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A Note on Global Smoothness Preservation by Boolean Sum Operators

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1. Introduction

Let (L_n) be a sequence of approximation operators. The degree of approximation $\|L_nf-f\|$ for some function f is usually estimated in terms of a certain "modulus of smoothness" of f. For several applications (see [3], [9], [10], and the references cited there) and as an interesting problem in itself one wants to find a good estimate for the modulus of smoothness of L_nf, or also for that of the remainder $f-L_nf$, in terms of the modulus of smoothness of f, i.e., a result on "global smoothness preservation". For operators defined on spaces of real-valued continuous functions $f \in C(X)$ on a compact metric space X and the ordinary modulus of continuity, this question was dealt with in [1], [2].

In this note, we consider approximation of $f \in C(R)$, R a compact rectangle, by *Boolean sum operators*

$$L_1^x \oplus L_2^y := L_1^x + L_2^y - L_1^x \circ L_2^y.$$

Here L_1, L_2 are univariate operators and L_1^x, L_2^y are their ***parametric extensions***; i.e., L_1^x is defined in such a way as if L_1 were applied to the bivariate function f with second variable fixed, and L_2^y is defined likewise.

In this context, it is quite appropriate (cf., e.g., [4], [7] and the references cited there) to use the *mixed modulus of continuity*

$$\omega_{mixed}(f;\delta_1,\delta_2) := \sup_{\substack{|x-s|\le\delta_1 \\ |y-t|\le\delta_2}} |\Delta f[(x,y),(s,t)]|$$

where

$$\Delta f[(x,y),(s,t)] := f(x,y) - f(x,t) - f(s,y) + f(s,t)$$

is a *mixed difference* of f. An equivalent measure of smoothness is the *mixed K-functional* introduced in [5]:

$$K_{mixed}(f;\delta_1,\delta_2) := \inf_{\substack{g_1\in C^{1,0}(R) \\ g_2\in C^{0,1}(R) \\ h\in C^{1,1}(R)}} \{\|f-g_1-g_2-h\| + \delta_1\|D^{1,0}g_1\| + \delta_2\|D^{0,1}g_2\| + \delta_1\delta_2\|D^{1,1}h\|\}$$

Here $C^{1,0}(R)$, $C^{0,1}(R)$, $C^{1,1}(R)$ are the subspaces of all functions f for which the (mixed) partial derivatives $D^{1,0}f$, $D^{0,1}f$, $D^{1,1}f$ up to the indicated order are in $C(R)$. Furthermore, $\|\cdot\|$ will always denote the max-norm, and all operator norms figuring here will be based upon it.

The derivation of the main results in [1], [2] relies heavily on the fact that the least concave majorant of an (ordinary) modulus of continuity can be expressed in terms of corresponding (ordinary) K-functionals. Unfortunately, up to now we do not have a similar representation for mixed K-functionals. Instead, it was only proved in [5] that there exist constants c_1, c_2 independent of f such that

$$c_1\cdot\omega_{mixed}(f;\delta_1,\delta_2) \le K_{mixed}(f;\delta_1,\delta_2) \le c_2\cdot\omega_{mixed}(f;\delta_1,\delta_2).$$

But we do not know any optimal constants in this equivalence relation. Therefore, we prefer to directly give estimates of $K_{mixed}((L_1^x\oplus L_2^y)f;\delta_1,\delta_2)$ and $K_{mixed}(f-(L_1^x\oplus L_2^y)f;\delta_1,\delta_2)$ in terms of $K_{mixed}(f;\delta_1,\delta_2)$ which are shown to be optimal in some sense. As a by-product of our considerations we also arrive at a certain optimal estimate for the modulus of continuity of the remainder for the univariate Bernstein polynomials.

2. Main Results

The announced result on global smoothness preservation by Boolean sum operators in terms of mixed K-functionals is as follows.

Theorem 1. *Let I_1, I_2 be two compact real intervals, $R = I_1 \times I_2$, and let $L_i : C(I_i) \to C(I_i)$, $i = 1, 2$, be continuous univariate operators, $L_i \neq 0$. Moreover, suppose that L_i maps $C^1(I_i)$ (functions with derivative $D^1\gamma \in C(I_i)$) into itself, satisfying an estimate*

$$\|D^1 L_i \gamma\| \leq c_i \cdot \|D^1\gamma\| \quad \text{for all } \gamma \in C^1(I_i).$$

Then, for all $f \in C(R), \delta_1, \delta_2 \geq 0$,

$$\text{(i)}\quad K_{mixed}((L_1^x \oplus L_2^y)f; \delta_1, \delta_2)$$

$$\leq \|L_1\| \cdot K_{mixed}\left(f; \frac{c_1\delta_1}{\|L_1\|}, \delta_2\right) + \|L_2\| \cdot K_{mixed}\left(f; \delta_1, \frac{c_2\delta_2}{\|L_2\|}\right)$$

$$+\|L_1\| \cdot \|L_2\| \cdot K_{mixed}\left(f; \frac{c_1\delta_1}{\|L_1\|}, \frac{c_2\delta_2}{\|L_2\|}\right);$$

$$\text{(ii)}\quad K_{mixed}(f - (L_1^x \oplus L_2^x)f; \delta_1, \delta_2)$$

$$\leq (1 + \|L_1\|)(1 + \|L_2\|) \cdot K_{mixed}\left(f; \frac{(1+c_1)\delta_1}{1+\|L_1\|}, \frac{(1+c_2)\delta_2}{1+\|L_2\|}\right),$$

where $\|L_1\|, \|L_2\|$ denote the corresponding operator norms.

Proof. We observe that $\|L_1^x\| = \|L_1\|$, $\|L_2^y\| = \|L_2\|$. Moreover, we have (see [8] and [6], respectively) the commutativity relations $L_1^x \circ L_2^y = L_2^y \circ L_1^x$ and $D^{1,0} \circ L_2^y = L_2^y \circ D^{1,0}$, $D^{0,1} \circ L_1^x = L_1^x \circ D^{0,1}$. Since L_i maps $C^1(I_i)$ into itself we obtain

$$K_{mixed}(L_1^x f; \delta_1, \delta_2)$$

$$\leq \inf_{\substack{g_1 \in C^{1,0}(R)\\ g_2 \in C^{0,1}(R)\\ h \in C^{1,1}(R)}} \{\|L_1^x(f - g_1 - g_2 - h)\| + \delta_1\|D^{1,0}(L_1^x g_1)\| + \delta_2\|D^{0,1}(L_1^x g_2)\| + \delta_1\delta_2\|D^{1,1}(L_1^x h)\|\}$$

$$\leq \inf_{\substack{g_1 \in C^{1,0}(R)\\ g_2 \in C^{0,1}(R)\\ h \in C^{1,1}(R)}} \{\|L_1\| \cdot \|f - g_1 - g_2 - h\| + \delta_1 c_1\|D^{1,0}g_1\| + \|L_1\| \cdot \delta_2\|D^{0,1}g_2\| + \delta_1\delta_2 c_1\|D^{1,1}h\|\}$$

$$= \|L_1\| \cdot K_{mixed}\left(f; \frac{c_1}{\|L_1\|}\delta_1, \delta_2\right).$$

In the same way, we get

$$K_{mixed}(L_2^y f; \delta_1, \delta_2) \le \|L_2\| \cdot K_{mixed}\left(f; \delta_1, \frac{c_2}{\|L_2\|}\delta_2\right),$$

$$K_{mixed}((L_1^x \circ L_2^y)f; \delta_1, \delta_2) \le \|L_1\| \cdot \|L_2\| \cdot K_{mixed}\left(f; \frac{c_1}{\|L_1\|}\delta_1, \frac{c_2}{\|L_2\|}\delta_2\right).$$

Since

$$K_{mixed}((L_1^x \oplus L_2^y)f; \delta_1, \delta_2)$$

$$\le K_{mixed}(L_1^x f; \delta_1, \delta_2) + K_{mixed}(L_2^y f; \delta_1, \delta_2) + K_{mixed}((L_1^x \circ L_2^y)f; \delta_1, \delta_2),$$

we arrive at part (i) .

Moreover, we have for all $g_1 \in C^{1,0}(R)$, $g_2 \in C^{0,1}(R)$, $h \in C^{1,1}(R)$, and $\delta_1, \delta_2 \ge 0$ that

$$K_{mixed}(f - (L_1^x \oplus L_2^y)f; \delta_1, \delta_2)$$

$$= \inf_{\substack{\tilde{g}_1 \in C^{1,0}(R) \\ \tilde{g}_2 \in C^{0,1}(R) \\ \tilde{h} \in C^{1,1}(R)}} \{\|f - (L_1^x \oplus L_2^y)f - \tilde{g}_1 - \tilde{g}_2 - \tilde{h}\| + \delta_1\|D^{0,1}\tilde{g}_1\| + \delta_2\|D^{0,1}\tilde{g}_2\| + \delta_1\delta_2\|D^{1,1}\tilde{h}\|\}$$

$$= \inf_{\substack{g_1 \in C^{1,0}(R) \\ g_2 \in C^{0,1}(R) \\ h \in C^{1,1}(R)}} \{\|(f - g_1 - g_2 - h) - (L_1^x \oplus L_2^y)(f - g_1 - g_2 - h)\|$$

$$+\delta_1\|D^{1,0}[g_1 - (L_1^x \oplus L_2^y)g_1]\| + \delta_2\|D^{0,1}[g_2 - (L_1^x \oplus L_2^y)g_2]\| + \delta_1\delta_2\|D^{1,1}[h - (L_1^x \oplus L_2^y)h]\|\}$$

$$\le \inf_{\substack{g_1 \in C^{1,0}(R) \\ g_2 \in C^{0,1}(R) \\ h \in C^{1,1}(R)}} \{(1 + \|L_1\|)(1 + \|L_2\|) \cdot \|f - g_1 - g_2 - h\|$$

$$+\delta_1 \cdot (1 + c_1)(1 + \|L_2\|) \cdot \|D^{1,0}g_1\| + \delta_2 \cdot (1 + c_2)(1 + \|L_1\|) \cdot \|D^{0,1}g_2\|$$

$$+\delta_1\delta_2 \cdot (1 + c_1)(1 + c_2) \cdot \|D^{1,1}h\|\},$$

from which we obtain part (ii) . □

For the well-known Bernstein operators $B_k : C[0,1] \to C^1[0,1]$ we have $\|B_k\| = 1$ and $\|D^1 B_k \gamma\| \le \|D^1 \gamma\|$ for all $\gamma \in C^1[0,1]$; these statements can easily be derived from the first pages of [11]. Therefore, as a typical application of Theorem 1, we obtain

Corollary 2. *For all* $f \in C([0,1]^2)$, $\delta_1, \delta_2 > 0$,

(i) $K_{mixed}((B_m^x \oplus B_n^y)f; \delta_1, \delta_2) \le 3K_{mixed}(f; \delta_1, \delta_2)$;

(ii) $K_{mixed}(f - (B_m^x \oplus B_n^y)f; \delta_1, \delta_2) \le 4K_{mixed}(f; \delta_1, \delta_2)$.

The estimates used in the proof of Theorem 1 are quite straight-forward and simple. Therefore one might expect that the given result is yet generally improvable. But in fact the following discussion shows that this is not the case.

We observe that for functions of product-type $f(x,y) = f_1(x) \cdot f_2(y)$ the identity

$$K_{mixed}(f;\delta_1,\delta_2) = K(f_1;\delta_1) \cdot K(f_2;\delta_2)$$

is valid (see [5; Lemma 12]). Here K denotes the usual univariate K-functional with respect to $C(I_i)$ and $C^1(I_i)$, $i = 1,2$. Thus, we obtain

$$K_{mixed}(f - (B_m^x \oplus B_n^y)f;\delta_1,\delta_2) = K(f_1 - B_m f_1;\delta_1) \cdot K(f_2 - B_n f_2;\delta_2).$$

Moreover,

$$K_{mixed}((B_m^x \oplus B_n^y)f;\delta_1,\delta_2) \geq K_{mixed}(f - (B_m^x \oplus B_n^y)f;\delta_1,\delta_2) - K_{mixed}(f;\delta_1,\delta_2).$$

Hence, the following lemma shows that neither estimate (i) nor (ii) of Corollary 2 (and thus also those of Theorem 1 either) can be improved in general.

Lemma 3. *For all $\epsilon > 0$ and $k \in \mathbb{N}$, there is a $\delta = \delta(\epsilon,k)$ and a function $\gamma = \gamma(\delta)$ such that*

$$K(\gamma - B_k\gamma;\delta) > (2-\epsilon) \cdot K(\gamma;\delta).$$

Proof. Because of the well-known identity (see [12])

$$K(\gamma;\delta) = \frac{1}{2}\tilde{\omega}(\gamma;2\delta),$$

it suffices to show the assertion for $\tilde{\omega}$ in place of K. Here $\tilde{\omega}$ denotes the least concave majorant of the (usual) modulus of continuity $\omega(\gamma;\delta)$.

We consider the function γ whose graph linearly connects the points $(0,0), (\delta,-\delta)$ and $(1,1)$ for $\delta < \frac{1}{2k}$; i.e., all points $\left(\frac{j}{n}, f\left(\frac{j}{n}\right)\right)$ lie on the line $\ell(x) = \frac{1}{1-\delta}[-\delta(1-x) + x - \delta]$ for $j \geq 1$, and we have $f(0) = 0$. Thus

$$\begin{aligned} B_k\gamma(x) &= \ell(x) - B_k(\ell - \gamma)(x) = \ell(x) - (\ell - \gamma)(0) \cdot (1-x)^k \\ &= \frac{-\delta(1-x)}{1-\delta} + \frac{x-\delta}{1-\delta} + \frac{2\delta}{1-\delta}(1-x)^k. \end{aligned}$$

Since $B_k\gamma$ is an increasing function, it is easily seen that

$$\tilde{\omega}(\gamma - B_k\gamma;\delta) = \omega(\gamma - B_k\gamma;\delta) = 2\delta(1-\delta)^{k-1}$$

while

$$\tilde{\omega}(\gamma,\delta) = \omega(\gamma,\delta) = \frac{1+\delta}{1-\delta}\delta.$$

Hence, using Bernoulli's inequality we see that the inequality of Lemma 3 is satisfied for all $\delta < \frac{\epsilon}{2(k+1)-\epsilon}$. □

The previous lemma also enables us to determine the optimal constant c in the estimate $K(\gamma - B_k\gamma;\delta) \le c \cdot K(\gamma;\delta)$, or equivalently,

$$\omega(\gamma - B_k\gamma;\delta) \le \tilde{\omega}(\gamma - B_k\gamma;\delta) \le c \cdot \tilde{\omega}(\gamma;\delta).$$

Since (cf. [1]) $\tilde{\omega}(B_k\gamma;\delta) \le \tilde{\omega}(\gamma;\delta)$, an application of the triangle inequality yields

Corollary 4. *For all $\gamma \in C[0,1]$, $k \in \mathbb{N}$, and $\delta > 0$,*

$$\tilde{\omega}(\gamma - B_k\gamma;\delta) \le 2\tilde{\omega}(\gamma;\delta).$$

The constant 2 *is optimal in the sense of Lemma* 3.

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