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A Jackson-type Theorem on Approximation by Bounded Trigonometric Blending Polynomials

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1. Introduction

Let $M_{2\pi}(\mathbb{R}^d)$ denote the space of d-variate real-valued bounded functions which are 2π-periodic with respect to each variable. Let $C_{2\pi}(\mathbb{R}^d)$ be the subspace of continuous elements of $M_{2\pi}(\mathbb{R}^d)$.

Given a d-tuple of directions $X = (\xi_1, \ldots, \xi_d)$, $\xi_i \in \mathbb{R}^d$, the *directional (mixed) modulus of smoothness* of $f \in M_{2\pi}(\mathbb{R}^d)$ is defined by

$$\omega_X(f; s) = \sup_{|\delta_i| \le s_i} \left\| \left(\prod_{i=1}^{d} \triangle_{\delta_i}^{\xi_i} \right) f \right\| \tag{1}$$

with $s = (s_1, \ldots, s_d)$, $s_i, \delta_i \in \mathbb{R}$, $s_i \ge 0$, $i = 1, \ldots, d$. Here $\| \cdot \|$ denotes the uniform norm and

$$\left(\triangle_\delta^\xi f \right)(x) = f(x + \delta\xi) - f(x), \quad \delta \in \mathbb{R}, \; x, \xi \in \mathbb{R}^d$$

is a *directional forward difference* of f.

Jetter [9] has proved a Jackson-type theorem for the approximation of functions from $C_{2\pi}(\mathbb{R}^d)$ by certain spaces of 'direction-dependent' blending polynomials in terms of the modulus (1). In this note, we extend his result (under some additional assumptions) to a much larger subset of $M_{2\pi}(\mathbb{R}^d)$, namely to the set $B_{2\pi}^X$ of all functions for which $\omega_X(f; s) \to 0$ as $s_i \to 0$, $i = 1, \ldots, d$. Obviously $B_{2\pi}^X$ is a vector space.

In the sequel, we consider $\xi_i \in \mathbb{Z}^d$ and *unimodular* matrices of directions X, i.e., $|\det X| = 1$. Equivalently, there exist (unique) vectors $\alpha_{\xi_i} \in \mathbb{Z}^d$ such that $\alpha_{\xi_i} \cdot \xi_j = \delta_{ij}$ (Kronecker symbol). Note that if X is the matrix formed by the ξ_i written as rows, then the matrix formed by the α_{ξ_i} written as columns is the inverse of X.

Examples of elements from $B_{2\pi}^X \setminus C_{2\pi}(\mathbb{R}^d)$ are given by functions of the form

$$b(x) = g(x) \cdot \gamma(x \cdot \alpha_\xi), \ \xi \in X, \tag{2}$$

where $\gamma \in C_{2\pi}(\mathbb{R})$, and $g \in M_{2\pi}(\mathbb{R}^d)$ is independent of ξ, i.e., $g(x) = g(x + \lambda\xi)$ for all $\lambda \in \mathbb{R}$.

The space of *blending polynomials* with respect to the directions $X = (\xi_1, \dots, \xi_d)$, $\xi_i \in \mathbb{Z}^d$, and degrees $N = (n_1, \dots, n_d)$, $n_i \in \mathbb{N}_0$, is given by

$$\mathcal{T}_{N,X} := \sum_{i=1}^{d} T_{n_i,\xi_i}.$$

Here the spaces $T_{n,\xi}$ consist of all functions t which can be written in the form

$$t(x) = \sum_{k=0}^{2n} \varphi_k(x - (x \cdot \alpha_\xi)\xi)\tau_k(x \cdot \alpha_\xi), \tag{3}$$

where $\varphi_k \in M_{2\pi}$, $k = 0, \dots, 2n$, and $(\tau_k)_{k=0,\dots,2n}$ is a basis of the space $T_n(\mathbb{R})$ of univariate trigonometric polynomials of degree n. Thus, t is a polynomial in direction ξ; the coefficient functions $\tilde{\varphi}_k(x) = \varphi_k(x - (x \cdot \alpha_\xi)\xi)$ are independent of ξ. The restriction to integer components of ξ is necessary in order to guarantee that the approximants are 2π-periodic with respect to each variable.

By the example given in (2) we have that $\mathcal{T}_{N,X} \subset B_{2\pi}^X$. If X consists of canonical unit vectors, $B_{2\pi}^X$ is a space of so-called *B-continuous* functions. The notion of B-continuity was introduced by K. Bögel [1], [2] (see also [3]) and studied later by several authors, especially from the Romanian school. A number of references, in particular connected with approximation theory, can be found, e.g., in [5]. For canonical unit vectors ξ_i the functions from $\mathcal{T}_{N,X}$ are sometimes called quasi- or pseudo-polynomials in the literature, see, e.g., [6] and the references cited there.

For the degree of approximation

$$E_{N,X}(f) := \inf_{t \in \mathcal{T}_{N,X}} \|f - t\|$$

of functions $f \in C_{2\pi}(\mathbb{R}^d)$ Jetter [9] showed the Jackson-type estimate

$$E_{N,X}(f) \le c_d \cdot \omega_X \left(f; \frac{\pi}{n_1+1}, \ldots, \frac{\pi}{n_d+1}\right) \tag{4}$$

with a constant c_d independent of f and N. Here Jetter restricts himself to approximation by *continuous* elements from $T_{N,X}$. (In fact, Jetter's theorem is more general than stated in (4): he considers directions $X = (\xi_1, \ldots, \xi_\nu)$ with also $\nu \ne d$, and a weaker assumption on X than unimodularity even for $\nu \ne d$.) In the next section we will show the validity of (4) for all $f \in B_{2\pi}^X$.

2. Main Result

Bojanic and Shisha [4] have constructed discretely defined trigonometric polynomial operators K_n which satisfy a Jackson-type estimate

$$\|\gamma - K_n\gamma\| \le c_1 \cdot \omega\left(\gamma; \frac{1}{n+1}\right). \tag{5}$$

Here $\omega(\gamma;\cdot)$ is the usual modulus of continuity of γ; i.e., the modulus (1) for $d = 1$ and $X = \{1\}$. The authors stated their result for continuous functions $\gamma \in C_{2\pi}(\mathbb{R})$ only, and indeed the uniform approximation of discontinuous functions by trigonometric polynomials is not very interesting in itself. However, we will need estimate (5) for all bounded functions, and we will shortly explain that the proof in this case is essentially the same as in [4].

Lemma 1. *There exist linear operators $K_n : M_{2\pi}(\mathbb{R}) \to T_n(\mathbb{R})$, $n \in \mathbb{N}_0$, such that estimate (5) holds for all $\gamma \in M_{2\pi}(\mathbb{R})$ with a constant c_1 independent of γ and n.*

Proof. In [4] it was shown that there exist positive constant-reproducing operators of the form

$$(K_n\gamma)(z) = \sum_{k=0}^{2n} c_{k,n}\gamma(t_{k,n})\tau_k(z) \quad (c_{k,n}, t_{k,n} \in \mathbb{R}) \tag{6}$$

which satisfy

$$\sqrt{K_n\left(\sin^2\frac{z-\cdot}{2}; z\right)} \le \frac{c_0}{n+1} \tag{7}$$

for some constant $c_0 > 0$. Due to the specific form (6), K_n can be applied to all $\gamma \in M_{2\pi}(\mathbb{R})$.

In [4] the authors used a quantitative Korovkin theorem [8; Theorem 2.4] to arrive at (5) for $\gamma \in C_{2\pi}(\mathbb{R})$. We may follow exactly the lines of [8] in order to establish (5) for all $\gamma \in M_{2\pi}(\mathbb{R})$:

For all $z, t \in \mathbb{R}$ there is a $z' = z + 2\ell\pi$, $\ell \in \mathbb{Z}$, such that $|z' - t| \leq \pi$. Then

$$|\gamma(z) - \gamma(t)| = |\gamma(z') - \gamma(t)|,$$

and

$$|z' - t| \leq \pi |\sin \frac{z-t}{2}|.$$

Consequently,

$$\begin{aligned} |\gamma(z) - \gamma(t)| &\leq \omega(\gamma; |z' - t|) \leq \omega\left(\gamma; \pi \left|\sin \frac{z-t}{2}\right|\right) \\ &\leq \left(1 + \frac{\pi}{\delta}\left|\sin \frac{z-t}{2}\right|\right) \cdot \omega(\gamma; \delta) \end{aligned}$$

for all $\delta > 0$. Thus,

$$\begin{aligned} |K_n\gamma(z) - \gamma(z)| &\leq K_n(|\gamma(z) - \gamma(\cdot)|; z) \\ &\leq \left[1 + \frac{\pi}{\delta} K_n\left(\left|\sin \frac{z - \cdot}{2}\right|; z\right)\right] \cdot \omega(\gamma; \delta) \\ &\leq \left[1 + \frac{\pi}{\delta}\sqrt{K_n\left(\sin^2 \frac{z - \cdot}{2}; z\right)}\right] \cdot \omega(\gamma; \delta), \end{aligned}$$

where in the last step we have used the Cauchy-Schwarz inequality for positive linear operators. Setting $\delta = \frac{1}{n+1}$ and applying (7), we arrive at (5). □

Now we extend the definition of the operators (6) to the multivariate setting. Let $E = \{e_1, \ldots, e_d\}$ denote the identity matrix consisting of the canonical unit vectors e_i. The *parametric extension* $K_{n_i}^{e_i} : M_{2\pi}(\mathbb{R}^d) \to T_{n_i, e_i}$ is defined in such a way that K_{n_i} acts on a d-variate function g as if all variables exept of the i-th one where fixed; i.e.,

$$K_{n_i}^{e_i} g(z_1, \ldots, z_d) := (K_{n_i}\gamma)(z_i)$$

with $\gamma(t) := g(z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_d)$.

For the operators $K_{n_1}^{e_1}, \ldots, K_{n_\nu}^{e_\nu}$ the Boolean sum $\bigoplus_{i=1}^{\nu} K_{n_i}^{e_i}$ is defined inductively by

$$\bigoplus_{i=1}^{1} K_{n_i}^{e_i} := K_{n_1}^{e_1}; \quad \bigoplus_{i=1}^{\nu} K_{n_i}^{e_i} = K_{n_\nu}^{e_\nu} \oplus \bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i},$$

where $A \oplus B := A + B - A \circ B$ for operators A, B. Note that for the specific form (6) of K_n the Boolean sum is commutative.

Obviously, $\bigoplus_{i=1}^{d} K_{n_i}^{e_i} g \in \mathcal{T}_{N,E}$ with $N = (n_1, \ldots, n_d)$ for all $g \in M_{2\pi}(\mathbb{R}^d)$.

Lemma 2. *For all $g \in M_{2\pi}(\mathbb{R}^d)$ we have that*

$$\left\| g - \bigoplus_{i=1}^{d} K_{n_i}^{e_i} g \right\| \leq c_1^d \cdot \omega_E \left(g; \frac{1}{n_1+1}, \ldots, \frac{1}{n_d+1} \right).$$

Proof. It follows from Lemma 1 that

$$\left\| g - K_{n_k}^{e_k} g \right\| \leq c_1 \cdot \sup_{|\delta_k| \leq \frac{1}{n_k+1}} \left\| \Delta_{\delta_k}^{e_k} g \right\| \tag{8}$$

for all $1 \leq k \leq d$.

We proceed inductively and assume that

$$\left\| g - \bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i} g \right\| \leq c_1^{\nu-1} \cdot \sup_{|\delta_i| \leq \frac{1}{n_i+1}} \left\| \prod_{i=1}^{\nu-1} \Delta_{\delta_i}^{e_i} g \right\| \tag{9}$$

for some $1 < \nu \leq \delta$. For $\nu = 2$ this is identical to assertion (8) with $k = 1$.

We obtain from (8) for $k = \nu$ that

$$\begin{aligned} \left\| g - \bigoplus_{i=1}^{\nu} K_{n_i}^{e_i} g \right\| &= \left\| \left(g - \bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i} g \right) - K_{n_\nu}^{e_\nu} \left(g - \bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i} g \right) \right\| \\ &\leq c_1 \cdot \sup_{|\delta_\nu| \leq \frac{1}{n_\nu+1}} \left\| \Delta_{\delta_\nu}^{e_\nu} \left(g - \bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i} g \right) \right\|. \end{aligned}$$

Obviously, $\Delta_{h_\nu}^{e_\nu}$ and $\bigoplus_{i=1}^{\nu-1} K_{n_i}^{e_i}$ commute. (Roughly speaking, this is due to the fact that the first operator acts on the ν-th variable while the second one acts only on the first $\nu - 1$ variables of g.) Moreover, $\Delta_{h_i}^{e_i}$ and $\Delta_{h_j}^{e_j}$ commute for all $1 \leq i, j \leq d$. Therefore, (9) yields that

$$\left\| g - \bigoplus_{i=1}^{\nu} K_{n_i}^{e_i} g \right\| \leq c_1^{\nu} \cdot \sup_{|\delta_i| \leq \frac{1}{n_i+1}} \left\| \prod_{i=1}^{\nu} \Delta_{\delta_i}^{e_i} g \right\|.$$

For $\nu = d$ we arrive at the assertion of the lemma. □

We are now in the state to prove our main theorem. For the case $d = 2$ the following result was obtained in [7] as a consequence of a certain bivariate Korovkin-type theorem.

Theorem 3. *Let $X = (\xi_1, \ldots, \xi_d)$ be unimodular. Then there exists a constant $c > 0$ depending only on d such that*

$$E_{N,X}(f) \leq c \cdot \omega_X \left(f; \frac{1}{n_1+1}, \ldots, \frac{1}{n_d+1} \right)$$

for all $f \in B_{2\pi}^X$ and $N = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$.

Proof. Since X is unimodular, each $x \in \mathbb{R}^d$ can uniquely be written in the form

$$x = \sum_{i=1}^{d} z_i \xi_i$$

where $z_i = x \cdot \alpha_{\xi_i}$.

For the function $g \in M_{2\pi}(\mathbb{R}^d)$ given by $g(z_1, \ldots, z_d) = f(x)$ we have that

$$\omega_E(g; s) = \omega_X(f; s). \tag{10}$$

Thus, we may approximate $f(x)$ by

$$t(x) := \bigoplus_{i=1}^{d} (K_{n_i}^{e_i} g)(z_1, \ldots, z_d).$$

It is easy to see that $t \in \mathcal{T}_{N,X}$ because of the unimodularity of X. Hence the assertion follows from Lemma 2 and (10). □

As an immediate consequence of Theorem 3 we obtain

Corollary 4. *For any unimodular matrix X the space $\bigcup_{N \in \mathbb{N}_0^d} \mathcal{T}_{N,X}$ is dense in $B_{2\pi}^X$.* □

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