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SOME PROPERTIES OF MONOTONE FUNCTIONS OF SEVERAL VARIABLES

M.I.DYACHENKO

§1. Introduction

We need some notations. Everywhere below we will consider a 2π-periodic on each variable function $f(t) = f(t_1,\dots,t_m)$. In the case where $f(t) \in L(T^m) = L([-\pi,\pi)^m)$ we will denote by $S_n(f,x)$ for $n = (n_1,\dots,n_m)$ the corresponding rectangular partial sum of the trigonometric Fourier series of $f(t)$ at the point x. Then if $x,y \in R^m$ and $x_j < y_j$ ($x_j \le y_j$) for $1 \le j \le m$ we will say that $x < y$ ($x \le y$). If α is a number we will also denote by α the vector with all coordinates equal to α. If $a,b \in \overline{T}^m$ and $a<b$ let $[a,b] = \prod_{j=1}^{m} [a_j,b_j]$, $[a,b) = \prod_{j=1}^{m} [a_j,b_j)$ etc.

<u>Definition</u>. Let $a,b \in \overline{T}^m$, $a<b$ and vector $\gamma \in \{0,1\}^m$. Then we will say that $f(t) \in M(a,b,\gamma)$ ($f(t)$ is monotone on (a,b) with vector γ) if $f(x) = 0$ for $x \in T^m \setminus [a,b]$ and $f(x) \ge f(y)$ for all $x,y \in (a,b)$ such that $(-1)^{\gamma_j} x_j \le (-1)^{\gamma_j} y_j$ for $1 \le j \le m$.

Furthermore, if

$T^m = \coprod_{l=1}^{r} [a(l),b(l))$ ($C = A \coprod B$ means that $C = A \cup B$ and $A \cap B = 0$)

and $f(x)\chi_{[a(l),b(l))}(x) \in M(a(l),b(l),\gamma(l))$ for $1 \le l \le r$, then we will say that $f(x) \in PM$ (here and below by $\chi_C(t)$ we denote the indicator of a set $C \subseteq T^m$).

In paper [1] we have proved that the Fourier series of functions $f(x) \in PM \cap L(T^m)$ possesses many good properties, for instance, they converge in Pringsheim sense almost everywhere on T^m, and besides, if $f(x)$ is bounded, they converge at every point of continuity of function $f(x)$. The purpose of this article is to get quantitative specifications of the last statement. We get quantitative specifications of the last statement. We introduce some more notations. We will say that function $\omega : [0,\infty) \to [0,\infty)$ belongs to the Q class (ω is a module of continuity) if:

1. $\omega(t) \in C([0,\infty))$;
2. $\omega(0) = 0$ and $\omega(t) > 0$ for $t>0$;
3. $\omega(t)\uparrow$ on $[0,\infty)$ and $\omega(t) = \omega(\pi)$ for all $t\geq\pi$;
4. for all $t_1, t_2 \geq 0$ we have $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.

Let us consider the Bary - Stechkin operator B on Q

$$B\omega(t) = \begin{cases} t\displaystyle\int_t^{\infty} u^{-2}\omega(u)du & \text{for } t>0, \\ 0 & \text{for } t=0. \end{cases}$$

We shall prove below that $B : Q \to Q$. By $A_1(m), A_2(m), \ldots$ we will denote the positive constans depending only on m , and etc. Finally, let $\|t\| = \max_{1\leq j\leq m} |t_j|$ and $K(t,\delta) = \{ x: \|t-x\|\leq\delta \}$ for $\delta>0$.

Theorem 1. Let $f(t) \in PM \cap L(T^m)$, $x \in T^m$, $\omega(u) \in Q$ and $|f(x)-f(t)| \leq \omega(\|t-x\|)$ for all t. Then for all $n \geq 0$ we have

$$\zeta_n(f,x) = |S_n(f,x)-f(x)| \leq rA_1(m)B^m\omega(\pi\nu^{-1}),$$

where $\nu = \min_{1\leq j\leq m} n_j+1$, r is the number of parallelepipeds of monotonicity of function $f(t)$, and $B^m\omega$ is the result of the m-time application of B to $\omega(u)$.

Corollary 1. If $\omega(u)$ in theorem 1 is such that $B\omega(u) = O(\omega(u))$ for $u\to_+0$, then $\zeta_n(f,x) \leq rA_1(m,\omega)\omega(\pi\nu^{-1})$,

Theorem 1 can't be strengthened. For $\omega(u) \sim B\omega(u)$ it is evident. Furthermore, we have

Theorem 2. If $\omega(u) \in Q$ is convex and $B\omega(u) \neq O(\omega(u))$ as $u\to_+0$, then there exist a function $f(t) \in PM \cap H^{\omega}$ such that

$$\zeta_{(n_k,\ldots,n_k)}(f,0) \geq A_2(m,\omega)B^m\omega(\pi(n_k+1)^{-1})$$

for some sequence $n_k\to\infty$.

Nevertheless, for fixed $f(x) \in PM \cap L^{\infty}(T^m)$ we can strengthen the estimate from theorem 1 almost everywhere. In connection with that we can point out that in paper [1] we have proved that if

$f(x) \in M(a,b,\gamma)$, then $f(x)$ is continuous almost everywhere. The same rezult for $m=2$ was independently proved by B. Lavric [2].

Theorem 3. If $f(x) \in M(a,b,\gamma)$, then $f(x)$ is differentiable almost everywhere on (a,b).

Applying theorem 1 we obtain

Corollary 2. If $f(x) \in PM \cap L^{\infty}(T^m)$, then almost everywhere we have $\zeta_n(f,x) = O_x((\ln\nu)^m \nu^{-1})$ as $\nu \to \infty$.

The question about the possibility of strenthening corollary 2 remains open. It is tightly connected with the maximal rate of growth almost everywhere of the partial sums of Fourier series. Nevertheless, it can be shown that even for $m=1$ we can't get the estimate $\zeta_n(f,x) = O_x(\nu^{-1})$ almost everywhere.

In §2 we prove theorem 1 and in §3 - theorem 3. We omit the proof of theorem 2 because of its technical complexity.

§2. Proof of theorem 1

Lemma 1. If $\omega(u) \in Q$, then $B\omega(u) \in Q$.

Proof. The continuity $B\omega(u)$ at $u \neq 0$ is obvious. Further for all $0 < u < s$ we have

$$B\omega(u) = u\int_u^s v^{-2}\omega(v)dv + u\int_s^{\infty} v^{-2}\omega(v)dv \le \omega(s) + us^{-1}\omega(\pi).$$

Since $\omega(s) \to 0$ as $s \to 0$, successively choosing suitable s and u we establish the continuity of $B\omega(u)$ at 0. The fulfilment of condition 2 is obvious. Further, for $u > 0$ we have

$$(B\omega(u))' = \int_u^{\infty} t^{-2}\omega(t)dt - u^{-1}\omega(u) \ge \omega(u)(\int_u^{\infty} t^{-2}dt - u^{-1}) = 0,$$

and so we have proved the monotonicity of $B\omega(u)$. Finally, for $u_1, u_2 \ge 0$ we have

$$B\omega(u_1 + u_2) = u_1 \int_{u_1+u_2}^{\infty} t^{-2}\omega(t)dt + u_2 \int_{u_1+u_2}^{\infty} t^{-2}\omega(t)dt \le B\omega(u_1) + B\omega(u_2).$$

This was to be proved.

The next lemma is considered to be the main one.

Lemma 2.(see [1,lemma 4]). If for some **a**,**b** and γ the function $f(\mathbf{t}) \in M(\mathbf{a},\mathbf{b},\gamma)$ and $|f(\mathbf{t})| \le D$ for all **t**, then

$$\sup_{x\in T^m} \sup_{\mathbf{n}\ge 0} |S_{\mathbf{n}}(f,x)| \le A_2(m)D .$$

Futhermore, if $\mathbf{t} \in T^m$ and $\mathbf{n} \ge 0$ are such that for some natural $j \in [1,m]$ and $k \in [1,n_j]$ we have

$$\rho_j(\mathbf{t};(\mathbf{a},\mathbf{b})) = \inf_{a_j \le y \le b_j} \min(|t_j-y|,|t_j-y-2\pi|,|t_j-y+2\pi|) \ge \pi k(n_j+1/2)^{-1},$$

then

$$|S_{\mathbf{n}}(f,\mathbf{t})| \le A_3(m)D(\ln(k+1))^{m-1}k^{-1} .$$

Let's turn to the proof of theorem 1. Without loss of generality we may assume that $x = f(x) = 0$. Let $T^m = \coprod_{q=1}^{r} R(q)$, where $R(q) = [\mathbf{a}(q),\mathbf{b}(q))$ and $f_q(\mathbf{x}) = f(\mathbf{x})\chi_{R(q)}(\mathbf{x}) \in M(\mathbf{a}(q),\mathbf{b}(q),\gamma(q))$ for $1 \le q \le r$. Then for each q and each $\mathbf{t} \in T^m$ we have $|f_q(\mathbf{t})| \le \omega(\|\mathbf{t}\|)$. Fix vector $\mathbf{n} \ge 0$. Let η be the number for which

$$2^{\eta} \le \nu = \min_{1\le j\le m} n_j + 1 < 2^{\eta+1}.$$

Denote

$$\Gamma_{\eta,q} = K(0,\pi 2^{-\eta}) \cap R(q) \quad\text{and}\quad \Gamma_{i,q} = R(q) \cap (K(0,\pi 2^{-i})\setminus K(0,\pi 2^{-i-1}))$$

for $0 \le i \le \eta-1$. For all $i \ne q$ we can represent $\Gamma_{i,q}$ as disjunct unification of no more than $2m$ non-empty parallelepipeds $\{R_{i,q,s}\}$ such as for each s there exists $j = j(s)$ for which $\rho_j(0;R_{i,q,j}) \ge \pi 2^{-(i+1)}$. Thus, using lemma 2, we have

$$\zeta_{\mathbf{n}}(f;0) \le \sum_{q=1}^{r}|S_{\mathbf{n}}(f_q,0)| \le \sum_{q=1}^{r}\Big(\sum_{i=0}^{\eta-1}\sum_{s=1}^{2m}|S_{\mathbf{n}}(f_q\chi_{R_{i,q,s}},0)| + |S_{\mathbf{n}}(f_q\chi_{\Gamma_{\eta,q}},0)|\Big) \le$$

$$\le r\Big(\sum_{i=0}^{\eta-1} 2mA_3(m)\omega(\pi 2^{-i})(\eta-i)^{m-1}2^{i-\eta} + A_2(m)\omega(\pi 2^{-\eta})\Big).$$

Successively using Abel transformation we get

$$\zeta_n(f;0) \le rA_4(m)\sum_{i=0}^{\eta}(\eta-i+1)^{m-1}2^{i-\eta}\omega(\pi 2^{-i}) =$$

$$= rA_4(m)\sum_{i=0}^{\eta}(\eta-i+1)^{m-2}(m-1)\sum_{j=0}^{i}\omega(\pi 2^{-j})2^{\eta-j} =$$

$$= rA_5(m)\sum_{i=0}^{\eta}(\eta-i+1)^{m-2}2^{i-\eta}\pi 2^{-i}\sum_{j=0}^{i}\omega(\pi 2^{-j})2^{j}\pi^{-1} \le$$

$$\le rA_5(m)\sum_{i=0}^{\eta}(\eta-i+1)^{m-2}2^{i-\eta}B\omega(\pi 2^{-i}) \le \ldots \le$$

$$\le rA_6(m)\sum_{i=0}^{\eta}2^{i-\eta}B^{m-1}\omega(\pi 2^{-i}) \le rA_6(m)B^m\omega(\pi 2^{-\eta}) \le 2rA_6(m)B^m\omega(\pi\nu^{-1}).$$

This was to be proved.

§3. Proof of the theorem 3

We need two theorems from book [3] by S.Saks.

Theorem A. (see [3,Ch.IX,Th.11.2]). If the finite function F of two variables is measurable on the set E, then its approximative partial derivative numbers are also measurable on E.

Theorem B. (V.V.Stepanoff, see [3,Ch.IX,Th.12.2]). The finite measurable on the set E function F of two variables is almost everywhere approximately differentiable on the set E if and only if it is almost everywhere on E approximately differentiable with respect to each variable.

S.Saks pointed out that these theorems remain valid for functions of any number of variables. Below we will use these statements for any dimension.

Lemma 3. Let $f(x) \in M(a,b,\gamma)$, $t \in (a,b)$ and $f'_{ap}(t)$ exist. Then also exists $f'(t) = f'_{ap}(t)$.

Proof. Without loss of generality we assume that $\gamma = 0$. Let E be the set from the definition of the approximative derivative at t. For given $\varepsilon \in (0.1)$ we can choose $\delta > 0$ so that for any $z \in K(t,s) \cap E$ we have

$$|\varphi(z,t)| = |f(z) - f(t) - (f'_{ap}(t), z-t)| \le \frac{\varepsilon}{8}\|x-t\| \qquad (1)$$

and for any $\delta_1 \in (0,\delta)$ the measure

$$\mu(E \cap K(t,\delta_1)) \ge \left(1 - \left[\frac{\varepsilon}{9C}\right]^m\right)\mu(K(t,\delta_1)), \text{ where } C = m\|f'_{ap}(t)\| + 1.$$

Fix a point $x \in K(t,\delta)$. Let $\Pi_1 = \{ y \in K(x,\varepsilon(2C)^{-1}\|x-t\|): y \ge x \}$ and $\Pi_2 = \{ y \in K(x,\varepsilon(2C)^{-1}\|x-t\|): y \le x \}$. Then $\Pi_1, \Pi_2 \subset K(t, 2\|x-t\|)$, and since

$$\mu(\Pi_1) = \mu(\Pi_2) = (\varepsilon(2C)^{-1}\|x-t\|)^m = (\varepsilon(8c)^{-1})^m \mu(K(t,2\|x-t\|)),$$

there exist points $y_1 \in \Pi_1 \cap E$ and $y_2 \in \Pi_2 \cap E$. Furthermore, due to monotonicity of function f we have $f(y_1) \le f(x) \le f(y_2)$. Then (see (1))

$$|\varphi(x,t)| \le \max_{i=1,2} (|\varphi(y_i,t)| + |(f'_{ap}(t), y_i - x)|) \le$$

$$\le \max_{i=1,2} \left(\frac{\varepsilon}{8}\|y_i - t\| + C\|y_i - x\|\right) \le \frac{\varepsilon}{4}\|x-t\| + C\,\varepsilon(2C)^{-1}\|x-t\| \le \varepsilon\,\|x-t\|$$

and so lemma 3 is proved.

Finally establish theorem 3. As we pointed out in the introduction, the function $f(t)$ is continuous almost everywhere on (a,b) and consequently it is measurable on (a,b). Then by theorem A the set where $f(t)$ has all partial approximative derivatives is measurable. Due to monotonicity of $f(t)$ we obtain that $f(t)$ has all partial approximative derivatives almost everywhere on (a,b). Using theorem B we can prove that almost everywhere on (a,b) $f(t)$ has approximative derivative, and using lemma 3 completes the proof.

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