CONSTRUCTIVE THEORY OF FUNCTIONS, Varna '91
Sofia, 1992, pp. 99-103

On Salzer's Approach to Trigonometric Hermite Interpolation

Franz J. Delvos
University of Siegen, Germany

1. Introduction Salzer [3] has investigated the problem on general trigonometric Hermite interpolation. Salzer reduced the interpolation problem to a system of auxiliary equations which he feeled always to be solvable but he gave no proof. In this paper we will show that these equations are always solvable by solving the problem the problem of osculatory trigonometric interpolation at one point. By the way it can be shown that Salzer's approach is extendable to trigonometric Hermite interpolation with arbitrary orders at the interpolation points.

2. Osculatory interpolation at one point

Let

$$e_r(x) = \exp(irx) \quad , \quad r \in \mathbb{Z} .$$

The algebra of trigonometric polynomials is given by

$$\tau = \langle\langle\, e_r : r \in \mathbb{Z} \,\rangle\rangle .$$

The linear space of trigonometric polynomials of degree m is given by

$$\tau_m = \langle\langle\, e_r : -m \le r \le m \,\rangle\rangle.$$

Note that τ_m possesses always odd dimension $2m+1$.

Proposition 1 Let $r = 2m+1$ with $m \in \mathbb{Z}_+$. Suppose that f is a periodic sufficiently differentiable function. Then there is a unique trigonometric polynomial $H_r(f)$ of degree m satisfying the interpolatory conditions

$$D^s H_r(f)(0) = D^s f(0) \qquad (s = 0,\dots, 2m) .$$

Proof: The result is standard [1] but we include it for completeness. It is sufficient to show that the homogenous problem has only the trivial solution. Note that for any trigonometric polynomial g(x) of degree m there is a unique algebraic polynomial q(z) of degree 2m such that

$$g(x) = z^{-m}q(z), \qquad z = \exp(ix).$$

It is easily seen that

$$D^s g(0) = 0 \ (s \le 2m) \quad \Leftrightarrow \quad D^s q(1) = 0 \quad (s \le 2m)$$

which implies $q(z) = 0$ and $g(x) = 0$. ∎

Salzer's approach uses implicitly antiperiodic trigonometric polynomials. A periodic function f is *antiperiodic* iff

$$f(x+\pi) = -f(x) \quad .$$

The linear space of anitperiodic trigonometric polynomials is given by

$$\omega = \langle\langle \{ e_{2r-1} : r \in \mathbb{Z} \} \rangle\rangle .$$

The linear space of antiperiodic trigonometric polynomials of degree 2m-1 is given by

$$\omega_m = \langle\langle \{ e_{2r-1} : -m < r \le m \} \rangle\rangle.$$

Note that ω_m possesses always even dimension 2m.

Proposition 2 Let $r = 2m$ with $m \in \mathbb{N}$. Suppose that f is a antiperiodic sufficiently differentiable function. Then there is a unique antiperiodic trigonometric polynomial $G_r(f)$ of degree 2m-1 satisfying the interpolatory conditions

$$D^s G_r(f)(0) = D^s f(0) \qquad (s = 0,\dots, 2m-1) .$$

Proof: We consider the homogeneous problem. Note first that

$$\omega_m \subseteq \tau_{2m-1} .$$

For any antiperiodic trigonometric polynomial h(x) of degree 2m-1 there is a unique algebraic polynomial of p(z) of degree 2(2m-1) such that

$$h(x) = z^{-2m+1}p(z) \quad , \quad z = \exp(ix) .$$

Since g(x) is antiperiodic we have

$$D^s h(0) = 0, \qquad D^s h(\pi) = 0 \qquad (s = 0,\dots, 2m-1) .$$

It is easily seen that

$$D^s h(0) = 0 \quad (s < 2m) \quad \Leftrightarrow \quad D^s p(1) = 0 \quad (s < 2m) ,$$

$$D^s h(\pi) = 0 \quad (s < 2m) \quad \Leftrightarrow \quad D^s p(-1) = 0 \quad (s < 2m).$$

Recall that p(z) is a polynomial of degree 2(2m-1) . Since

$$D^s p(-1) = 0 \quad (s < 2m), \qquad D^s p(1) = 0 \quad (s < 2m)$$

we obtain $p(z) = 0$ and $h(x) = 0$. ■

3. Osculatory interpolation of arbitrary order at any number of points

The interpolation points x_j are assumed to satisfy

$$0 \le x_1 < \dots < x_n < 2\pi .$$

We consider the most general Hermite trigonometric interpolation problem where the derivatives of the smooth periodic function f are prescribed as

$$D^k f(x_j) \quad (0 \le k < r_j, \quad 1 \le j \le n) .$$

Next we introduce the positive intergers

$$R = r_1 + \dots + r_n , \qquad R_j = R - r_j \quad (1 \le j \le n).$$

Then we define the trigonometric functions

$$U(x) = \sin((x-x_1)/2)^{r_1} \dots \sin((x-x_n)/2)^{r_n} ,$$

$$U_j(x) = U(x)/\sin((x-x_j)/2)^{r_j} .$$

It is easily seen that for even R_j the trigonometric function $U_j(x)$ is a trigonometric polynomial of degree $R_j/2$:

(1) $$U_j(x) \in \tau_{R_j/2} .$$

If R_j is odd then the trigonometric function $U_j(2x)$ is an antiperiodic trigonometric polynomial of degree R_j :

(2) $$U_j(2x) \in \omega_{R_j} .$$

Proposition 3 Let $R = r_1 + \dots + r_n$ be odd and let f be a periodic sufficiently differentiable function. Then there exists a unique trigonometric polynomial $H_R(f)$ of degree $(R-1)/2$ satisfying

$$D^k H_R(f)(x_j) = D^k f(x_j) \quad (0 \le k < r_j, \quad 1 \le j \le n) .$$

Proof: Salzer searched for a Lagrange type representation of the interpolant:

(3) $$H_R(f)(x) = \sum_{j=1}^{n} \sum_{k<r_j} D^k f(x_j) L_{j,k}(x) .$$

The construction of the basis function $L_{j,k}(x)$ depends on R_j and r_j. Suppose first

(4) $$R_j = 2m_j , \qquad r_j = 2n_j + 1 .$$

The basis function $L_{j,k}(x)$ is defined by

(5) $$L_{j,k}(x) = U_j(x) v_k(x-x_j)$$

where $v_k(x)$ is the unique trigonometric polynomial of degree n_j such that

(6) $$D^t[U_j(x)v_k(x-x_j)]|_{x=x_j} = \delta_{k,t} \qquad (0 \le k,t < r_j) .$$

The existence of $v_k(x)$ follows from Proposition 1 and the triangular form of the linear system (6) :

(7) $$U_j(x_j)\cdot D^t v_k(0) + \sum_{u<t} b_{u,t} D^u v_k(0) = \delta_{k,t} \qquad (0 \le t < r_j) .$$

Since $U_j(x) \in \tau_{R_j/2}$ and $v_k(x-x_j) \in \tau_{(r_j-1)/2}$ it follows

$$L_{j,k}(x) = U_j(x)v_k(x-x_j) \in \tau_{(R-1)/2} .$$

Assume next

(8) $$R_j = 2m_j + 1 , \qquad r_j = 2n_j .$$

In this case $V_j(x) = U_j(2x)$ is an antiperiodic polynomial of degree R_j. The basis function $L_{j,k}(x)$ is now defined by

(9) $$L_{j,k}(x) = U_j(x)v_k((x-x_j)/2)$$

where $v_k(x)$ is the unique antiperiodic polynomial of degree r_j-1 with

(10) $$D^t[U_j(x)v_k((x-x_j)/2)]|_{x=x_j} = \delta_{k,t} \qquad (0 \le k,t < r_j) .$$

The existence of $v_k(x)$ follows now from Proposition 2 and the triangular form of the linear system (10) :

(11) $$2^{-t}U_j(x_j)\cdot D^t v_k(0) + \sum_{u<t} c_{u,t} D^u v_k(0) = \delta_{k,t} \qquad (0 \le t < r_j) .$$

Since $U_j(2x) \in \omega_{R_j}$ and $v_k(x-x_j) \in \omega_{r_j-1}$ it follows again

$$L_{j,k}(x) = U_j(x)v_k((x-x_j)/2) \in \tau_{(R-1)/2} .$$

Thus the general trigonometric Hermite interpolant (3) can be generated with the aid of (5) and (9) . The uniqueness of $H_R(f)$ follows from dimension arguments. ■

Salzer's approach is given by the equations (3), (5) , and (6). He considered only the case $r_j = r$ which implies $R = n \cdot r$. Therefore the case (8) is not contained in Salzer's approach .

References

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Franz J. Delvos
Lehrstuhl fuer Mathematik I
University of Siegen
D - 5900 Siegen (Germany)