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# HERMITE INTERPOLATION ON TWO NEW NODES MATRICES[1]

GIULIANA CRISCUOLO* - BIANCAMARIA DELLA VECCHIA* - GIUSEPPE MASTROIANNI**

**Abstract.** The authors establish simultaneous approximation theorems, by using Hermite interpolation on two new nodes matrices.

## 1. Introduction

Given a differentiable function $f$ on $[-1,1]$ and a matrix $V$ of nodes in $[-1,1]$, we denote by $H_m(V;f)$ the corresponding Hermite interpolating polynomial of degree $2m-1$.

The problem of uniform convergence of Hermite interpolating polynomials on the zeros of Jacobi polynomials was studied by several authors and some classical results can be found for example in [17].

A complete theorem on the convergence of the derivatives of Hermite polynomials on the zeros of Jacobi polynomials was also given by Neckermann and Runk in [11].

Some years ago Pottinger [12] and Steinhaus [15] proved that, if $x_i, i=1,...,m$, are the Chebyshev nodes of first kind, then the estimate

$$||H_m||_1 = O(m), \quad ||H_m||_1 = \sup_{||f||_1=1} ||H_m f||_1,$$

holds, where $||f||_1 = \max\{||f||, ||f'||\}$.

Recently Xu in [18] obtained the estimate

$$\text{const} \leq m^{-2-2\gamma}||H_m||_1 \leq \text{const}, \quad \gamma = \max\{\alpha,\beta\},$$

for the more general weight

$$w(x) = \phi(x)(1-x)^{\alpha}(1+x)^{\beta}, \alpha, \beta > -1, 0 < \phi, \phi' \in \text{Lip } 1.$$

However, from all these papers it follows that the convergence conditions for Hermite interpolation consist in some restrictions for the Jacobi parameters, namely they must be less than 0.

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Recently the authors in [3] showed that the restrictions on Jacobi parameters can be relaxed, provided that we add a suitable number of points near $\pm 1$.

This procedure of adding points near $\pm 1$ was introduced by Szabados [16] and Runk and Vertesi [13] and used in several papers; among the others we mention [1,5-7,9,10].

Later the authors in [2] considered Hermite interpolation on matrices generated by the product of two polynomials orthogonal with respect to two different but correlated in some sense weights. Such matrices are called extended matrices. Examples of extended matrices are the following

$$X_1 = \{ \text{ zeros of } p_{m+1}^{(\alpha,\beta)} p_m^{(\alpha+1,\beta+1)} \},$$
$$X_2 = \{ \text{ zeros of } p_m^{(\alpha,\beta+1)} p_m^{(\alpha+1,\beta)} \},$$

where $p_m^{(\alpha,\beta)}$ is the $m$-th orthonormal Jacobi polynomial with parameters $\alpha$ and $\beta$.

Extended matrices were used also by Srivastava in [14], but for quasi Hermite-Fejér interpolation (if the Jacobi parameters in $X_1$ are equal 0).

Unfortunately the extended matrices are not generally good matrices, in the sense that the corresponding Lebesgue constants of Hermite interpolation are high.

However, if we add suitable points near $\pm 1$ to the above extended matrix, then the corresponding Hermite interpolating polynomial realizes simultaneous approximation optimal in some sense. Obviously this favourable situation does not happen for all extended matrices.

Very recently Mastroianni and Prossdorf in [8] introduced a collocation method for solving numerically singular integral equations and they proved that the zeros of the polynomials $p_m^{(\alpha,-\alpha)}(x)p_m^{(-\alpha,\alpha)}(x)$, $-1 < \alpha < 1$, or $p_m^{(\alpha,1-\alpha)}(x)p_{m+1}^{(-\alpha,\alpha-1)}(x)$, $0 < \alpha < 1$ have an arcsin distribution. This property allowed to consider the following extended matrices

$$Y_1 = \{t_{k,2m}, k = 1, ..., 2m / t_{k,2m} \text{ zeros of } p_m^{(\alpha,-\alpha)}(x)p_m^{(-\alpha,\alpha)}(x)\}$$

$$Y_2 = \{t_{k,2m+1}, k = 1, ..., 2m+1 / t_{k,2m+1} \text{ zeros of } p_m^{(\alpha,1-\alpha)}(x)p_{m+1}^{(-\alpha,\alpha-1)}(x)\}.$$

In the present paper we consider Hermite interpolation on the previous two matrices plus some points near $\pm 1$ and we obtain pointwise simultaneous approximation estimates, which are optimal in a certain sense. This interpolation process has the remarkable property that the convergence conditions are independent on the Jacobi parameters.

## 2. Main results

If $a$ and $b$ are two quantities depending on some parameters, we write $a \sim b$, if $a/b \leq$ const and $b/a \leq$ const, uniformly for the parameters in question.

Together with the first matrix $Y_1$, we can consider the following $r+s$ additional points $y_{j,m} = -1 + \frac{j-1}{s}(1+t_{1,2m})$, $j = 1, ..., s$ and $z_{i,m} = t_{2m,2m} + \frac{i}{r}(1-t_{2m,2m})$, $i = 1, ..., r$, distributed on $[-1,1]$ as follows

$$-1 = y_{1,m} < ... < y_{s,m} < t_{1,2m} < t_{2,2m} < ... < t_{2m,2m} < z_{1,m} < ... < z_{r,m} = 1.$$

This is not the only,possible choise, as we will see later on. Here we consider the equispaced case just for sake of simplification.
Then, we denote by $H_{m,r,s}(f)$ the Hermite interpolating polynomial of degree $4m+r+s-1$ interpolating $f$ and $f'$ on the zeros of $p_m^{(\alpha,-\alpha)}p_m^{(-\alpha,\alpha)}$ and on the additional points $y_{j,m}, j = 1,...,m$ and $z_{i,m}, i = 1,...,r$, defined by

$$H_{m,r,s}^{(i)}(f;t_{k,2m}) = f^{(i)}(t_{k,2m}), \quad i = 0,1, \quad k = 1,...,2m,$$

$$H_{m,r,s}(f;y_{k,m}) = f(y_{k,m}), \quad k = 1,...,s, \tag{2.1}$$

$$H_{m,r,s}(f;z_{k,m}) = f(z_{k,m}), \quad k = 1,...,r.$$

We complete the definition by putting $H_{m,0,0}(f) = H_m(f)$.

We will call $H_{m,r,s}(f)$ the *extended Hermite interpolating polynomial.*

Now, denoting by $\mathcal{P}_n$ the set of algebraic polynomials of degree at most $n$, we let

$$E_n(f) = \min_{P\in\mathcal{P}_n} ||f-P||, \quad f \in C([-1,1]), \quad \text{with } ||f-P|| = \max_{|x|\le 1}|f(x)-P(x)|.$$

Then we establish the following

**Theorem 2.1.** *For all $f \in C([-1,1])$ and for all $\alpha \in (-1,1)$,*

$$||f - H_{m,1,1}(f)|| \le \text{const } E_{m-1}(f')\frac{logm}{m}, \tag{2.2}$$

*with some constant independent of $f$ and $m \ge 9$. If in addition $f$ is an analytic function, then (2.2) can be replaced by*

$$\frac{|f(x) - H_{m,1,1}(f;x)|}{\Delta_m(x)} \le C(\alpha)\frac{||f^{(4m+2)}||}{2^{4m+1}(4m+2)!}, \quad \Delta_m(x) = \sqrt{1-x^2}+m^{-2}, \tag{2.3}$$

*where $C(\alpha)$ is a constant depending only on $\alpha$.*

Estimate (2.3) is optimal; infact Erdos and Turan proved that, for every nodes matrix $\{x_k, k = 1,...,m, m \in N\}$,

$$\max_{|x|\le 1}\sum_{k=1}^{m} l_{m,k}^2(x)|x-x_k| \ge \frac{1}{m}[\frac{2}{\pi}logm - \text{const } log(logm)],$$

where $l_{m,k}(x)$ is the $k$-th fundamental Lagrange polynomial. (See also [17, pag. 348]).

Now, letting

$$||f||_k = \max_{0\le i\le k}||f^{(i)}||$$

, with $||.||_0 = ||.||$, we get the following simultaneous approximation theorem.

**Theorem 2.2.** *Let $f \in C^q([-1,1])$, $q \geq 1$ and $-1 < \alpha < 1$. Then, for $h = 0, ..., q$,*

$$\|f - H_{m,r,s}(f)\|_h \leq \text{const } E_{m-q}(f^{(q)}) \frac{logm}{m^{q-h}}, \tag{2.4}$$

*with some constant independent of $f$ and $m \geq 4q+5$, whenever the integers $r$ and $s$ fulfill*

$$\frac{h}{2} + 1 \leq r < \frac{h}{2} + 2, \tag{2.5}$$

$$\frac{h}{2} + 1 \leq s < \frac{h}{2} + 2. \tag{2.6}$$

To complete the previous theorem, we observe that, if we interpolate $f$ and $f'$ only on the matrix $Y_1$ and $H_m(f)$ is the corresponding extended Hermite interpolating polynomial, then it is not hard to prove that

$$\|H_m\| \sim m^2.$$

Therefore, the matrix having as knots the zeros of the polynomial $p_m^{(\alpha,-\alpha)} p_m^{(-\alpha,\alpha)}$ is not a good interpolation matrix generally.
Then Theorem 2.2 assures us that, by adding suitable $s$ knots near -1 and $r$ knots near 1, we transform the "bad" extended Jacobi matrix into a good one, i.e. the extended Hermite polynomial $H_{m,r,s}(f)$ is a good approximation of $f$ and its derivatives simultaneously.

From (2.5) and (2.6) it follows that the number of the additional nodes depends only on the order of differentiation $h$, but is independent on the Jacobi parameter $\alpha$ and there exist infinitely many good matrices for which (2.4) holds true.

Untill now we assumed that the additional points are simple, but we can consider in particular the case $y_{j,m} = -1, j = 1, ..., s$ and $z_{i,m} = 1, i = 1, ..., r$. Obviously now we must assume $f$ smooth enough, i.e. $r, s \leq q+1$. Then $H_{m,r,s}(f)$ is the extended Hermite polynomial defined by

$$H^{(i)}_{m,r,s}(f; t_{k,2m}) = f^{(i)}(t_{k,2m}), \quad i = 0, 1, \quad k = 1, ..., 2m,$$

$$H^{(j)}_{m,r,s}(f; -1) = f^{(j)}(-1), \quad j = 0, ..., s-1, \tag{2.7}$$

$$H^{(i)}_{m,r,s}(f; +1) = f^{(j)}(+1), \quad i = 0, ..., r-1.$$

For the polynomial defined by (2.7), we can obtain pointwise simultaneous approximation estimates of Gopengauz-Telyakovskii type. Indeed we have

**Theorem 2.3.** *Let $f \in C^q([-1,1])$, $q \geq 1$ and $f^{(q)} \in Lip_M \lambda$, $0 < \lambda \leq 1$. Assume that the values $f^{(i)}(-1), i = 0, ..., s-1$, and $f^{(j)}(1), j = 1, ..., r-1$, with $r, s \leq q+1$ are* *known. Then there exist infinitely many Jacobi weights* $w^{(\alpha,-\alpha)}$*, such that, for* $|x| \leq 1$ *and* $h = 0, ..., \min\{r, s\}$,

$$\left| [f(x) - H_{m,r,s}(f;x)]^{(h)} \right| \leq \text{const } \left[\frac{\sqrt{1-x^2}}{m}\right]^{q+\lambda-h} \log m,$$

*for some constant independent of* $f$ *and* $m$.

Finally we remark that the same theorems can be obtained, if we consider the extended Hermite interpolating polynomial on the matrix $Y_2$.

We omit the proofs of Theorems 2.1, 2.2 and 2.3 for sake of brevity. The demonstration techniques are similar to those used in [2] and in [3]. Here we only mention that we used classical estimates of orthogonal polynomials, of Christoffel functions and of their first derivatives, Gopengauz-Telyakovskii and Dzjadik Lemmas and some properties of the matrices $Y_1$ and $Y_2$ showed in [8].

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* Istituto per Applicazioni della Matematica, C.N.R.
Via P. Castellino 111, 80131 Napoli

** Dipartimento di Matematica, Università della Basilicata
Via N.Sauro 85, 85100 Potenza
Istituto per Applicazioni della Matematica, C.N.R.
Via P. Castellino 111, 80131 Napoli