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# ERROR ESTIMATES OF THE LOWEST ORDER MIXED FINITE ELEMENT METHOD ON GRIDS WITH REGULAR LOCAL REFINEMENT

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**1. Introduction.** Let $\Omega \subset\subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega$. Assume that the Dirichlet problem

$$(1.1)\quad \begin{aligned} &\text{(a)}\quad Lp = -div(a(x)\nabla p + \underline{b}(x)p) + c(x)p = f(x), \quad x \in \Omega \\ &\text{(b)}\quad p = -g(x), \quad x \in \Omega \end{aligned}$$

is uniquely solvable for $\{f,g\} \in L^2(\Omega) \times H^{3\backslash 2}(\partial\Omega)$ and that

$$(1,2)\qquad \|p\|_2 \le Q\left(\|f\|_0 + \|g\|_{3/2;\partial\Omega}\right),$$

where $\underline{\nabla}w$ denotes the gradient of a scalar function $w$, $div\ \underline{v} = \underline{\nabla}.\underline{v}$ denotes the divergence of a vector function $\underline{v}$, and $a(x) : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the requirement $a(x) \ge a_0 > 0$. We set

$$(1.3)\qquad \underline{u} \equiv (u_1, u_2) = (a(x)\underline{\nabla}p + \underline{b}(x)p)$$

$$(1.4)\qquad \alpha(x) = a(x)^{-1}, \quad \underline{\beta}(x) = \alpha(x)\underline{b}(x)$$

Let

$$(1.5)\quad \begin{aligned} &\text{(a)}\quad \underline{V} = \underline{H}(div;\Omega) = \left\{\underline{u} \in L^2(\Omega)^2 : div\ \underline{u} \in L^2(\Omega)\right\}, \\ &\text{(b)}\quad W = L^2(\Omega) \end{aligned}$$

Then the weak form of (1.1) reads as follows: find a pair $(u,p) \in V \times W$ such that

$$(1.6)\quad \begin{aligned} &\text{(a)}\quad (\alpha\underline{u},\underline{v}) - (div\ \underline{v},p) + (\underline{\beta}p,\underline{v}) = \langle g,\underline{v},\underline{\nu}\rangle, \quad \underline{v} \in \underline{V}, \\ &\text{(b)}\quad (div\ \underline{u},w) + (cp,w) = (f,w), \quad w \in W, \end{aligned}$$

where $\underline{\nu}$ is the outer normal to $\Omega$ and the inner product in $L^2(\Omega)^2$ is indicated by $(.,.)$ and in $L^2(\partial\Omega)$ by $\langle .,.\rangle$ respectively.

For the sake of simplicity we take the domain $\Omega$ to be the square $(1,0)^2$. The results that we dereve in this paper can be easily extendet to more general domains.

The existence, uniqueness and global rate of convergence of the finite element solution to (1.5) $(\underline{u}_h, p_h)$, using Raviart-Thomas spaces [2] have been established in [3]. For regular finite elements there have been derived global $L^2$-estimates, which for the polynomials of lowest degree are of the following type:

$$(1.7)\qquad \begin{aligned} &(a) \quad \|p-p_h\|_{0,\Omega} \le Qh\|p\|_{2,\Omega}, \\ &(b) \quad \|\underline{u}-\underline{u}_h\|_{0,\Omega} \le Qh\|p\|_{2,\Omega}, \\ &(c) \quad \|div(\underline{u}-\underline{u}_h)\|_{0,\Omega} \le Qh\|p\|_{3,\Omega}. \end{aligned}$$

Error analysis of the finite element solution of (1.5) in the case $\underline{b}=\underline{0}$ and $c=0$ with regular local refinement of rectangular elements has been considered in [4]. Efficient iterative methods for the system algebraic equations on the composite grid have been discussed in [5].

In this paper , using modification of Raviart-Thomas projection [6] we establish error estimates of lowest order mixed finite element method on grid with regular local refinement . These estimates are optimal for $\|p-p_h\|_{0,\Omega}$ and $\|div(\underline{u}-\underline{u}_h)\|_{0,\Omega}$.

The plan of the paper is as follows. In section 2. we give the grid with regular local refinement and construct the Raviart-Thomas finite element spaces. The important part here is the construction of the modified Raviart-Thomas projection on the composite grid.Then we present local estimates. In section 3. we give the duality Lemma which is essential for the error analysis. Finally, in section 4. we establish the main result of this paper - convergence of the mixed finite element solution in $L^2$ norm.

**2.Mixed finite element approximation.** Our goal is to construct the mixed element approximations to the problem (1.5) on rectangular grids with regular local refinement.

First, we introduce the coarse-grid partition of the rectangle $\Omega$ denoted by $\tau_c$ (see , for instance [4]). The partition $\tau_c$ in general is nonuniform but regular [1]. with a characteristic parameter $h_c$. Let $\Omega_1$ , a sub domain of $\Omega$ , be a union of a certain number of coarse finite elements. We partition the elements in $\Omega_1$ ,introducing a finer mesh, denoted by $\tau_f$, as shown in Figure 1. We suppose that the refinement is uniform with parameter $h_f$ and $h_c = nh_f$ . We set

$$
(2.1)\quad
\begin{aligned}
&(a)\quad \Omega_2 = \Omega \setminus \Omega_1 \\
&(b)\quad I_f = \left\{ T \in \tau_f \in \Omega_1 : \bar{T} \cap \bar{\Omega}_2 \neq \emptyset \right\} \\
&(c)\quad I_c = \left\{ T \in \tau_c \in \Omega_1 : \bar{T} \cap \bar{\Omega}_2 \neq \emptyset \right\} \\
&(d)\quad \tau_h = \tau_c \cup \tau_f
\end{aligned}
$$

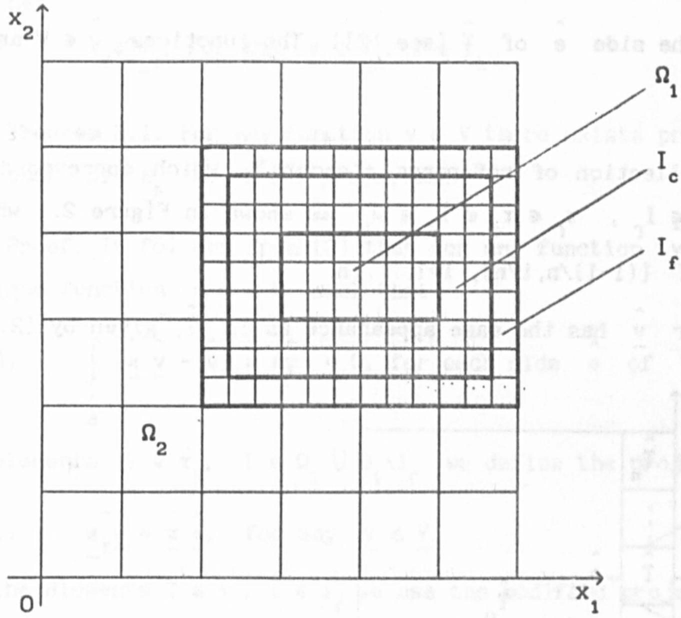


**Figure 1. Grid with local refinement**

Let $\underline{V}_c$ be Raviart-Thomas space in $\Omega$ with index r=0 for $\tau_c$, andlet $\underline{V}_f(\Omega_1)$ be Raviart-Thomas space in $\Omega_1$ with index r=0for $\tau_f$ such that the normal component of the vector function on the $\Omega_1$ is zero. Following [4],[5] we define the Raviart-Thomas spaces on the composite grid $\tau_h$ as follow:

$$
(2.2)\quad
\begin{aligned}
&(a)\quad \underline{V} = \underline{V}_c + \underline{V}_f(\Omega_1); \\
&(b)\quad W_h = \left\{ w \in L^2(\Omega),\ w_{|T \in \tau_h} = \text{const} \right\}.
\end{aligned}
$$

The mixed finite element approximating to (1.1) is defined by determining a pair $\{\underline{u}_h, p_h\} \in \underline{V}_h \times W_h$ such that

$$
(2.3)\quad
\begin{aligned}
&(a)\quad (\alpha \underline{u}_h, \underline{v}) - (\mathit{div}\ \underline{v}, p_h) + (\beta p_h, \underline{v}) = \langle g, \underline{v}.\nu \rangle\ ,\ \underline{v} \in \underline{V}_h \\
&(b)\quad (\mathit{div}\ \underline{u}_h, w) + (c p_h, w) = (f, w)\ ,\ w \in W_h
\end{aligned}
$$

Now we introduce the modified Raviart-Thomas projections [6]

$$\underline{\pi}_h \times P_h\ :\ \underline{V} \times W \longrightarrow \underline{V}_h \times W_h .$$

Let $\hat{T} = [0,1]$ be the unit square in the $(\hat{x}_1,\hat{x}_2)$-plane. The function $\underline{v} = (\hat{v}_1,\hat{v}_2) \in \underline{\hat{V}}$ is

$$(2.4) \quad \begin{aligned} &(a) \quad \hat{v}_1 = a_0 + a_1\hat{x}_1 \\ &(b) \quad \hat{v}_2 = b_0 + b_1\hat{x}_2 \end{aligned}$$

and the degrees of freedom of $\hat{v}$ are uniquely determined by the moments $\int_{\hat{e}} \underline{\hat{v}}.\underline{\hat{\nu}}\, d\gamma$ of $\underline{\hat{v}}.\underline{\hat{\nu}}$ on the side $\hat{e}$ of $\hat{T}$ (see [2]). The functions $\hat{w} \in \hat{W}$ are constant on $\hat{T}$.

We introduce a collection of reference elements , which correspond to interface elements $\tau_f \in I_f$ , $\tau_f \in \tau_c \in I_c \in \Omega_1$ as shown in Figure 2., where $\hat{T}_0 = [0,1]^2$ , $\hat{T}_i = [0,1/n]$ $[(i-1)/n, i/n]$, $i=1,\dots,n$.

In $\hat{T}_i$ the vector $\underline{\hat{v}}$ has the same appearance as in $\hat{T}$, given by (2.4).

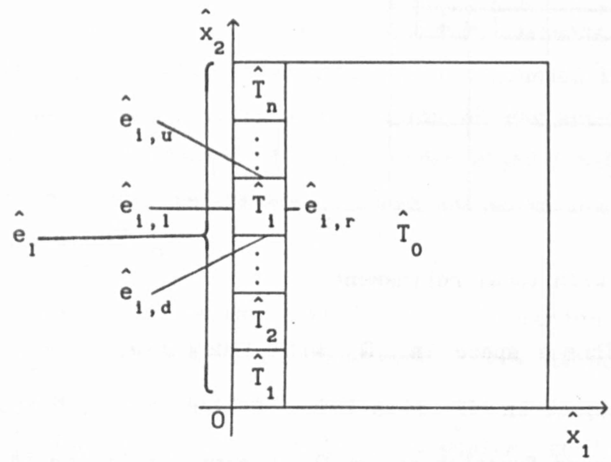

**Figure 2.** Basic collection of elements.

The degrees of freedom of $\underline{\hat{v}}$ on $\hat{T}_i$ are determined by

$$(2.5) \quad \begin{aligned} &(a) \quad \int_{\hat{e}} \underline{\hat{v}}.\underline{\hat{\nu}}\, d\gamma, \\ &(b) \quad \int_{\hat{e}_{i,p}} \underline{\hat{v}}.\underline{\hat{\nu}}\, d\gamma \quad , \ i = 1, \dots ,n \ ; \ p = d,r,u \end{aligned}$$

Let $F_T : \hat{x} \longrightarrow x = F_T(\hat{x}) = B_T\hat{x} + \underline{b}_T$, $B_T \in L(\mathbb{R}^2)$, $\underline{b}_T \in \mathbb{R}^2$, be the unique linear mapping of $\hat{T}$ into $T$. With any scalar function $w$ on $T$ we define[2]

$$(2.6) \qquad w = \hat{w} \circ F_T^{-1} \qquad ( \hat{w} = w \circ F_T)$$

and for any vector-valued function $\underline{v}$ on $T$

$$(2.7) \qquad \underline{v} = \frac{1}{J_T} B_T \hat{\underline{v}} \circ F_T^{-1} \qquad ( \hat{\underline{v}} = J_T B_T^{-1} \underline{v} \circ F_T),$$

where $J_T = \det(B_T)$.

With the rectangle $T$ we associate the space

$$(2.8) \qquad \underline{V}_T = \left\{ \underline{v} \in \underline{H}(div;T),\ \hat{\underline{v}} \in \hat{\underline{V}} \right\}.$$

Let

$$(2.9) \qquad \underline{v}_T \in \underline{V}_T, \text{ for any } \underline{v} \in \underline{V}_h,\ T \in \tau_h.$$

**Theorem 2.1.** For any function $\underline{v} \in \underline{V}$ there exists projection $\underline{\pi}_h : \underline{V} \longrightarrow \underline{V}_h$ such that *div* $\underline{\pi}_h$ is $L^2(\Omega)$-projection.

Proof. It follows from [2] that for any function $\hat{\underline{v}} \in H^1(\hat{T})^2$ there exists a unique function $\hat{\underline{\pi}}\, \hat{\underline{v}} \in \hat{\underline{V}}$ such that

$$(2.10) \qquad \int_{\hat{e}} (\hat{\underline{\pi}}\, \hat{\underline{v}} - \hat{\underline{v}}).\hat{\underline{\nu}}\, d\gamma = 0, \text{ for each side } \hat{e} \text{ of } \hat{T}.$$

For elements $T \in \tau_h$, $T \in \Omega_2 \cup \Omega_1 \backslash I_f$ we define the projection $\underline{\pi}_T$ by

$$(2.11) \qquad \widehat{\underline{\pi}_T \underline{v}} = \hat{\underline{\pi}}\, \hat{\underline{v}}, \text{ for any } \underline{v} \in \underline{V}.$$

For the elements $T \in \tau_f$, $T \in I_f$ we use the modified projection [6].

Since on $\hat{e}_1$ $\hat{\pi}\hat{v}_1$ is a constant by (2.5.a) we get

$$(2.12) \qquad \hat{\pi}\, \hat{v}_1 \big|_{\hat{x}_1=0} = \int_0^1 \hat{v}_1\, d\hat{x}_2.$$

The requirement that *div* $\underline{\pi}_h$ is $L^2$- projection written for all $\hat{T}_i$, $i=1,\ldots,n$, leads to the system

$$(2.13) \qquad \begin{cases} M_i = \sum_{j=1}^{i} E_j \\ 0 = \sum_{i=1}^{n} E_i, \end{cases}$$

where

$$M_i = \int_{\hat{e}_{i,u}} ( \hat{v}_2 - \hat{\pi}\, \hat{v}_2)\, d\hat{x}_1, \qquad i=1,\ldots,n-1,$$

$$E_j = \int_{\hat{e}_{j,l}} ( \hat{v}_1 - \hat{\pi}\, \hat{v}_1)\, d\hat{x}_2, \qquad j=1,\ldots,n.$$

The system (2.13) determine a unique function $\hat{\underline{\pi}}\, \hat{\underline{v}} \in \hat{\underline{V}}(\hat{T}_1)$. For the elements

$T \in \tau_f$, $T \in I_f$ we define $\underline{\pi}_T$ again by (2.11) and we set $\underline{\pi}_h|_T = \underline{\pi}_T$. Thus by construction $\underline{\pi}_h: \underline{V} \longrightarrow \underline{V}_h$ and $div\ \underline{\pi}_h$ is $L^2(\Omega)$-projection.

In the case when the refined element has two adjacent coarse grids elements the construction of $\underline{\pi}_h$ is similar.

**Theorem 2.2.** Let $\underline{\pi}_h \times P_h: \underline{V} \times W \longrightarrow \underline{V}_h \times W_h$, where $\underline{\pi}_h$ is defined in Theorem 2.1. and $P_h$ is $L^2(\Omega)$-projection. Then the following approximation properties hold

(2.14)

(a) $\|w - P_h w\|_{-s,T} \le Q\ \|w\|_{k,T}\ h_T^{k+s}$, $0 \le k,s \le 1$, $T \in \tau_h$

(b) $\|div(\underline{\pi} - \underline{\pi}_h \underline{v})\|_{-s,T} \le Q\ \|div\ \underline{v}\|_{k,T}\ h_T^{k+s}$, $0 \le k,s \le 1$, $T \in \tau_h$

(c) $\|\underline{v} - \underline{\pi}_h \underline{v}\|_{0,T} \le Q\ \|\underline{v}\|_{1,T}\ h_T$, $T \in \tau_h \backslash I_f$

(d) $\|\underline{v} - \underline{\pi}_h \underline{v}\|_{0,T_f} \le Q\ \|\underline{v}\|_{1,T_c} h_c$, $T_f \in I_f$, $T_f \in T_c \in I_c$

Proof. Inequality (2.14a),(2.14b) and (2.14c) follow from [2]. Now, let us consider $T_f \in I_f$, with corresponding reference element $\hat{T}_1$. Let $\hat{\underline{v}} = (c_1, c_2)$ on $\hat{T}_1$. We continue $\hat{\underline{v}}$ from $\hat{T}_1$ in $\hat{T}_0$. Then $\hat{\underline{\pi}}\, \hat{\underline{v}} = \hat{\underline{v}}$ and (see [1], Theorem 3.1.4. in the vector form ) we obtain (2.14d).

**3. The Duality Lemma** . We say that $\Omega$ is 2-regular if the Dirichlet problem

(3.1)

(a) $L^* \varphi = \psi$, $x \in \Omega$,

(b) $\varphi = 0$, $x \in \partial\Omega$

is uniquely solvable for $\psi \in L^2(\Omega)$ and if

(3.2) $\|\varphi\|_2 \le Q\ \|\psi\|_0$, for all $\varphi \in L^2(\Omega)$.

For any $\underline{f} \in L^2(\Omega)^2$ we define

(3.3) $\underline{f}(\underline{v}) = (\underline{f}, \underline{v})$, $\underline{v} \in \underline{V}$

**Lemma 3.1.** Let $\underline{V}_h \times \underline{W}_h$ is defined as in (2.2) Assume that $\Omega$ is 2-regular. Let $\underline{\zeta} \in \underline{V}$, $\underline{f} \in \underline{V}$ and $g \in L^2(\Omega)$. If $z \in W_h$ satisfies the relations

(3.4)

(a) $(\alpha\underline{\zeta}, \underline{v}) - (div\ \underline{v}, z) + (\underline{\beta} z, \underline{v}) = \underline{f}(\underline{v})$, $\underline{v} \in \underline{V}_h$

(b) $(div\ \underline{\zeta}, w) + (cz, w) = g(w)$, $w \in W_h$

then for sufficiently small $h_c$

$$\|z\|_{0,\Omega} \le Q\Big[\|\underline{f}\|_{-1,\Omega} + \|g\|_{-2,\Omega} + \|\underline{f}\|_{0,\Omega_1}h_f + \|\underline{f}\|_{0,\Omega_2}h_c + \|\underline{f}\|_{0,I_f}n^{1\backslash 2}h_c \tag{3.1}$$

$$+ \|\underline{\zeta}\|_{0,\Omega_1}h_f + \|\underline{\zeta}\|_{0,\Omega_1}h_c + \|\underline{\zeta}\|_{0,I_f}n^{1\backslash 2}h_c + \|div\ \underline{\zeta}\|_{0,\Omega_1}h_f + \|div\ \underline{\zeta}\|_{0,\Omega_2}h_c\Big].$$

Proof. Let $\psi \in L^2(\Omega)$ and let $\varphi \in H^2(\Omega) \cap H_1^0(\Omega)$ be the solution of (3.1). From [3] follows that

$$(z,\psi) = \underline{f}(\underline{\pi}_h(a\underline{\nabla}\varphi)) + g(P_h\varphi) + (\alpha\underline{\zeta} + \underline{\beta}z, a\underline{\nabla}\varphi - \underline{\pi}_h(a\underline{\nabla}\varphi)) + (div\ \underline{\zeta} + cz, \varphi - P_h\varphi).$$

Using (2.14) and (3.2) we consequently estimate the right hand side terms:

$$|\underline{f}(\underline{\pi}_h(a\underline{\nabla}\varphi))| \le |\underline{f}(a\underline{\nabla}\varphi)| + |\underline{f}(\underline{\pi}_h(a\underline{\nabla}\varphi) - a\underline{\nabla}\varphi)|$$

$$\le Q\Big[\|\underline{f}\|_{-1,\Omega}\|a\underline{\nabla}\varphi\|_{1,\Omega} + \sum_{T\in\tau}\|\underline{f}\|_{0,T}\|\underline{\pi}_h(a\underline{\nabla}\varphi) - a\underline{\nabla}\varphi)\|_{0,T}\Big]$$

$$\le Q\Big[\|\underline{f}\|_{-1,\Omega} + \|\underline{f}\|_{0,\Omega_1}h_f + \|\underline{f}\|_{0,\Omega_2}h_c + \|\underline{f}\|_{0,I_f}n^{1\backslash 2}h_c\Big]\|\psi\|_{0,\Omega};$$

Similarly

$$|g(P_h(\varphi)| \le Q\Big[\|g\|_{-2,\Omega} + \|g\|_{0,\Omega_1}h_f + \|g\|_{0,\Omega_2}h_c\Big]\|\psi\|_{0,\Omega};$$

$$(\alpha\underline{\zeta} + \underline{\beta}z,\ a\underline{\nabla}\varphi - \pi_h(a\underline{\nabla}\varphi)) \le Q\Big[\|\underline{\zeta}\|_{0,\Omega_1}h_f + \|\underline{\zeta}\|_{0,\Omega_2}h_c + \|\underline{\zeta}\|_{0,I_f}n^{1\backslash 2}h_c$$

$$+ \|z\|_{0,\Omega_1}h_f + \|z\|_{0,\Omega_2}h_c + \|z\|_{0,I_f}n^{1\backslash 2}h_c\Big]\|\psi\|_{0,\Omega};$$

$$(div\ \underline{\zeta} + cz,\ \varphi - P_h\varphi)$$

$$\le Q\Big[\|div\ \underline{\zeta}\|_{0,\Omega_1}h_f + \|div\ \underline{\zeta}\|_{0,\Omega_2}h_c + \|z\|_{0,\Omega_1}h_f + \|z\|_{0,\Omega_2}h_c\Big]\|\psi\|_{0,\Omega}.$$

This sequence of inequalities completes the proof.

**4. Error estimates in $L^2(\Omega)$.** Assume for a moment that (2.3) has a unique solution for sufficiently small $h_c$ (which will be an easy consequence from the convergence).

We set

$$\text{(a)}\quad \underline{\xi} = \underline{u} - \underline{u}_h,\ \underline{\sigma} = \underline{\pi}_h\underline{u} - \underline{u}_h \tag{4.1}$$

$$\text{(b)}\quad \eta = p - p_h,\ \tau = P_h p - p_h,\ \rho = p - P_h p$$

Then from (1.5) and (2.3) we get

$$(4.2)\quad \begin{aligned} &\text{(a)}\quad (\alpha\underline{\xi},\underline{v}) - (div\ \underline{v},\eta) + (\beta\eta,\underline{v}) = 0, \quad \underline{v}\in \underline{V}_h, \\ &\text{(b)}\quad (div\ \underline{\xi},w) + (c\eta,w) = 0, \quad w\in W_h, \end{aligned}$$

or equivalently

$$(4.3)\quad \begin{aligned} &\text{(a)}\quad (\alpha\underline{\xi},\underline{v}) - (div\ \underline{v},\tau) + (\beta\tau,\underline{v}) = -(\beta\rho,\underline{v}), \quad \underline{v}\in \underline{V}_h, \\ &\text{(b)}\quad (div\ \underline{\xi},w) + (c\tau,w) = -(c\rho,w)\ , \quad w\in W_h. \end{aligned}$$

Then for sufficiently small $h_c$ from the duality Lemma we get

$$(4.4)\quad \begin{aligned} \|\tau\|_{0,\Omega} \le Q\Big[&\|\rho\|_{-1,\Omega} + \|\rho\|_{0,\Omega_1}h_f + \|\rho\|_{0,\Omega_2}h_c + \|\rho\ \|_{0,I_f}h_c n^{1/2} \\ &+ \|\underline{\xi}\|_{0,\Omega_1}h_f + \|\underline{\xi}\|_{0,\Omega_2}h_c + \|\underline{\xi}\ \|_{0,I_f}h_c n^{1/2} + \|div\ \underline{\xi}\|_{0,\Omega_1}h_f + \|div\ \underline{\xi}\|_{0,\Omega_2}h_c\Big]. \end{aligned}$$

From (2.14a) we have

$$(4.5)\quad \begin{aligned} &\text{(a)}\quad \|\rho\|_{0,\Omega_1} \le Q\|p\|_{k,\Omega_1}h_f^{\,k}, \qquad 0\le k\le 1 \\ &\text{(b)}\quad \|\rho\|_{0,\Omega_2} \le Q\|p\|_{k,\Omega_2}h_c^{\,k}, \qquad 0\le k\le 1 \\ &\text{(c)}\quad \|\rho\|_{0,I_f} \le Q\|p\|_{k,I_f}h_f^{\,k}, \qquad 0\le k\le 1 \end{aligned}$$

On the other hand

$$(4.6)\quad \|\rho\|_{-1,\Omega} \le \Big(\sum_{T\in\tau_h}\|\rho\|^2_{-1,T}\Big)^{1/2} \le Q\Big[\|p\|_{t,\Omega_1}h_f^{t+1} + \|p\|_{t,\Omega_2}h_c^{t+1}\Big], \quad 0\le t\le 1.$$

From (4.4), (4.5) and (4.6) for $t = k$ and since $\eta = \rho + \tau$ we get

$$(4.7)\quad \begin{aligned} \|\eta\|_{0,\Omega} \le Q\Big[&\|p\|_{s,\Omega_1}h_f^{\,s} + \|p\|_{s,\Omega_2}h_c^{\,s} + \|p\|_{k,I_f}h_f^{\,k}h_c n^{1/2} \\ &+ \|\underline{\xi}\|_{0,\Omega_1}h_f + \|\underline{\xi}\|_{0,\Omega_2}h_c + \|\underline{\xi}\|_{0,I_f}h_c n^{1/2} \\ &+ \|div\ \underline{\xi}\|_{0,\Omega_1}h_f + \|div\ \underline{\xi}\|_{0,\Omega_2}h_c\Big], \quad 0\le s,t\le 1 \end{aligned}$$

Since $(div\ \underline{\sigma},w) = (div\ \underline{\xi},w)$ for $w\in W_h$, it follows from (4.2b) for $w = div\ \underline{\sigma}.\chi_T$ that

$$(4.8)\quad \|div\ \underline{\sigma}\|_{0,T} \le Q\ \|\eta\|_{0.T},$$

where $\chi_T$ is characterastic fufction of $T\in\tau_h$. Therefore

(4.9) $$\|div\ \underline{\xi}\|_{0,T} \le Q\left[\|\eta\|_{0,T} + \|div\ \underline{u}\|_{q,T}\ h_T^q\ \right], \qquad 0 \le q \le 1$$

If in (4.2a) we choose the test function $\underline{v} = \underline{\sigma}$ then

$$(\alpha\underline{\sigma},\underline{\sigma}) = (div\ \underline{\sigma},\eta) - (\underline{\beta}\eta,\underline{\sigma}) - (\alpha\underline{u} - \underline{\pi}_h\underline{u}),\underline{\sigma});$$

Therefore

(4.10) $$\|\underline{\sigma}\|_{0,\Omega} \le Q\left[\|\eta\|_{0,\Omega} + \|\underline{u} - \underline{\pi}_h\underline{u}\|_{0,\Omega}\right]$$

and since $\underline{\xi} = \underline{\sigma} + \underline{u} - \underline{\pi}_h\underline{u}$, we get

(4.11) $$\|\underline{\xi}\|_{0,\Omega} \le Q\left[\|\eta\|_{0,\Omega} + \|\underline{u} - \underline{\pi}_h\underline{u}\|_{0,\Omega}\right]$$

If we replace (4.9) and (4.12) in (4.7) choose k=s=1 and q=1, since $\|\underline{u}\|_1 + \|div\ \underline{u}\|_0 \le Q\|p\|_2$ then for sufficiently small $h_c$ we shall obtain

(4.13) $$\|\eta\|_{0,\Omega} \le Q\left[\|p\|_{2,\Omega_1} h_f + \|p\|_{2,\Omega_2} h_c\right]$$

From (4.12) and (4.9) we get consequently

(4.14) $$\|\underline{\xi}\|_{0,\Omega} \le Q\left[\|p\|_{2,\Omega_1} h_f + \|p\|_{2,\Omega_2} h_c + \|p\|_{2,I_f} h_c n^{1/2}\right]$$

and

(4.15) $$\|div\underline{\xi}\|_{0,\Omega} \le Q\left[\|p\|_{3,\Omega_1} h_f + \|p\|_{3,\Omega_2} h_c\right].$$

Before completing the error analysis we demonstrate the existence and uniqueness of the solution of (2.3). Since (2.3) is linear, it suffices to establish uniqueness. Let the functions h and g are zero. Then the choice $w = div\ \underline{u}_h\chi_T$ in (2.3b) get

(4.16) $$\|div\ \underline{u}_h\|_{0,T} \le Q\|p_h\|_{0,T}$$

Now, Lemma 3.1. implies that

(4.17) $$\|p_h\|_{0,\Omega} \le Q\Big[(\|\underline{u}_h\|_{0,\Omega_1} + \|div\ \underline{u}_h\|_{0,\Omega_1})h_f + (\|\underline{u}_h\|_{0,\Omega_2} + \|div\ \underline{u}_h\|_{0,\Omega_2} + \|\underline{u}_h\|_{0,I_f}.n^{1/2})h_c\Big].$$

From (4.16) and (4.17) it follows that for sufficiently small $h_c$

(4.18) $$\|p_h\|_{0,\Omega} \le Q\left[\|\underline{u}_h\|_{0,\Omega_1} h_f + (\|\underline{u}_h\|_{0,\Omega_2} + \|\underline{u}_h\|_{0,I_f})h_c\right].$$

If in (2.3a) we take the test function $\underline{v} = \underline{u}_h$, then

(4.19) $\|\underline{u}_h\|_{0,\Omega} \le Q\,\|p_h\|_{0,\Omega}$

Inequalities (4.16) and (4.19) show that for sufficiently small $h_c$ $\underline{u}_h = \underline{0}$ and $p_h = 0$, i.e. the system (2.3) has unique solution.

Thus we have proved the following theorem:

**Theorem 4.1.** Assume that the Dirichlet problem (1.1) has a unique solution for $\{\underline{f}, g\} \in L^2(\Omega) \times H^{3/2}(\partial\Omega)$ and that $\Omega$ is 2-regular. Then, for $h_c$ sufficiently small there exists a unique solution $\{\underline{u}_h, p_h\} \in \underline{V}_h \times W_h$ of the mixed finite element method with regular local refinement given by (2.3). Moreover, the error $\{\underline{u} - \underline{u}_h, p - p_h\}$ can be estimated as follows

$$\text{(a)} \quad \|p - p_h\|_{0,\Omega} \le Q\Big[\|p\|_{2,\Omega_1} h_f + \|p\|_{2,\Omega_2} h_c\Big]$$

$$\text{(4.20)} \quad \text{(b)} \quad \|\underline{u} - \underline{u}_h\|_{0,\Omega} \le Q\Big[\|p\|_{2,\Omega_1} h_f + (\|p\|_{2,\Omega_2} + \|p\|_{2,I_f} n^{1/2}) h_c\Big]$$

$$\text{(c)} \quad \|div(\underline{u} - \underline{u}_h)\|_{0,\Omega} \le Q\Big[\|p\|_{3,\Omega_1} h_f + \|p\|_{3,\Omega_2} h_f\Big].$$

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