}{2}}) \ \text{ as } \ \delta \to 0_+\} = 1,$$

and

$$P\{\omega_p(X(\cdot);\delta) = o(\delta^{\frac{1}{2}}) \ \text{ as } \ \delta \to 0_+\} = 0.$$

This result was announced in [3].

2. The proof

The main tool in the proof is the so called *Ciesielski transform* (c.f. [6]) between a given Lipszchitz class and suitable sequence space. One instance of such transform is realized by means of the Franklin orthonormal system $(f_0, f_1, \ldots)$ (see [1], [2]) and it can be formulated as follows

LEMMA 2.1. *For given α, $0 < \alpha < 1$ and p, $1 \le p < \infty$ and for each $f \in L^p(I)$ the following conditions are equivalent:*

(i) $$\omega_p(f;\delta) = O(\delta^\alpha) \quad \text{as} \quad \delta \to 0_+,$$

(ii) $$M^{\frac{1}{2}-\frac{1}{p}+\alpha}\Big(\sum_{n=M+1}^{2M} |a_n|^p\Big)^{\frac{1}{p}} = O(1) \quad \text{as} \quad m \to \infty,$$

where $M = 2^m$ and a_n is the Fourier-Frankklin coefficient of f i.e.

$$a_n = (f, f_n) = \int_I f(s) f_n(s)\, ds, \quad n = 0, 1, \ldots$$

Moreover, the statement remains true after replacing the O by o in both conditions (i) *and* (ii).

The Brownian path $X = X(\cdot, \omega)$ is with probability 1 in $C(I)$ and therefore in $L^p(I)$. Thus, since (f_n) is a Schauder basis in $L^p(I)$, we have in this space with probability 1

$$X = \sum_{n=0}^{\infty} (X, f_n) f_n.$$

For later convenience let $b_n = (X, f_n)$. The sequence $(b_n,\ n = 0, 1, \ldots)$ is random and by the Lemma 2.1 it is sufficient to prove for $p = 2k$, with an arbitrary positive integer k, that

(2.2) $$P\{M^{p-1} \sum_{M+1}^{2M} |b_n|^p = O(1) \quad \text{as} \quad m \to \infty\} = 1,$$

where $M = 2^m$. Below, for the class of all Gaussian-random variables with mean zero and variance 1 we use the customary notation $N(0,1)$. To prove (2.2) we need the following

LEMMA 2.3. *Let $(X_i, i = 1,\dots)$ be a sequence of $X_i \in N(0,1)$ such that for some constants $A < \infty$ and q, $0 < q < 1$,*

$$|EX_iX_j| \le Aq^{|i-j|} \quad \text{for} \quad i,j = 1,2,\dots$$

Then for each positive integer k

$$E\Big(\frac{1}{N}\sum_{i=1}^{N} |X_i|^{2k} - \mu(2k)\Big)^2 \le \frac{C}{N} \quad \text{for} \quad N \ge 1, \tag{2.4}$$

where C i some constant independent of N and $\mu(k) = EX_i^k$.

PROOF: For the time being let $Y, Z \in N(0,1)$ and let $r = EY\,Z$. Clearly there is $U \in N(0,1)$ stochastically independent of Z i.e. such that $EZU = 0$. Now, since $E(\sqrt{1-r^2}U + rZ)Z = r$, the probability distributions in R^2 of (Y,Z) and $(\sqrt{1-r^2}U + rZ, Z)$ are the same. Thus, in particular we obtain

$$E(YZ)^{2k} = E((\sqrt{1-r^2}U + rZ)Z)^{2k}.$$

This implies

$$E(YZ)^{2k} = (\mu(2k))^2 + O(r^2),$$

where the O is independent of r. Application of this inequality to each (X_i, X_j) with $r_{i,j} = EX_iX_j$ and then summing up these new inequalities implies that the left hand side of (2.4) equals to

$$O\Big(\frac{1}{N^2}\sum_{i,j=1}^{N} r_{i,j}^2\Big) = O\Big(\frac{1}{N}\Big).$$

To proceed with the proof of the Theorem some more auxiliary results are needed. To this end let

$$F_n(t) = \int_t^1 f_n(s)\,ds \quad \text{for} \quad n = 0,1,\dots$$

Now, for $n = 2^m + j$ where m, j are integers such that $m \ge 0$ and $1 \le j \le 2^m$, and for $t_n = \frac{2j-1}{2^{m+1}}$ it was proved in [2] that

$$|f_n(t)| \le C\sqrt{n}q^{n|t-t_n|} \quad \text{for} \quad n = 2,3,\dots; \quad t \in I,$$

where C is a finite constant and the constant q is such that $0 < q < 1$. This and the orthogonality of f_n to 1 for $n > 0$ imply

$$|F_n(t)| \le C\frac{1}{\sqrt{n}}q^{n|t-t_n|} \quad \text{for} \quad n = 2,3,\dots; \quad t \in I.$$

As a consequence from the last inequality we obtain

$$|(F_{2^m+j}, F_{2^m+i})| \leq C\frac{1}{n^2}q^{|i-j|} \quad \text{for} \quad i,j = 1,\ldots,2^m; \quad m \geq 0. \tag{2.5}$$

It also follows that

$$\|F_n\|_2 \sim \frac{1}{n}. \tag{2.6}$$

Now, for fixed m define the Gausian random variables

$$X_j = \frac{b_{2^m+j}}{\|F_n\|_2} \quad \text{for} \quad j = 1,\ldots,2^m.$$

Since

$$b_n = \int_I F_n(t)\, dX(t)$$

it follows that

$$Eb_ib_j = (F_i, F_j).$$

Thus,(2.5) and (2.6) imply that the hypothesis of Lemma 2.3 are satisfied with $N = 2^m$ and therefore (2.2) follows and this completes the proof of the theorem.

3. Comments

The result presented here has a nice implementation. Consider the *Besov space* (generalized Lipschitz class) $B^\alpha_{p,q}(I)$ over I corresponding to the exponent of integration p, to the avareging exponent q, and to the real smothness parameter α, where $1 \leq p < \infty$ and $1 \leq q \leq \infty$. For the detailed definitions, in particular in the case of negative α, we refer to [4]. Our theorem claims that the trajectories of the Brownian Motion for each $p \in [1,\infty)$ are in $B^{\frac{1}{2}}_{p,\infty}(I)$ with probability 1. Consequently, for each $f \in B^{-\frac{1}{2}}_{p',1}(I)$, where $1/p' + 1/p = 1$, the following functional has prescribed sense

$$\int_0^1 f(s)X(s)\, ds.$$

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Instytut Matematyczny PAN
ul. Abrahama 18
81-825 Sopot, POLAND