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Irregular distribution of $\{n\beta\}$, $n = 1, 2, 3, \ldots$ and curious basic hypergeometric series

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Abstract

Let $\beta\varepsilon(0,1)$ be irrational and $\{n\beta\}$ denote the fractional part of $n\beta$, $n \geq 1$. The uniform distribution of $\{n\beta\}$, $n \geq 1$, implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt\,,$$

for each bounded and Riemann-integrable g. Hardy and Littlewood and Sobol investigated this relation for singular g.

We show that by choosing suitable β, and g with an arbitrarily weak singularity at a suitable interior point $\alpha\varepsilon(0,1)$, one can ensure that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = \infty\,.$$

On the other hand, if the singularity lies at zero, at least

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt\,.$$

The motivation for these results lies in determination of the radius of convergence of the q or basic hypergeometric series

$$f(z) := 1 + \sum_{j=1}^{\infty} \left(\prod_{k=0}^{j-1} (A - q^k) \right) z^j ,$$

which is of interest in Padé approximation. For most $|A| = |q| = 1$, the radius of convergence is 1, but we show that for suitable $|A| = |q| = 1$, f may be an entire function.

1. Introduction

Consider the power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \tag{1.1}$$

where

$$a_0 = 1 \; ; \; a_j = \prod_{\ell=0}^{j-1} (A - q^{\ell+\alpha}), \quad j \geq 1 \, . \tag{1.2}$$

Here, A and $q \varepsilon \mathbb{C}$, $\alpha \varepsilon \mathbb{R}$ and to avoid trivialities, we assume

$$A \neq 0, \qquad A \neq q^{\ell+\alpha}, \qquad \ell \geq 0 \, . \tag{1.3}$$

This power series, or q-hypergeometric series, satisfies a functional equation (cf. [2])

$$f(z)(1 - zA) = 1 - zq^{\alpha} f(qz) \tag{1.4}$$

and many of the analytic properties of $f(z)$ can be obtained from (1.4) (cf.[4]). The case $|q| = 1$, say $q = e^{2\pi i\theta}$, $\theta \varepsilon (0, 1)$ where θ is irrational, is of particular interest in Padé approximation, since, in this case, $f(z)$ has a natural boundary on its circle of convergence (cf.[2]). While there are general theorems on convergence of Padé approximants for functions with essential singularities or branch points (cf.[10, 11]), there are not many examples of functions with natural boundaries for which convergence of Padé approximants have been proven (cf.[1, 5, 9]). In the 1960's, Wynn [12] obtained explicit formulæ for the Padé denominators of this (and other) class of q-hypergeometric series, thus making it possible to investigate the convergence of diagonal and other Padé sequences for these power series. Analysis of the zero distribution of the Padé denominators along similar lines to those used by Lubinsky and Saff [9] in their investigation of the partial theta function, enabled us to establish convergence results for the Padé approximants to the power series $f(z)$, defined by (1.1) and (1.2), in the case where $f(z)$ has a natural boundary (cf. [2]).

The obvious question that arises is: What is the radius of convergence of $f(z)$, particularly in the interesting case when $f(z)$ has a natural boundary, namely, when

$$q = e^{2\pi i\theta}, \quad \theta\varepsilon(0,1), \quad \theta \ \text{ irrational.} \tag{1.5}$$

The answer to this question turned out to be more interesting than was at first envisaged. It is well known that if $\beta\varepsilon(0,1)$ is irrational and $\{n\beta\}$ denotes the fractional part of $n\beta$, $n \geq 1$, then the sequence $\{n\beta\}$, $n \geq 1$ is uniformly distributed in [0,1], that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt, \tag{1.6}$$

for each g bounded and Riemann integrable on [0,1]. In investigating certain power series, Hardy and Littlewood [6] extended (1.6) to g with suitably integrable singularities at 0 and 1. We shall investigate (1.6) for all irrational $\beta\varepsilon(0,1)$ for two singular functions g; in the first instance the singularity of g is at the origin, and in the second instance, the singularity of g lies at an interior point of [0,1].

2. The Radius of Convergence of $f(z)$

Theorem 1

Let $f(z)$ be defined by (1.1) and (1.2) and assume (1.3) is satisfied. Let q be as in (1.5) and denote the radius of convergence of $f(z)$ by $R(A;q)$. Then

(i) for $|A| \neq 1$, $\quad R(A;q) = \frac{1}{\max\{1,|A|\}}$;

(ii) for $|A| = 1$, there exists a set $\mathcal{S}_q$, depending on q, with $\operatorname{cap}(\mathcal{S}_q) = 0$ such that $R(A;q) = 1$ whenever $A \not\in \mathcal{S}_q$;

(iii) if $A = 1$, $R(A;q) = 1$;

(iv) there exist A and q with $|A| = 1$ and $R(A;q) = \infty$.

Proof

From (1.1) and (1.2) it follows that

$$R^{-1} = \limsup_{k\to\infty} \mid a_k \mid^{\frac{1}{k}} = \limsup_{k\to\infty} \left[\left| \prod_{\ell=1}^{k} (A - q^{\ell+\alpha-1}) \right|^{\frac{1}{k}} \right] . \tag{2.1}$$

(i) From (1.2), for $k \geq 1$,

$$\log \mid a_k \mid^{\frac{1}{k}} = \frac{1}{k} \sum_{\ell=1}^{k} \log \mid A - q^{\ell+\alpha-1} \mid . \tag{2.2}$$

Define, for $k = 1,2,3,\ldots$ the sequence of functions

$$f_k(z) := \frac{1}{k} \sum_{\ell=1}^{k} \log \mid A - zq^{\ell} \mid . \tag{2.3}$$

Then $\{f_k(z)\}_{k=1}^{\infty}$ forms an equicontinuous sequence of uniformly bounded functions on any compact subset K of $\mathbb{C}$ not intersecting $\{z : |z| = |A|\}$. Since $\{q^{\ell}\}_{\ell=1}^{\infty}$ is uniformly distributed on the unit circle (cf.[7]), we have

$$\begin{aligned} \lim_{k\to\infty} f_k(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \mid A - ze^{it} \mid \mathrm{d}t \\ &= \max\{\log |z|, \log |A|\} . \end{aligned} \tag{2.4}$$

By the Arzela-Ascoli theorem, the convergence is uniform in K. For $|A| \neq 1$ and with $z = q^{\alpha-1}$, we obtain from (2.2), (2.3) and (2.4),

$$\lim_{k\to\infty} \log \mid a_k \mid^{\frac{1}{k}} = \max\{0, \log |A|\} .$$

Then, from (2.1), for $|A| \neq 1$, $\quad R(A;q) = \frac{1}{\max\{1,|A|\}}$.

(ii) For $|A| = 1$, we introduce for $k \geq 1$, the probability measure μ_k assigning mass $\frac{1}{k}$ to $q^{\ell+\alpha}$, $\quad 0 \leq \ell \leq k-1$. Then, as $\{q^{\ell+\alpha}\}_{\ell=0}^{\infty}$ is uniformly distributed on the unit circle, μ_k converges weakly to normalized Lebesgue measure on the unit circle as $k \to \infty$. Thus

$$\begin{aligned} \log \mid a_k \mid^{\frac{1}{k}} &= \frac{1}{k} \sum_{\ell=0}^{k-1} \log \mid A - q^{\ell+\alpha} \mid \\ &= \int \log \mid A - t \mid \mathrm{d}\mu_k(t) . \end{aligned}$$

By the Lower Envelope Theorem in potential theory, (cf.[8]),

$$\limsup_{k\to\infty} \log | a_k |^{\frac{1}{k}} = \liminf_{k\to\infty} \int \log\left(\frac{1}{| A-t |}\right) d\mu_k(t)$$
$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{| A-e^{is} |}\right) ds ,$$

except for A in a set of capacity zero. Thus, except for A in a set of capacity zero, $|A| = 1$,

$$\limsup_{k\to\infty} \log | a_k |^{\frac{1}{k}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log | A - e^{is} | ds = 0 .$$

(iii) For simplicity, we assume that $\alpha = 1$. Then, with $A = 1$, we have from (2.1) and (1.5),

$$R^{-1} = \limsup_{k\to\infty} \left[\prod_{\ell=1}^{k} \left|1 - e^{2\pi i\ell\theta}\right|\right]^{\frac{1}{k}} . \tag{2.5}$$

By Dirichlet's Box Principle, there are infinitely many positive integers p and q such that

$$(p, q) = 1 , \tag{2.6}$$

and

$$| \theta - p/q | < \frac{1}{q^2} . \tag{2.7}$$

Let

$$e(z) := \exp(2\pi iz) , \tag{2.8}$$

and consider

$$\sum_{j=1}^{q-1} \log | 1 - e(j\theta) | = \sum_{j=1}^{q-1} \log\left|(1 - e(jp/q))\left(1 + \frac{e(jp/q) - e(j\theta)}{1 - e(jp/q)}\right)\right|$$
$$= \sum_{j=1}^{q-1} \log\left|1 - e(jp/q)\right| + \sum_{j=1}^{q-1} \log\left|1 + \frac{e(jp/q) - e(j\theta)}{1 - e(jp/q)}\right|$$

By (2.8),

$$\left|e(jp/q) - e(j\theta)\right| = \left|2\sin \pi j(\theta - p/q)\right|$$
$$\le 2\pi j/q^2 ,$$

so that we obtain from (2.7) and the above,

$$\left|\sum_{j=1}^{q-1}\log\left|1-e(j\theta)\right|\right| \le \left|\sum_{j=1}^{q-1}\log\left|1-e(jp/q)\right|\right| + \frac{2\pi}{q^2}\sum_{j=1}^{q-1}\frac{j}{\left|1-e(jp/q)\right|}$$

$$\le \left|\sum_{j=1}^{q-1}\log\left|1-e(jp/q)\right|\right| + \frac{2\pi}{q}\sum_{j=1}^{q-1}\frac{1}{\left|1-e(jp/q)\right|}\,. \tag{2.9}$$

Further, using (2.6), we may deduce from (2.9) that

$$\left|\sum_{j=1}^{q-1}\log\left|1-e(j\theta)\right|\right| \le \left|\sum_{j=1}^{q-1}\log\left|1-e(j/q)\right|\right| + \frac{2\pi}{q}\sum_{j=1}^{q-1}\frac{1}{\left|1-e(j/q)\right|}\,. \tag{2.10}$$

Consideration of the second sum on the right hand side of (2.10) shows that

$$\frac{2\pi}{q}\sum_{j=1}^{q-1}\frac{1}{\left|1-e(j/q)\right|} = \mathrm{O}\left(\log q\right), \quad q\to\infty\,,$$

so that from (2.10), for q large, we have

$$\left|\frac{1}{q}\sum_{j=1}^{q-1}\log\left|1-e(j\theta)\right|\right| \le \left|\frac{1}{q}\sum_{j=1}^{q-1}\log\left|1-e(j/q)\right|\right| + \mathrm{O}\left(\frac{\log q}{q}\right). \tag{2.11}$$

It remains to show that

$$\lim_{q\to\infty}\frac{1}{q}\sum_{j=1}^{q-1}\log\left|1-e(j/q)\right| = 0\,.$$

Let $\eta > 0$ and $0 < \varepsilon < \frac{1}{2}$. Define

$$f_\varepsilon(t) := \begin{cases}\log\left|1-e(t)\right| \quad , & \varepsilon\le t\le 1-\varepsilon\\ 0 \quad , & \text{otherwise}\end{cases}$$

Then $f_\varepsilon(t)$ is a bounded Riemann-integrable function and we have

$$\lim_{q\to\infty}\frac{1}{q}\sum_{\substack{1\le j\le q-1\\ j/q\,\varepsilon\,[\varepsilon,1-\varepsilon]}}\log\left|1-e(j/q)\right| = \lim_{q\to\infty}\frac{1}{q}\sum_{j=1}^{q-1}f_\varepsilon(j/q)$$

$$= \int_0^1 f_\varepsilon(t)\,\mathrm{d}t\,.$$

Further, since $\int_0^1\log\left|1-e(t)\right|\mathrm{d}t = 0$, we can choose $\varepsilon = \varepsilon(\eta)$, $\varepsilon\,\varepsilon\left(0,\frac{1}{2}\right)$ so small that

$$\left|\int_\varepsilon^{1-\varepsilon}\log\left|1-e(t)\right|\mathrm{d}t\right| \le \eta\,.$$

Then, for $q \geq q_0 = n_0(\eta)$ say, we have for $q \geq q_0(\eta)$,

$$\left| \frac{1}{q} \sum_{\substack{1 \leq j \leq q-1 \\ j/q \, \varepsilon [\varepsilon, 1-\varepsilon]}} \log|1 - e(j/q)| \right| \leq 2\eta \,. \tag{2.12}$$

Next, for $j/q \, \varepsilon [0, \varepsilon]$ and for $j/q \, \varepsilon [1-\varepsilon, 1]$, it is not difficult to show (cf[1, p108]) that we can choose ε sufficiently small to ensure that

$$\left| \frac{1}{q} \sum_{\substack{1 \leq j \leq q-1 \\ j/q \, \varepsilon [0, \varepsilon]}} \log|1 - e(j/q)| \right| < \eta \,, \tag{2.13}$$

and

$$\left| \frac{1}{q} \sum_{\substack{1 \leq j \leq q-1 \\ j/q \, \varepsilon [1-\varepsilon, 1]}} \log|1 - e(j/q)| \right| < \eta \,. \tag{2.14}$$

As $\eta > 0$ is arbitrary we have proved (iii).

(iv) In the interests of brevity, we shall only state a sufficient set of conditions for A and q which ensure $R(A; q) = \infty$ and refer the interested reader to [3] for a complete proof. If $q = e^{2\pi i \theta}$, $\theta \varepsilon (0, 1)$, θ irrational and $A = e^{2\pi i \alpha}$, $\alpha \varepsilon (0, 1)$, then the pair (α, θ) may be any pair satisfying the following: Let p_j/q_j, $j = 1, 2, 3, \ldots$, denote the convergents of the continued fraction of θ and assume that

$$\lim_{k \to \infty} \log q_{k+1} \Big/ \sum_{j=0}^{k} q_j = \infty \,.$$

Define α by

$$\alpha := \theta + \sum_{k=1}^{\infty} (q_k \theta - p_k) \,.$$

Then $R(A; q) = \infty$. ■

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