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EQUICONVERGENCE AND EQUIAPPROXIMATION FOR ENTIRE FUNCTIONS

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1. INTRODUCTION

Equiapproximation results are obtained for the best onesided approximation with entire functions in L_p, $0<p\le\infty$, approximation with interpolation operators and the best approximation with entire functions in $L_{p,\delta}$, $0<p\le\infty$ (Section 2).

These results imply characterization of the orders of convergence for the above mentioned approximation processes.

Definition 1.1. Let $R^1(f)_\sigma$ and $R^2(f)_\sigma$ be two approximation processes. We say that $R^1_\sigma(f)$ and $R^2_\sigma(f)$ are equiapproximation processes in Ω if there exist constants $c_1>0$ and $c_2>0$ such that

$$c_1R^1_\sigma(f) \le R^2_\sigma(f) \le c_2R^1_\sigma(f) \text{ for every } f \in \Omega.$$

Definition 1.2. $R^1_\sigma(f)$ and $R^2_\sigma(f)$ are equiapproximation processes with a given order $\psi(\sigma)$ in Ω if

$$R^1_\sigma(f) = O(\psi(\sigma)) \iff R^2_\sigma(f) = O(\psi(\sigma)) \text{ for } f \in \Omega.$$

2. NOTATIONS

Let $\mathbb{R}$ be the real line and denote by B_σ the set of all entire functions of exponential type σ.

The spaces L_p, $0<p\le\infty$ are equipped with the (quasi-)norms

$$\|f\|_{L_p} := \|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p\,dt\right)^{1/p}, 0<p<\infty;$$

$$\|f\|_{L_\infty} := \|f\|_\infty := \sup\left\{ |f(t)|, t \in \mathbb{R} \right\}.$$

For a positive, integer number r and $\omega>0$ we denote

$D_{\omega,r}(x) := d(\omega,r)\left[\mathrm{SINC}(\omega x/2)\right]^{2r}$, $\|D_{\omega,r}\|_1 = 1$ - the modifyed approximation kernel, where $\mathrm{SINC}(x) := \sin(x)/x$.

$\omega_1(f,x,\gamma)$ is the local modulus of smoothness [16, p.6].

We will use also the following (local-global) (quasi-)norms [8] for $\delta>0$

and $0<p\leq\infty$

$$\|f\|_{L_{p,\delta}} := \|f\|_{p,\delta} := \|f_\delta\|_p;\ f_\delta(x) := \sup\left(|f(t)|:|x-t|\leq\delta/2\right),$$

and the following discrete l_p (quasi-)norms /k integer/

$$\|f\|_{l_{p,\delta}} := \left[\ \delta\sum_k |f(x_0+k\delta)|^p\ \right]^{1/p},\ x_0\in\mathbb{R},\ 0<p<\infty \text{ and}$$

$$\|f\|_{l_{\infty,\delta}} := \sup\left[|f(k\delta)|:k\ \right].$$

Let $\chi_{[a,b]}(t)$ denote the characteristic function of the interval [a,b].

3.DEFINITIONS

Definition 3.1. The best approximation of f with B_σ in L_p is given by

$$E_\sigma(f)_p := \inf\left[\|f-g\|_p:\ g\in B_\sigma\right].$$

Definition 3.2. The best approximation of f with B_σ in $L_{p,\delta}$ is given by

$$E_\sigma(f)_{p,\delta} := \inf\left[\|f-g\|_{p,\delta}:\ g\in B_\sigma\right].$$

Definition 3.3. The best onesided approximation of a function f with B_σ in the metrics of the spaces L_p or $L_{p,\delta}$ is respectively given by

$$\tilde{E}_\sigma(f)_p := \inf\left[\|g^+-g^-\|_p : g^+,\ g^-\in B_\sigma,\ g^-(x)\leq f(x)\leq g^+(x)\right].$$

$$\tilde{E}_\sigma(f)_{p,\delta} := \inf\left[\|g^+-g^-\|_{p,\delta} : g^+,g^-\in B_\sigma,\ g^-(x)\leq f(x)\leq g^+(x)\right].$$

4. AUXILIARY RESULTS

Lemma 4.1. *If* $\|D_{\omega,r}\|_1=1$, *then*

$$d(\omega,r)\ \leq\ (\omega/4)\left(\pi/2\right)^{2r-1}.$$

The proof is based on the fact $SINC(x)\ \geq\ 2/\pi$ for $x\in[0,\pi/2]$.

Lemma 4.2. *If* $g(x)\geq 0$ *and* $R_{\omega,r}(g,x) := \int_{-\infty}^{\infty} g(x+t)D_{\omega,r}(t)dt$, *then for* $\gamma>0$ *we have*

$$|g(x)-R_{\omega,r}(g,x)|\ \leq\ \|g\|_\infty(2r-1)^{-1}\left(\pi/(\omega\gamma)\right)^{2r-1} + \omega_1\left(g,x,\gamma\right).$$

Proof. Applying Lemma 4.1 we obtain

$$|g(x)-R_{\omega,r}(g,x)|\ \leq\ \left[\int_{|t|>\gamma} + \int_{|t|\leq\gamma}\right]|g(x+t)-g(x)|D_{\omega,r}(t)dt$$

$$\leq\ 2\|g\|_\infty d(\omega,r)\left(2/\omega\right)^{2r}\int_{t>\gamma} t^{-2r}dt + \omega_1\left(g,x,\gamma\right)$$

$$\leq \|g\|_\infty 2(\omega/4)(\pi/2)^{2r-1}(2/\omega)^{2r}(2r-1)^{-1}\gamma^{-2r+1} + \omega_1(g,x,\gamma). \square$$

Lemma 4.3. *Let* $r>(1+1/p)/2$, $\delta>0$ *and* $0<p\leq 1$. *Then the convolution*

$$R_{\pi/\delta,r}(x) := [(2r-1)/(2r-2)] \int_{-\infty}^{\infty} \chi_{[-3\delta/2,3\delta/2]}(t) D_{\pi/\delta,r}(x-t)dt$$

possesses the properties

a) $0<R_{\pi/\delta,r}(x)<(2r-1)/(2r-2)$, $x \in \mathbb{R}$;

b) $R_{\pi/\delta,r}(x) \geq 1$ *for* $x \in [-\delta/2,\delta/2]$;

c) $R_{\pi/\delta,r} \in B_{\pi r/\delta}$ *and* $\|R_{\pi/\delta,r}\|_p \leq c(p,r)\delta^{1/p}$;

d) $\sum_k R^p_{\pi/\delta,r}(x+k\delta) \leq c(p,r)$ *for every* $x \in \mathbb{R}$

and $c(p,r)$ *does not depend on* δ.

Proof. Let $g(x) := [(2r-1)/(2r-2)]\chi_{[-3\delta/2,3\delta/2]}(x)$. The property a) is evident. We prove b). Let $x\in[-\delta/2,\delta/2]$. Applying Lemma 4.2 for $g(x)$, $\omega=\pi/\delta$ and $\gamma=\delta$ we obtain

$$(2r-1)/(2r-2) - R_{\pi/\delta,r}(x)$$

$$\leq (2r-1)/(2r-2)(2r-1)^{-1}\left[\pi/((\pi/\delta)\delta)\right]^{2r-1} + \omega_1(g,x,\delta) = (2r-2)^{-1},$$

which proves b). Proof of c). $R_{\pi/\delta,r} \in B_{\pi r/\delta} \cap L_1$ [11, Th.3.6.2]. The function $R_{\pi/\delta,r}(x)$ is even and it is enough to consider $x\geq 0$. Applying Lemma 4.2 for $x\geq 5\delta/2$, $g(x)$, $\omega=\pi/\delta$ and $\gamma=x-3\delta/2$ we get

$$R_{\pi/\delta,r}(x)\leq(2r-2)^{-1}\left[\pi/((\pi/\delta)(x-3\delta/2))\right]^{2r-1} = (2r-2)^{-1}(\delta/(x-3\delta/2))^{2r-1}.$$

From the above estimate and the property a) it follows

$$\|R_{\pi/\delta,r}\|_p^p \leq \left\{5\left[(2r-1)/(2r-2)\right]^p + 2(2r-2)^{-p}\left[(2r-1)p-1\right]^{-1}\right\}\delta$$

for $r>(1+1/p)/2$, which completes the proof of the property c). Proof of the property d). The function

$$\Phi_{r,p,\delta}(x) := \sum_k R^p_{\pi/\delta,r}(x+k\delta)$$

is even and δ-periodic and it is enough to consider $x\in[0,\delta/2]$. Let $|k|\geq 3$. Then $|x+k\delta|\geq(|k|-1/2)\delta$. Applying Lemma 4.2 for $g(x)$, $\omega=\pi/\delta$ and $\gamma=|x+k\delta|-3\delta/2$ we obtain

$$R_{\pi/\delta,r}(x+k\delta) \le (2r-2)^{-1}\Big[\pi/\big((\pi/\delta)(|x+k\delta|-3\delta/2)\big)\Big]^{2r-1}$$

$$+\ \omega_1\big(g, x+k\delta, |x+k\delta|-3\delta/2\ \big)\ \le\ (2r-2)^{-1}\Big[|k|-2\Big]^{-2r+1}\quad \text{for } |k|\ge 3.$$

From the above estimate and the property a) it follows

$$\Phi_{r,p,\delta}(x)\ \le\ 5\Big[(2r-1)/(2r-2)\Big]^{p}\ +\ 2(2r-1)^{-p}\sum_{k\ge 3}(k-2)^{-(2r-1)p}\ \le\ c(p,r)$$

if $(2r-1)p>1$. This completes the proof of Lemma 4.3.$_\square$

5. EQUICONVERGENCE THEOREMS FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE IN L_p, $0<p\le\infty$

We consider three cases with respect to p. The first one is $p=\infty$.

Theorem A (M.Cartwright[5]). *If* $f \in B_\sigma$, *then*

a) $\sup_k |f(k\delta)|\ \le\ \|f\|_\infty,\ \delta>0;$

b) $\|f\|_\infty\ \le\ c(\eta)\sup_k |f(k\pi/(\sigma(1+\eta)))|$ *for* $\eta>0$ *and*

$c(\eta)\le\ c(1+\ln(1+\eta^{-1}))$.

Remark A. S.Bernstein [2] has proved that

$$c(\eta)=\ (2/\pi)\ln(1+\eta^{-1})\ +\ O(1),\ \eta\to 0+.$$

The second case is $1<p<\infty$.

Theorem B. *If* $f \in B_\sigma$, *then*

a) (S.M.Nikol'skii [11, p.122]). *if* $1\le p<\infty$, *then*

$$\|f\|_{l_{p,\delta}}\ \le\ (1+\sigma\delta)\|f\|_p;$$

b) (B.Levin [10, p.109]). *if* $1<p<\infty$, *then*

$$\|f\|_p\ \le\ c(p)\|f\|_{l_{p,\pi/\sigma}}.$$

The third case is $0<p\le 1$.

THEOREM 5.1. *If* $f \in B_\sigma$, $0<p\le 1$, *then*

a) $\|f\|_{l_{p,\delta}}\ \le\ c(p)\Big[1+(1+\sigma\delta)^{1-p}(\sigma\delta)^{p}\Big]^{1/p}\|f\|_p$ *for* $\delta>0$;

b) $\|f\|_p\ \le\ c(\eta,p)\|f\|_{l_{p,\pi/(\sigma(1+\eta))}}$ *for* $\eta>0$ *and*

$$c(\eta,p)\le\begin{cases} c(p)\,\big(1+\eta^{-1}\big)^{1/p-1}, & 0<p<1,\\ c\big(1+\ln(1+\eta^{-1})\big),\ p=1\,. \end{cases}$$

Proof. a). Let ξ_k, $|\xi_k - k\delta|\le\delta/2$ be such that

$$\int_{(2k-1)\delta/2}^{(2k+1)\delta/2} |f(t)|^p\,dt=\delta|f(\xi_k)|^p .$$

Applying the inequality $|x|^p-|y|^p\le|x-y|^p$ for $x,y\in\mathbb{R}$ and $0<p\le1$ we obtain

$$\sum_k |f((2k+1)\delta/2)|^p-\sum_k|f(\xi_k)|^p\le\sum_k|f((2k+1)\delta/2)-f(\xi_k)|^p$$

(5.1)
$$=\sum_k\left|\int_{\xi_k}^{(2k+1)\delta/2} f'(t)\,dt\right|^p\le\sum_k\left(\int_{(2k-1)\delta/2}^{(2k+1)\delta/2}|f'(t)|dt\right)^p$$

$$\le\sum_k\left(\int_{-\infty}^{\infty}\chi_{[-\delta/2,\delta/2]}(t-k\delta)|f'(t)|dt\right)^p.$$

Using the properties a), b) from Lemma 4.3 we obtain

(5.2)
$$R_{\pi/\delta,r}(t-k\delta)f'(t)\in B_{(\pi r/\delta)+\sigma}\cap L_p \quad\text{and}$$

(5.3)
$$|R_{\pi/\delta,r}(t-k\delta)f'(t)| = R_{\pi/\delta,r}(t-k\delta)|f'(t)|$$
$$\ge \chi_{[-\delta/2,\delta/2]}(t-k\delta)|f'(t)| .$$

From (5.1) and (5.3) we obtain the inequality

(5.4)
$$\sum_k |f((2k+1)\delta/2)|^p-\delta^{-1}\|f\|_p^p\le\sum_k\left(\int_{-\infty}^{\infty}R_{\pi/\delta,r}(t-k\delta)|f'(t)|dt\right)^p.$$

Applying the inequality between different metrics [12] we obtain

(5.5)
$$\left(\int_{-\infty}^{\infty}R_{\pi/\delta,r}(t-k\delta)|f'(t)|dt\right)^p$$
$$\le c(p)\big((\pi r/\delta)+\sigma\big)^{1-p}\int_{-\infty}^{\infty}R^p_{\pi/\delta,r}(t-k\delta)|f'(t)|^p\,dt.$$

Using the property d), Lemma 4.3 and Bernstein's inequality [15] (the case $0<p<1$), we reach a conclusion that

$$\sum_k \int_{-\infty}^{\infty} R^p_{\pi/\delta,r}(t-k\delta)|f'(t)|^p dt \le \int_{-\infty}^{\infty} \left[\sum_k R^p_{\pi/\delta,r}(t-k\delta)\right] |f'(t)|^p dt$$

(5.6)

$$\le c(p,r) \int_{-\infty}^{\infty} |f'(t)|^p dt \le c(p,r)\sigma^p \|f\|^p_p.$$

From (5.4) - (5.6) it follows, that

$$\delta \sum_k |f((2k+1)\delta/2)|^p \le \|f\|^p_p + c(p,r)\left((\pi r/\delta)+\sigma\right)^{1-p} \delta\sigma^p \|f\|^p_p \tag{5.7}$$

which proves the estimate a).

We prove b). Applying Whittaker-Kotelnikov-Shannon sampling theorem [4] for

$$f(t)\left[\mathrm{SINC}\left(\sigma\eta(x-t)/(2r)\right)\right]^{2r} \in B_{\sigma(1+\eta)} \cap L_2(t)$$

we obtain

$$f(x) = \sum_k f\left(k\pi/(\sigma(1+\eta))\right) \left[\mathrm{SINC}\left[\sigma\eta\left(x-k\pi/(\sigma(1+\eta))\right)/(2r)\right]\right]^{2r}$$

(5.8)

$$\times \mathrm{SINC}\left[\sigma(1+\eta)\left(x-k\pi/(\sigma(1+\eta))\right)\right].$$

Using the inequality $\left(\sum_k a_k\right)^p \le \sum_k a_k^p$, $0<p<1$, $a_k \ge 0$ we arrive at

$$\|f\|_p \le \left[(\sigma(1+\eta))^{-1} \sum_k \left|f\left(k\pi/(\sigma(1+\eta))\right)\right|^p\right]^{1/p}$$

(5.9)

$$\times \left[\int_{\infty}^{\infty} \left|\mathrm{SINC}\left[\eta v/((1+\eta)2r)\right]\right|^{2rp} \left|\mathrm{SINC}(v)\right|^p dv\right]^{1/p}.$$

Estimating the integral on the right in (5.9) we obtain the consants in

b), namely

$$\int_{-\infty}^{\infty} \left|\mathrm{SINC}\left[\eta v/((1+\eta)2r)\right]\right|^{2rp} \left|\mathrm{SINC}(v)\right|^p dv$$

$$\le \begin{cases} \left[(\pi r)^{1-p}(1-p)^{-1} + \left[2r(2/\pi)^{(2r+1)p-1}\right] / ((2r+1)p-1)\right] (1+\eta^{-1})^{1-p}, & 0<p<1 \\ (2/\pi)^{2r} + \pi/2 + \ln(2r) + \ln(1+\eta^{-1}), & p=1, \end{cases}$$

if $(2r+1)p>1$ and this completes the proof of b).□

Remark B. M.Planscherel and G.Polya [13] and R.P.Boas [3, p.101, p.197] had proved analogous results by using another methods.

6. THE SPACES $L_{p,\delta}$, $0<p<\infty$

Lemma 6.1. *The quasi-normed space* $L_{p,\delta}$ *possesses the properties*

a) $\|f+g\|_{p,\delta} \le \max(1,2^{1/p-1})\left[\|f\|_{p,\delta} + \|g\|_{p,\delta}\right]$;

b) $\|f\|_{p,\delta_1} \le \|f\|_{p,\delta_2}$, $\delta_1<\delta_2$; c) $\|f\|_p \le \|f\|_{p,\delta}$, $\|f\|_\infty = \|f\|_{\infty,\delta}$;

d) $\|f(.+h)\|_{p,\delta} = \|f(.)\|_{p,\delta}$, $h\in\mathbb{R}$;

e) $\|f\|_{p,m\delta} \le m^{1/p}\|f\|_{p,\delta}$*(natural* m); f) $\|f\|_{1_{p,\delta}} \le \|f\|_{p,\delta}$.

Proof. The property d) follows from $f_\delta(.+h)(x) = f_\delta(.)(x+h)$.

The property e) is proved in [8, p.425] by using an interpolation ptoperty of the spaces $L_{p,\delta}$, $1\le p\le\infty$. In the case $0<p\le 1$ this property follows from the inequality

$$f_{m\delta}(x) \le \sum_{k=0}^{m-1} f_\delta\left[x+(2k-(m-1))\delta/2\right].$$

The property f) follows from the inequality

$$f_\delta(t+x) \ge |f(x)| \text{ for } t \in [-\delta/2,\delta/2].$$

The other properties are more or less obvious.□

Remark C. The space $L_{p,\delta}$ (2π-periodic, $1\le p\le\infty$ case) has been considered in [8], where equiapproximation results are obtained in 2π-periodic case.

Lemma 6.2. *If* $f(x) \in B_\sigma$, *then*

a) $\|f\|_{\infty,\delta} = \|f\|_\infty$ *for* $\delta>0$;

b) $\|f\|_{p,\delta} \le (1+\sigma\delta)\|f\|_p$ *for* $\delta>0$ *and* $1\le p<\infty$;

c) $\|f\|_{p,\delta} \le c(p)\left[1+(1+\sigma\delta)^{1-p}(\sigma\delta)^p\right]^{1/p}\|f\|_p$ *for* $\delta>0$ *and* $0<p\le 1$.

Proof. a) follows from Lemma 6.1, c). The proof of b) is similarly to this in 2π-periodic case [8]. We will prove c). Let

$$\sup\left[|f(y)|,|x-y|\le\delta/2\right] = |f(\xi_x)|,\ |x-\xi_x| \le \delta/2.$$

Applying the inequality $|x|^p-|y|^p\le|x-y|^p$ for $x,y \in \mathbb{R}$ and $0<p\le 1$ we obtain

$$\int_{-\infty}^{\infty} |f_\delta(x)|^p dx - \int_{-\infty}^{\infty} |f(x)|^p dx = \int_{-\infty}^{\infty} |f(\xi_x)|^p dx - \int_{-\infty}^{\infty} |f(x)|^p dx$$

$$(6.1)\qquad \le \int_{-\infty}^{\infty}\Big(|f(\xi_x)| - |f(x)|\Big)^p dx \le \int_{-\infty}^{\infty}\Big|f(\xi_x) - f(x)\Big|^p dx$$

$$\le \int_{-\infty}^{\infty}\Big|\int_x^{\xi_x}\big|f'(t)\big|\,dt\Big|^p dx \le \int_{-\infty}^{\infty}\Big(\int_{x-\delta/2}^{x+\delta/2}\big|f'(t)\big|\,dt\Big)^p. dx$$

$$= \int_{-\infty}^{\infty}\Big(\int_{-\delta/2}^{\delta/2}\big|f'(x+t)\big|dt\Big)^p dx = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\chi_{[-\delta/2,\delta/2]}(t)\big|f'(x+t)\big|dt\Big)^p dx.$$

Using Lemma 4.3, a), b), c) and the inequality between different metrics [12] we obtain

$$\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\chi_{[-\delta/2,\delta/2]}(t)\big|f'(x+t)\big|\,dt\Big)^p dx$$

$$\le \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}R_{\pi/\delta,r}(t)\,\big|f'(x+t)\big|\,dt\Big)^p dx$$

$$(6.2)\qquad \le c(p)\big((\pi r/\delta)+\sigma\big)^{1-p}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R^p_{\pi/\delta,r}(t)\,\big|f'(x+t)\big|^p dtdx$$

$$\le c(p,r)\big((\pi r/\delta)+\sigma\big)^{1-p}\,\delta\int_{-\infty}^{\infty}\big|f'(x)\big|^p dx$$

because of $R_{\pi/\delta,r}(t)f'(x+t) \in B_{(\pi r/\delta)+\sigma} \cap L_p$ for a fixed x.

Applying Bernstein's inequality [15] in the case $0<p\le 1$ one may write from (6.1) and (6.2)

$$\|f\|^p_{p,\delta} \le \|f\|^p_p + c(p,r)\big((\pi r/\delta)+\sigma\big)^{1-p}\,\delta\sigma^p\,\|f\|^p_p$$

for $r>(1+1/p)/2$.□

Corollary 6.1. *If* $f \in B_\sigma$ *and* $\sigma\delta\le\eta$, *then*

$$\|f\|_p \le \|f\|_{p,\delta} \le c(\eta,p)\|f\|_p,\quad 0<p\le\infty.$$

Corollary 6.2(Bernstein inequality). *If* $f \in B_\sigma$ *and* $\sigma\delta\le\eta$, *then*

$$\|f^{(k)}\|_{p,\delta} \le c(\eta,p)\sigma^k\|f\|_{p,\delta},\quad 0<p\le\infty.$$

7. EQUIAPPROXIMATION THEOREM

Lemma 6.2 and a modification of a method which is developed in [9], [8], [1] will be a basis in the proof of the next result.

THEOREM 7.1. *If* $f(x)\in L_\infty$, *then* $E_\sigma(f)_p$ *and* $\tilde{E}_\sigma(f)_{p,1/\sigma}$ *are equiapproximation processes for* $0<p\le\infty$. *In other words, there exist constants* $c_1(p)$ *and* $c_2(p)$, *such that*

$$\tilde{E}_\sigma(f)_p \le c_1(p)E_\sigma(f)_{p,1/\sigma} \le c_2(p)\tilde{E}_\sigma(f)_p.$$

Proof. Let $\tilde{E}_\sigma(f)_p = \|g^+-g^-\|$; $g^+,g^-\in B_\sigma$, $g^-(x)\le f(x)\le g^+(x)$. From Corollary 6.1 we obtain

$$E_\sigma(f)_{p,1/\sigma} \le \tilde{E}_\sigma(f)_{p,1/\sigma} \le \|g^+-g^-\|_{p,1/\sigma}$$
$$\le c(p)\|g^+-g^-\|_p = c(p)\tilde{E}_\sigma(f)_p.$$

We prove the opposite inequality by using the kernel

$$\Phi_{r,\sigma}(t) := (2/\pi)^{2r}(\sigma/(2\pi r))\left[\mathrm{SINC}(\sigma t/(2r))\right]^{2r},$$

which has the properties

a) $\Phi_{r,\sigma}(t) \ge 0$; b) $\Phi_{r,\sigma}(t) \in B_\sigma \cap L_p$ if $r>1/(2p)$;

c) $\|\Phi_{r,\sigma}\|_1 = c(r)$; d) $\Phi_{r,\sigma}(t) \ge \sigma/(2\pi r)$ if $|t|\le(\pi r)/\sigma$;

e) $\sup\left\{\Phi_{r,\sigma}(t) : t\in[(2m-1)\pi r/\sigma,(2m+1)\pi r/\sigma]\right\} \le c(r)|m|^{-2r}\sigma$ if $|m|\ge 1$.

Let $g \in B_\sigma$ and $E_\sigma(f)_{p,1/\sigma} = \|f-g\|_{p,1/\sigma}$. If $(f-g) \in L_{p,1/\sigma}$, then $(f-g) \in L_\infty$ and $(f-g)_{2\pi r/\sigma}(x) \in L_\infty$. It follows [11, Theorem 3.6.2, p.135], that

$$\Phi_{r,\sigma}*\left[(f-g)_{2\pi r/\sigma}\right](x) := \int_{-\infty}^{\infty}\Phi_{r,\sigma}(t)\left[(f-g)_{2\pi r/\sigma}\right](x-t)dt \in B_\sigma \cap L_\infty.$$

We define the functions

$$f^+(x) := f^+(f,x) := g(x) + \Phi_{r,\sigma}*\left[(f-g)_{2\pi r/\sigma}\right](x) \in B_\sigma \cap L_\infty,$$
$$f^-(x) := f^-(f,x) := g(x) - \Phi_{r,\sigma}*\left[(f-g)_{2\pi r/\sigma}\right](x) \in B_\sigma \cap L_\infty.$$

For $f^-(x)$ one may write the following esimate

$$f(x)-f^-(x) \ge f(x)-g(x) + \int_{-\pi r/\sigma}^{\pi r/\sigma}\Phi_{r,\sigma}(t)\left[(f-g)_{2\pi r/\sigma}\right](x-t)dt$$
$$\ge f(x)-g(x) + |f(x)-g(x)| \ge 0.$$

We have $f^+(x) \ge f(x)$ using the same arguments.

We will prove $\|f^+-f^-\|_p \le c(p)E_\sigma(f)_{p,1/\sigma}$, if $r=r(p)>1/2p$.

The case $1\le p\le\infty$ can be done more or less as 2π-periodic, $1\le p\le\infty$ case [8]. We prove only the case $0<p\le 1$. Using the definitions of $f^+(x)$ and $f^-(x)$ we obtain

$$f^+(x)-f^-(x) = 2\Phi_{r,\sigma}*\left[(f-g)_{2\pi r/\sigma}\right](x)$$

$$= 2\sum_{m=-\infty}^{\infty}\int_{(2m-1)\pi r/\sigma}^{(2m+1)\pi r/\sigma}\Phi_{r,\sigma}(t)\left[(f-g)_{2\pi r/\sigma}\right](x-t)dt$$

(7.1)
$$\le c(r)\sigma\sum_{m=-\infty}^{\infty}(|m|+1)^{-2r}\int_{(2m-1)\pi r/\sigma}^{(2m+1)\pi r/\sigma}\left[(f-g)_{2\pi r/\sigma}\right](x-t)dt$$

$$\le c(r)\sigma\sum_{m=-\infty}^{\infty}(|m|+1)^{-2r}\int_{(2m-1)\pi r/\sigma}^{(2m+1)\pi r/\sigma}\left[(f-g)_{4\pi r/\sigma}\right](x-2m\pi r/\sigma)dt$$

$$= c(r)\sum_{m=-\infty}^{\infty}(|m|+1)^{-2r}(f-g)_{4\pi r/\sigma}(x-2m\pi r/\sigma) .$$

Applying $\left(\sum_k a_k\right)^p \le \sum_k a_k^p$, $0<p\le 1$, $a_k \ge 0$ to (7.1) and taking into account Lemma 6.1, d), e) we arrive at

$$\tilde{E}_\sigma(f)_p \le \|f^+ - f^-\|_p \le c(p)\|f-g\|_{p,1/\sigma} = c(p)E_\sigma(f)_{p,1/\sigma}. \square$$

8. APPROXIMATION VIA INTERPOLATION OPERATORS OF WHITTAKER-KOTELNIKOV-SCHANNON TYPE

Let k and $r\ge 0$ be integer, $\eta\ge 0$, $h_\sigma:=\pi/\sigma$ and denote

$$V_{\sigma,r,\eta}(x) := \left[\mathrm{SINC}(\sigma\eta x/r)\right]^r\mathrm{SINC}(\sigma(1+\eta)x);\ V_\sigma(x) := \mathrm{SINC}(\sigma x).$$

We have

$$V_{\sigma,r,\eta}(kh_{\sigma(1+\eta)}) = V_\sigma(kh_\sigma) = \delta_{k,0}\ \left[\text{Kronecker delta}\right].$$

We consider the discrete linear operators

$$I_{\sigma,r,\eta}(f,x) := \sum_k f(kh_{\sigma(1+\eta)})V_{\sigma,r,\eta}(x-kh_{\sigma(1+\eta)}),$$

$$I_\sigma(f,x) := \sum_k f(kh_\sigma)V_\sigma(x-kh_\sigma),$$

which possess the following interpolation properties

$$I_{\sigma,r,\eta}(f,kh_{\sigma(1+\eta)}) = f(kh_{\sigma(1+\eta)});\ I_\sigma(f,kh_\sigma) = f(kh_\sigma).$$

The next two theorems are consequences from Wittaker-Kotelnikov-Shannon samling theorem [4]

Theorem C [6]. *If* $f \in B_\sigma \cap L_p$, $0<p<\infty$, *then*

$$f(x) = I_\sigma(f,x), \ x \in \mathbb{R}$$

the series being absolutely and uniformly convergent on $\mathbb{R}$.

Theorem D [4]. *If* $f \in B_\sigma \cap L_\infty$ *and* $r \geq 1$, *then*

$$f(x) = I_{\sigma,r,\eta}(f,x), \ x \in \mathbb{R},$$

the series being absolutely and uniformly convergent in every compact domain in the complex plane.

Lemma 8.1. *The following estimates hold*

a) *if* $f \in l_{p,h_{\sigma(1+\eta)}}$, $r > \max(1/p-1,0)$, *then* $I_{\sigma,r,\eta}(f) \in B_{\sigma(1+2\eta)}$ *and*

$$\|I_{\sigma,r,\eta}(f,x)\|_p \leq c(p,r,\eta)\|f\|_{l_{p,h_{\sigma(1+\eta)}}}, \ 0<p\leq\infty;$$

b) [10, Theorem 21, p.108]. *if* $f \in l_{p,h_\sigma}$, then $I_\sigma(f) \in B_\sigma$ *and*

$$\|I_\sigma(f,x)\|_p \leq c(p)\|f\|_{l_{p,h_\sigma}}, \ 1<p<\infty.$$

Proof. The proof of a) is similar to the proof of [7, Lemma B and Lemma 4] and it is based on Theorem B, a) and [11, p.115(5)], when $1\leq p\leq\infty$. The case $0<p<1$ can be done making use of $\left(\sum_k a_k\right)^p \leq \sum_k a_k^p$, $0<p<1$, $a_k\geq 0$. The proof of b) is based on Theorem B, a) and b).□

Lemma 8.2. *The following estimates hold*

a) *if* $f \in L_{p,h_{\sigma(1+\eta)}}$, $r > \max(1/p-1,0)$, *then* $I_{\sigma,r,\eta}(f) \in B_{\sigma(1+2\eta)}$ *and*

$$\|I_{\sigma,r,\eta}(f,x)\|_{p,h_{\sigma(1+\eta)}} \leq c(p,r,\eta)\|f\|_{p,h_{\sigma(1+\eta)}}, \quad 0<p\leq\infty;$$

b) *if* $f \in L_{p,h_\sigma}$, *then* $I_\sigma(f) \in B_\sigma$ *and*

$$\|I_\sigma(f,x)\|_{p,h_\sigma} \leq c(p)\|f\|_{p,h_\sigma}, \ 1<p<\infty.$$

Proof. a). The case $p=\infty$ is a trivial corollary from Lemma 8.1, a) and Lemma 6.1, c). The case $0<p<\infty$ is a consequence from Lemma 6.1, f), Lemma 6.2, b), c) and Lemma 8.1, a). We have

$$\|I_{\sigma,r,\eta}(f,x)\|_{p,h_{\sigma(1+\eta)}} \le c_1\|I_{\sigma,r,\eta}(f,x)\|_p$$

$$\le c_2\|f\|_{l_{p,h_{\sigma(1+\eta)}}} \le c_3\|f\|_{p,h_{\sigma(1+\eta)}} .$$

The proof of b) follows from Lemma 6.1, f), Lemma 6.2, b) and Lemma 8.1, b).$_\square$

THEOREM 8.1. *If* $f(x) \in L_\infty$, $0<p\le\infty$, $r>\max(1/p-1,0)$, *then*

$$\tilde{E}_{\sigma(1+2\eta)}(f)_p \le c_1\|f-I_{\sigma,r,\eta}(f)\|_{p,1/\sigma} \le c_2\tilde{E}_\sigma(f)_p .$$

and the constants c_1 *and* c_2 *depend only on* p,r *and* η.

Proof. From Theorem 7.1 and Lemma 6.1 we obtain

$$\tilde{E}_{\sigma(1+2\eta)}(f)_p \le c_1E_{\sigma(1+2\eta)}(f)_{p,(\sigma(1+2\eta))^{-1}}$$

$$\le c_2E_{\sigma(1+2\eta)}(f)_{p,1/\sigma} \le c_3\|f-I_{\sigma,r,\eta}(f)\|_{p,1/\sigma}.$$

Conversely, let g(x) be such that $E_\sigma(f)_{p,1/\sigma} = \|f-g\|_{p,1/\sigma}$. From Theorem D and Lemma 8.2,a) we obtain

$$\|f-I_{\sigma,r,\eta}(f)\|_{p,1/\sigma} \le \|f-g\|_{p,1/\sigma} + \|I_{\sigma,r,\eta}(f-g)\|_{p,1/\sigma}$$

$$\le c_1\|f-g\|_{p,1/\sigma} \le c_1E_\sigma(f)_{p,1/\sigma} \le c_2\tilde{E}_\sigma(f)_p .\square$$

Corollarry 8.1. *If* $f(x) \in L_\infty$, $r\ge1$, *then*

$$E_{\sigma(1+2\eta)}(f)_\infty \le c_1\|f-I_{\sigma,r,\eta}(f)\|_\infty \le c_2E_\sigma(f)_\infty .$$

Proof. It follows from Theorem 8.1, the fact that $E_\sigma(f)_\infty$ and $E_\sigma(f)_\infty$ are equiapproximation in L_∞ [16, p.163] and Lemma 6.1, c).$_\square$

Theorem 8.2. *If* $f(x) \in L_{p,\delta}$ *and* $1<p<\infty$, *then*

$$\tilde{E}_\sigma(f)_p \le c_1(p)\|f-I_\sigma(f)\|_{p,1/\sigma} \le c_2(p)\tilde{E}_\sigma(f)_p$$

and the constants c_1 *and* c_2 *depend only on* p.

Proof. It is based on the same method (Theorem 8.1), Theorem C, Theorem 7.1 and Lemma 8.2, b).$_\square$

9. DIRECT AND CONVERSE THEOREMS AND EQUIAPPROXIMATION WITH GIVEN ORDER

Applying the averaged moduli of smoothness $\tau_k(f,\gamma)_p$ [16,pp.6] Professor V.Popov proved direct and converse theorems for the best onesided approximation [16, pp.162-172], [14]. His ideas were used in the proofs of many results of this type: for example direct and converse theorems for onesided approximation by means of entire functions [9], [1]. In this part of the paper we give three theorems, which are consequences from the results of Professor V.Popov and Theorem 8.1, Theorem 8.2.

Theorem 9.1. *If* $f \in L_\infty$, *then*

$$\|f-I_{\sigma,r,\eta}(f)\|_\infty = O(\sigma^{-\alpha}) \iff E_\sigma(f)_\infty = O(\sigma^{-\alpha}) \iff \omega_k(f,\sigma^{-1})_\infty = O(\sigma^{-\alpha}),$$

$k>\alpha$ *and* $r\geq 1$. (One may compare with [4].)

Theorem 9.2. *If* $f \in L_{p,\delta}$ *and* $1<p<\infty$, *then*

$$\|f-I_\sigma(f)\|_{p,1/\sigma} = O(\sigma^{-\alpha}) \iff \tilde{E}_\sigma(f)_p = O(\sigma^{-\alpha}) \iff \tau_k(f,\sigma^{-1})_p = O(\sigma^{-\alpha}),$$

$k>\alpha$. (One may compare with [8].)

Theorem 9.3. *If* $f \in L_\infty \cap L_p$ *and* $0<p\leq\infty$, *then*

$$\|f-I_{\sigma,r,\eta}(f)\|_{p,1/\sigma} = O(\sigma^{-\alpha}) \iff \tilde{E}_\sigma(f)_p = O(\sigma^{-\alpha}) \iff \tau_k(f,\sigma^{-1})_p = O(\sigma^{-\alpha}),$$

$k>\alpha$ *and* $r>\max(1/p-1,0)$.

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