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On the Multivariate Spline Approximation in C-metric

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Dedicated to the memory of Vasil Popov

1. Introduction

In this paper we obtain a characterization of the best uniform approximation of functions on the unit cube $\Omega = [0,1]^d$ by a kind of multivariate splines. We consider splines which are linear combinations of tensor product of univariate **B**-splines and its dyadic dilates and translates. These splines have been introduced as a tool for approximation by R. DeVore and V. Popov [4,5]. To characterize the approximations considered we shall apply a well known approach: first, to prove theorems of Jackson and Bernstein type and then to use the general method given in [1,6,10].

In what follows $r, r \in \mathbf{N}, r > d$ will be a fixed number. Denote

$$(1) \qquad N(t) := [0, \dots, r](\cdot - t)_+^{r-1}, \qquad t \in \mathbf{R},$$

with usual divided difference notation. The function N is the univariate **B**-spline of order r with knots at the points $0, \dots, r$. It is piece-wise polynomial (of degree $r-1$) and non-negative function supported on the interval $[0, r]$. For higher dimension, we define N by

$$N(x) := N(x_1) \dots N(x_d), \qquad x = (x_1, \dots, x_d) \in \mathbf{R}^d.$$

Let us also denote

$$N_{j,k}(x) := N(2^k x - j); \qquad k = 0, 1, \dots, \ j \in \mathbf{Z}^d,$$

$$\mathbf{B} := \left\{ N_{j,k} : k = 0, 1, \dots, \ j \in \mathbf{Z}^d \right\},$$

$$\Sigma_n := \left\{ \varphi : \varphi = \sum_{i=1}^{n} \mu_i b_i, \ \mu_1, \dots, \mu_n \in \mathbf{R}, \ b_1, \dots, b_n \in \mathbf{B} \right\}.$$

Any $\varphi \in \Sigma_n$ will be called free spline of freedom n and order r.

We let

$$S_n(f) := S_n(f)_{C(\Omega)} := \inf\left\{\| f - s \|_{C(\Omega)} :\ s \in \Sigma_n\right\},$$

$n = 0, 1, \ldots$ denote the error in approximating $f \in C(\Omega)$ by splines of freedom n. We suppose that $S_0(f) :=\| f \|_{C(\Omega)}$. We are interested in characterizing functions f for which $S_n(f)$ has a prescribed rate of convergence. Let us introduce the so called approximation classes (see [4,10]) $S_q^\alpha := S_q^\alpha(C)$ of all functions $f \in C(\Omega)$ for which

$$\| f \|_{S_q^\alpha} := \left(\sum_{\nu=-1}^{\infty} ((2^{\alpha\nu} S_{2^\nu}(f))^q \right)^{1/q} < \infty$$

with $S_{1/2} := S_0(f)$, $0 < \alpha < \infty$, $0 < q < \infty$. When $q = \infty$, S_∞^α consists of all functions f for which $S_n(f) = O(n^{-\alpha})$.

Our goal is to characterize S_q^α as a certain function space. This characterization involve the classical Besov spaces. To define the Besov spaces $B_{p,q}^\alpha$, we need the modulus of smoothness of order k of $f \in L_p(\Omega)$,

$$\omega_k(f,t)_p := \sup\left\{\| \Delta_h^k(f,\cdot) \|_{L_p(\Omega(kh))} : | h |\le t\right\}, \tag{2}$$

where $| h |$ is the Euclidean length of the vector h; Δ_h^k is k–th order difference with step $h \in \mathbf{R}^d$ and the norm in (2) is the L_p norm on the set $\Omega(kh) := \{x : x, x + kh \in \Omega\}$. Of course, when $p < 1$ (we further have exactly this case) this is not really a norm, it is only a quasi-norm. If $\alpha, p, q > 0$, we say f is in the Besov space $B_{p,q}^\alpha$ whenever

$$| f |_{B_{p,q}^\alpha} := \left(\int_0^1 (t^{-\alpha}\, \omega_k(f,t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty . \tag{3}$$

Here k is any integer greater than α. As usual, if $q = \infty$, the integral in (3) is replaced by $\sup_{t>0}$. For more details about Besov spaces see [1]. When $p = q$, we denote the space $B_{p,q}^\alpha$ simply by B_p^α.

In §**2** and **3**, we will prove the following theorems.

Theorem 1 (Jackson type inequality) *Let $f \in B_\sigma^\alpha$, $d \le \alpha < r$, $\sigma = d/\alpha$, d is the dimension of the Euclidean space, then $f \in C(\Omega)$ and*

$$S_n(f)_{C(\Omega)} \le c n^{-\alpha/d} \, | f |_{B_\sigma^\alpha\ (\Omega)} , \tag{4}$$

with[1] $n \ge r^d$.

Theorem 2 (Bernstein type inequality) *If $s \in \Sigma_n$, $n = 0, 1, \ldots$, then*

$$| s |_{B_\sigma^\alpha(\Omega)} \le c n^{\alpha/d} \, \| s \|_{C(\Omega)} , \tag{5}$$

whenever $d \le \alpha < r$, $\sigma = d/\alpha$.

These two inequalities allow us using the general principles given in [1,6,10] to obtain our main result.

[1]Here and throughout the paper c denotes a constant which may depend on α, d and r, but does not depend on f and n. The constant c can also vary at each appearance.

Theorem 3 *If $d \le \alpha < r$, $\sigma = d/\alpha$, $0 < \gamma < \alpha$, $0 < q \le \infty$, $C = C(\Omega)$, then*

$$S_q^{\gamma/d} = (C,\ B_\sigma^\alpha)_{\gamma/\alpha,q}\ , \tag{6}$$

Moreover if $q = d/\gamma$, then

$$S_q^{\gamma/d} = B_q^\gamma$$

with equivalent quasinorms. Here $(X, Y)_{\theta,q}$ is the corresponding real interpolation space between X and Y, see [1].

The assertion in (6) corresponds to a result of P. Petrushev [9] concerning the approximation by means of univariate splines with free knots in L_p norm , $0 < p < \infty$.

> $S_q^\gamma(L_p)$ (univariate splines with free knots of order r) $= (L_p, B_\sigma^\alpha)_{\gamma/\alpha,q}$
> whenever $0 < \gamma < \alpha < r$, $1/\sigma = \alpha + 1/p$.

R. DeVore and V. Popov [5] have obtained a similar result for multivariate free splines of order $r = 1$, namely

> $S_q^{\gamma/d}(L_p)$ (multivariate free splines of order $r = 1$) $= (L_p, B_\sigma^\alpha)_{\gamma/\alpha,q}$
> with $0 < p < \infty$, $1/\sigma = \alpha/d + 1/p$, $0 < \gamma < \alpha < \min(1, d/(p(d-1))$.

The remainder of this paper will be devoted to proving the basic estimates (4) and (5).

2. Jackson inequality

In order to prove Theorem 1 let us first give some facts concerning the Besov spaces and some properties of **B**-splines.

Using the monotonicity properties of ω_k, we get a discrete quasi-norm which is equivalent to (3)

$$|\ f\ |_{B_{p,q}^\alpha} := \left(\sum_{i=0}^{\infty} (2^{i\alpha}\omega_k(f, 2^{-i})_p)^q\right)^{1/q} \tag{7}$$

We can also consider the following quasi-norm in $B_{p,q}^\alpha$

$$\|\ f\ \|_{B_{p,q}^\alpha} :=\|\ f\ \|_{L_p(\Omega)} + |\ f\ |_{B_{p,q}^\alpha}\ .$$

This is really a norm when $p, q \ge 1$.

There are many properties of **B**-splines (see [2,3]). One of them we need is that a **B**-spline of certain "level" can be represented as linear combination of B-splines of lower "level", i.e. for any $k \in \mathbf{N}$, $i \in \mathbf{Z}^d$, we have

$$N_{i,k} = \sum_{j \in Z^d} \mu_j N_{j,k+1} \tag{8}$$

with some $\mu_j \in \mathbf{R}$. Moreover $\mu_j \ne 0$ only for those $N_{j,k+1}$ which satisfy supp $N_{j,k+1} \subset$ supp $N_{i,k}$.

B-splines can also recover the polynomials of coordinate degree no more than $r-1$, i.e. for any such a polynomial P and any $k \in \mathbf{N}$, there exist $\lambda_j \in \mathbf{Z}^d$ such that

$$(9) \qquad P(x) = \sum_{j \in Z^d} \lambda_j N_{j,k}(x), \qquad x \in \mathbf{R}^d .$$

It is clear that for every $x \in \mathbf{R}^d$, the sum in (9) is finite. Another fact we need is that there exists a constant c depending on d and r such that

$$(10) \qquad \| \varphi \|_{C(R^d)} \leq c \| \varphi \|_{C\left([0,r]^d\right)}$$

for any $\varphi = \sum_{j:\, supp\, N_{j,0} \cap (0,r)^d \neq \emptyset} \mu_j N_{j,0},\ \mu \in \mathbf{R},\ j \in \mathbf{Z}^d$. The above assertion follows from the linear independence of $N_{j,0}$, on $[0,r]^d$ and the fact that any two norms in a linear space with a finite dimension are equivalent.

Some notations. Since B-splines $N_{j,k},\ k \in \mathbf{Z},\ j \in \mathbf{Z}^d$ are countable, we can rearrange them to depend only on one index $i = 0,1,\ldots$; i.e. we can assume that $\mathbf{B} = \{N_i : i = 0,1,\ldots\}$. Using (1) we have that, if $b \in \mathbf{B}$ and $b = N_{j,k}$, then $\operatorname{supp} b = 2^{-k}\left(j + [0,r]^d\right) \subset 2^{-k}\left(j + [0,q]^d\right)$, where $q = 2r+1$. Obviously q is an odd number. With each $b \in \mathbf{B},\ b = N_{j,k}$, we associate the cube $\mathrm{s}\, b := 2^{-k}\left(j + [0,q]^d\right)$. We also denote $\Lambda := \{I : I = \mathrm{s}\, b,\ b \in \mathbf{B}\}$. The most of the norms of this paper are taken over Ω and when it is clear, we shall simply write $\|\cdot\|_C,\ \|\cdot\|_P,\ \|\cdot\|_{B^\alpha_\sigma}$ instead of $\|\cdot\|_{C(\Omega)},\ \|\cdot\|_{L_p(\Omega)},\ \|\cdot\|_{B^\alpha_\sigma(\Omega)}$.

Theorem 1 is proved by applying an atomic decomposition of the functions in the space B^α_σ. Namely

Lemma 1 (*R. DeVore, V. Popov* [4]) *If $f \in B^\alpha_\sigma,\ 0 < \alpha < r,\ \sigma = d/\alpha$, then f can be represented as*

$$f = \sum_{i=1}^{\infty} \lambda_i N_i, \qquad \lambda_i \in \mathbf{R}$$

with convergence in the sense of L_σ and $\| f \|_{B^\alpha_\sigma}$ is equivalent to the quasi-norm

$$^1\| f \|_{B^\alpha_\sigma} := \left(\sum_{i=1}^{\infty} |\lambda_i|^\sigma \right)^{1/\sigma} .$$

Let now $d \leq \alpha < r$ and $f \in B^\alpha_\sigma$. Since $\sigma \leq 1$ by Lemma 1 it follows that $f = \sum_{i=1}^\infty \lambda_i N_i$ with convergence in the sense of $C-$norm. It allows us to choose m such that

$$(11) \qquad \| f - \sum_{i=1}^{m} \lambda_i N_i \|_C \leq n^{-\alpha/d} \| f \|_{B^\alpha_\sigma}$$

In what follows $N_1,\ldots,N_m$ will be fixed. It will be convenient when $I = \mathrm{s}\, N_i$ to denote $\lambda_I := \lambda_i,\ N_I := N_i$.

Let $\Gamma \subset \Lambda$ and for any $I, J \in \Gamma$ only one of the following three cases is possible

i) $I \subset J$

ii) $J \subset I$

iii) $\operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset$

Here int (I) means the interior of the cube I. Then for $I, J \in \Gamma$ we say J is a child of I with respect to Γ if $J \subset I$, $J \neq I$ and J is maximal i.e. J is not contain in another cube of Γ with these properties.

We also need the following combinatorial assertion.

Lemma 2 *The set Λ can be divided into $q^d = (2r+1)^d$ sets $\Gamma_1, \ldots, \Gamma_{q^d}$, such that each $\Gamma := \Gamma_i$, $i = i, ..., q^d$, satisfies the following property*

$$\text{(12)} \qquad \begin{array}{l} \textit{For any } I, J \in \Gamma \textit{ we have} \\ \operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset, \quad \textit{or } I \subset J, \quad \textit{or } J \subset I. \end{array}$$

Proof. Case $d = 1$. Denote $\Lambda_k := \{I \in \Lambda : l(I) = 2^{-k} q\}$, $k \in \mathbf{Z}$, $q = 2r + 1$, $l(I)$ means the Euclidean length of I. At first, one can easily see that the elements of Λ_0 can be split into sets $\Gamma_1, \ldots, \Gamma_q$ satisfying (12). We intend to join consecutively the rest of Λ to some Γ_i holding on (12). Let us assume that such a procedure is carried out with the set $\cup_{i=0}^{\nu} \Lambda_i$.

$$\text{(13)} \qquad \begin{array}{l} \text{If } I \in \Lambda_{\nu+1}, \text{ then there exists unique } J, \text{ such that} \\ J \in \Lambda_\nu,\ I \subset J, \\ \text{and } I \text{ and } J \text{ have common beginning or end.} \end{array}$$

Here (13) holds because q is an odd number. Then we join I to the same Γ_i which J belongs to. We repeat this procedure with each $I \in \Lambda_{\nu+1}$. It should be mentioned that $\Gamma_1, \ldots, \Gamma_q$ again satisfy (12). Continuing in such a maner we obtain Lemma 2 for $\cup_{i=0}^{\infty} \Lambda_i$. Similar arguments cover the case for negative ν.

Case $d > 1$. This case will be obtained as a consequence of the case $d = 1$. If $I \in \Lambda$, then $I = I^1 \times \cdots \times I^d$, where $I^1, \ldots, I^d$ are projects of I on the coordinate axes. Let us consider the sets $\Lambda^j := \{I^j : I \in \Lambda\}$, $j = 1, \ldots, d$. Since Lemma 2 was establish when $d = 1$, each of sets Λ^j, $j = 1, \ldots, d$ can be split into classes $\Gamma_1^j, \ldots, \Gamma_q^j$ satisfying (12). Now, we define classes $\Gamma_{i_1,\ldots,i_d}$, $1 \leq i_k \leq q$, $k = 1, \ldots, d$ in the following way. If $I \in \Lambda$, $I = I^1 \times \ldots \times I^d$ and $I^k \in \Gamma_{i_k}^k$, $k = 1, \ldots, d$, then $I \in \Gamma_{i_1,\ldots,i_d}$. Obviously these classes satisfy the conditions of Lemma 2. ∎

Proof of Theorem 1. Let $\overline{\Gamma}_i := \{I_1, \ldots, I_m\} \cap \Gamma_i$, $i = 1, \ldots, (2r+1)^d$, (see (11)), where Γ_i are the sets from Lemma 2. Denote $\psi_i := \sum_{I \in \overline{\Gamma}_i} \lambda_I N_I$, $i = 1, \ldots, (2r+1)^d$. From (11) it is clear that

$$\| f - \sum_{i=1}^{(2r+1)^d} \psi_i \| \leq n^{-\alpha/d} \| f \|_{B_\sigma^\alpha}.$$

Our purpose is to approximate appropriately each ψ_i with linear combination of no more than n terms N_I, $I \in \Lambda$. We will discuss it only for ψ_1. We shall apply the technique introduced by E. Moskona and P. Petrushev in [8]. With each chain $I_1 \supset \cdots \supset I_k$, $I_j \in \overline{\Gamma}_1$, $j = 1, \ldots, k$ we associate the number $\sum_{s=1}^{k} | \lambda_s |^\sigma$. Let $\lambda := \max \sum_{s=1}^{k} | \lambda_s |^\sigma$ where max is taken over all such chains. Let $I_1^* \supset \cdots \supset I_q^*$ be a chain for which $\sum_{i=1}^{q} | \lambda_i^* |^\sigma = \lambda$. We denote $G_1 = \{I_1^*, \ldots, I_q^*\}$ and apply for $\overline{\Gamma}_1 \backslash G_1$ same procedure as above with $\overline{\Gamma}_1 \backslash G_1$ instead of $\overline{\Gamma}_1$. It can be repeated n

times and as a result we obtain the sets $G_1, \dots, G_n$. Let $G := \cup_{i=1}^n G_i$, $\Gamma_1^* := \overline{\Gamma}_1 \backslash G$. Since $\sum_{I \in G_1} \mid \lambda_I \mid^\sigma \geq \dots \geq \sum_{I \in G_n} \mid \lambda_I \mid^\sigma$, we have

$$(14) \qquad \max \sum_{i=1}^{s} \mid \lambda_i \mid^\sigma \leq \frac{1}{n} \sum_{I \in \overline{\Gamma}_1} \mid \lambda_I \mid^\sigma \leq \frac{1}{n} {}^1\| f \|_{B_\sigma^\alpha}^\sigma ,$$

where the max in the last inequality is taken over all chains $I_1 \supset \dots \supset I_s$, such that $I_1, \dots, I_s \in \Gamma_1^*$. Using the structure of Γ_1, (14) and the fact that $\sigma \leq 1$, we obtain

$$(15) \qquad \| \sum_{I \in \overline{\Gamma}_1} \lambda_I N_I - \sum_{I \in G} \lambda_I N_I \|_C = \| \sum_{I \in \Gamma_1^*} \lambda_I N_I \|_C$$

$$\leq c \max \sum_{i=1}^{s} \mid \lambda_i \mid \leq c \max \left(\sum_{i=1}^{s} \mid \lambda_i \mid^\sigma \right)^{1/\sigma}$$

$$\leq c n^{-1/\sigma} {}^1\| f \|_{B_\sigma^\alpha} = c n^{-\alpha/d} {}^1\| f \|_{B_\sigma^\alpha} .$$

Here max is taken on the same chains as in (14). According to above, we say that $\mathbf{V} = \{I_1, \dots, I_k\}$, $I_1 \supset \dots \supset I_k$, $I_1, \dots, I_k \in \Lambda$ is a chain (of cubes) and denote $\underline{V} := I_1$, $\overline{V} = I_k$. There exist no more than n cubes $I \in G$ which have at least two children with respect to G. Indeed, with each I which has more than one child, we can associate certain G_i, $1 \leq i \leq n$, satisfying $\underline{G_i} = I$. Since the case $\underline{G_i} = \underline{G_j}$, $i \neq j$ is impossible, the above statement follows immediately. Similar arguments give that there exist no more than n cubes $I \in G$ with no children. Now one can see that G can be split into no more than $2n$ chains $V_1, \dots, V_l$, $l \leq 2n$, with property that for any V_i, V_j, $i \neq j$, the following opportunities are only possible

$$(16) \qquad \begin{aligned} &i) \quad \underline{V}_i \subset \overline{V}_j \\ &ii) \quad \underline{V}_j \subset \overline{V}_i \\ &iii) \quad \operatorname{int} \underline{V}_i \cap \operatorname{int} \underline{V}_j = \emptyset. \end{aligned}$$

Indeed, let us consider the set of all chains of the type $I_1 \supset \dots \supset I_k$, where each I_s, $s = 1, \dots, k-1$, has only one child I_{s+1} and the chain $I_1 \supset \dots \supset I_k$ is not a part of another chain of the same kind. These chains are no more than $2n$ because for each of them $I_1 \supset \dots \supset I_k$, I_k has either no children or at least two children. We denote them by $V_1, \dots, V_l$, $l \leq 2n$, and evidently they satisfy (16). Now, we shall split each chain V_i, $1 \leq i \leq l$, into "short subchains". Let $\epsilon := \frac{1}{n} \sum_{I \in G} \mid \lambda_I \mid^\sigma$. Every chain V_i, $1 \leq i \leq l$, which consists of $I_1 \supset \dots \supset I_k$ can be divided into subchains $I_1 \supset \dots \supset I_{p_1}$, $I_{p_1+1} \supset \dots \supset I_{p_2}, \cdots$, $I_{p_\nu+1} \supset \ldots \supset I_k$ satisfying one of the following three possibilities (we use notations $p_0 = 0$, $p_{\nu+1} = k$)

$$(17) \qquad i) \qquad \epsilon \leq \sum_{i=p_j+1}^{p_{j+1}} \mid \lambda_{I_i} \mid^\sigma \leq 2\epsilon,$$

$$ii) \qquad \sum_{i=p_j+1}^{p_{j+1}} \mid \lambda_{I_i} \mid^\sigma < \epsilon \text{ and } \sum_{i=p_j+1}^{p_{j+1}+1} \mid \lambda_{I_i} \mid^\sigma > 2\epsilon, \text{ or } j = \nu$$

$$iii) \qquad p_j + 1 = p_{j+1} =: q \text{ and } \mid \lambda_q \mid^\sigma > 2\epsilon,$$

with $j = 0, \dots, \nu$. One can see that the number of the subchains satisfying condition $ii)$ in (17) is no more than $3n$ and the number of the subchains submitted to $i)$ or *iii*) does not exceed n. Thus, the total number of the subchains is no more than $4n$. We again denote them by $V_1, \ldots, V_l$, $l \leq 4n$. They obviously satisfy (16). Moreover, each V_i, $i = 1, \ldots, l$, satisfies one of the following possibilities

$$\text{(18)} \qquad i) \quad \sum_{I \in V_i} |\lambda_I|^\sigma \leq 2\epsilon,$$

$$ii) \quad V_i \text{ consists of only one cube } I \text{ and } |\lambda_I|^\sigma > 2\epsilon.$$

Let $\varphi_i := \sum_{I \in V_i} \lambda_I N_I$, $i = 1, \ldots, l$. Evidently

$$\text{(19)} \qquad \sum_{I \in G} \lambda_I N_I = \sum_{i=1}^{l} \varphi_i, \quad l \leq 4n.$$

Further, each φ_i will be approximated by linear combination of no more than c terms N_I, $I \in \Lambda$. Let φ_i be fixed $1 \leq i \leq l$.

Case 1. V_i satisfies the condition *i*) in (18). For every $J \in V_i$, N_J admits the following representation (see (8))

$$\text{(20)} \qquad N_J = \sum_{I \in M_1} \mu_I N_I$$

where $\mu_I \in \mathbf{R}$ and $M_1 = \{I : I \in \Lambda, \ |I| = |\overline{V}_i|, \ I \subset J\}$ ($|I|$ is the volume of the cube I)

Applying (20) to each N_J, $J \in V_i$, we get

$$\text{(21)} \qquad \varphi_i := \sum_{I \in V_i} \lambda_I N_I = \sum_{I \in M_2} \tilde{\mu}_I N_I$$

for some $\tilde{\mu}_I \in \mathbf{R}$ and $M_2 = \{I : I \in \Lambda, |I| = |\overline{V}_i|, \ I \subset \underline{V}_i\}$. We set

$$\tilde{\varphi}_i := \sum_{I \in M_3} \tilde{\mu}_I N_I$$

with $M_3 := \{I : I \in M_2, I \cap \operatorname{int}(\overline{V}_i) \neq \emptyset\}$ and $\tilde{\mu}_I$ are the same as in (21). Let us notice that $\tilde{\varphi}_i$ have the following properties

$$\text{(22)} \qquad \begin{aligned} &i) \quad \operatorname{supp}(\tilde{\varphi}_i) \subset \underline{V}_i \\ &ii) \quad \tilde{\varphi}_i(x) = \varphi_i(x) \ \text{ whenever } x \in \overline{V}_i \\ &iii) \quad \| \tilde{\varphi}_i \|_C \leq c \, \| \varphi_i \|_C \end{aligned}$$

The construction of $\tilde{\varphi}_i$ implies i) and ii); iii) follows from (10) by changing the scale. It is also clear that $\#M_3 \leq$ c. ($\#M$ denotes the number of the elements of M)

Case 2. V_i satisfies the condition *ii*) in (18). Then V_i consists of only one cube I. In this case we set

$$\text{(23)} \qquad \tilde{\varphi}_i := \varphi_i = \lambda_I N_I .$$

Now, by (16), (18) and (22) respectively (23), we get

$$\text{(24)} \quad \| \sum_{i=1}^{l} \varphi_i - \sum_{i=1}^{l} \tilde{\varphi}_i \|_C \leq \max_{1 \leq i \leq l} \| \varphi_i - \tilde{\varphi}_i \|_{C(\underline{V}_i \setminus \overline{V}_i)}$$

$$\leq \max_{i\in F}(\| \varphi_i \|_C + \| \tilde{\varphi}_i \|_C) \leq \max_{i\in F} \| \varphi_i \|_C \leq c \max_{i\in F} \sum_{I\in V_i} | \lambda_I |$$

$$\leq c \max_{i\in F} \left(\sum_{I\in V_i} | \lambda_I |^\sigma \right)^{1/\sigma} \leq c\, n^{-\alpha/d\, 1} \| f \|_{B^\alpha_\sigma},$$

where F is the set of those i, $1 \leq i \leq l$, for which V_i satisfies case i) in (18). Using (15), (19) and (24), we obtain

$$\| \sum_{I\in\overline{\Gamma}_1} \lambda_I N_I - \sum_{i=1}^{l} \tilde{\varphi}_i \|_C \leq \; cn^{-\alpha/d\, 1} \| f \|_{B^\alpha_\sigma} \; . \tag{25}$$

We can find similar to (25) estimates for each $\overline{\Gamma}_2, \ldots, \overline{\Gamma}_{(2r+1)^d}$ with corresponding $\tilde{\varphi}_i$. Since $l \leq 4n$ and each $\tilde{\varphi}_i$ is a linear combination of no more than c elements of $\mathbf{B}$ then there exist $\tilde{N}_1, \ldots, \tilde{N}_t \in \mathbf{B}$; $\mu_1, \ldots, \mu_t \in \mathbf{R}$ such that

$$\| \sum_{i=1}^{m} \lambda_i N_i - \sum_{i=1}^{t} \mu_i \tilde{N}_i \|_C \leq cn^{-\alpha/d\, 1} \| f \|_{B^\alpha_\sigma} \tag{26}$$

with $t \leq cn$. By (11), Lemma 1 and (26), we get

$$\| f - \sum_{i=1}^{t} \mu_i \tilde{N}_i \|_C \leq cn^{-\alpha/d} \;\; \| f \|_{B^\alpha_\sigma} \; .$$

Therefore

$$S_n(f)_C \leq cn^{-\alpha/d} \;\; \| f \|_{B^\alpha_\sigma} \tag{27}$$

To prove Theorem 1 it remains to replace $\| f \|_{B^\alpha_\sigma}$ in (27) with $| f |_{B^\alpha_\sigma}$. We use Whithney's theorem (see [10, 11]).

$$E_{r-1}(f)_{\sigma(\Omega)} \leq c\omega_r(f,1)_{\sigma(\Omega)} \; ,$$

i.e. there exists a polynomial of total degree $\leq r-1$ such that

$$\| f - P \|_\sigma \leq c\,\omega_r(f,1)_\sigma \leq c \left(\sum_{i=1}^{\infty} \left(2^{i\alpha}\, \omega_r(f, 2^{-i}) \right)^\sigma \right)^{1/\sigma} \leq c \, | f |_{B^\alpha_\sigma} \; . \tag{28}$$

In (28) we used (7) and the fact that $\alpha < r$. By (27), (28) and the simple observation that $| f - P |_{B^\alpha_\sigma} = | f |_{B^\alpha_\sigma}$, we obtain

$$\begin{aligned} S_n(f-P)_C \;\; &\leq \;\; cn^{-\alpha/d} \, \| f - P \|_{B^\alpha_\sigma} \\ &= \;\; cn^{-\alpha/d} \left(| f - P |_{B^\alpha_\sigma} + \| f - P \|_\sigma \right) \leq cn^{-\alpha/d} \, | f |_{B^\alpha_\sigma} \; . \end{aligned} \tag{29}$$

But according to (9) the restriction of P on Ω can be represented as follows

$$P(x) = \sum \nu_j N_{j,0}(x), \qquad \nu_j \in \mathbf{R}, \tag{30}$$

where the last sum is taken on $j: \; j \in \mathbf{Z}^d$, supp $N_{j,0} \cap \Omega \neq \emptyset$. By (29) and since the number of the terms in (30) is less than constant, depending only on r and d , we finally obtain $S_n(f)_C \leq cn^{-\alpha/d} \, | f |_{B^\alpha_\sigma}$ which proves Theorem 1. ∎

3. Bernstein inequality

We need some notations. Let $l(I)$ denotes the side length of a cube I. We say Q is dyadic cube if $Q = 2^{-k}\Omega + 2^{-k}j$ for some $k \in \mathbf{N}$, $j \in \mathbf{Z}^d$. If I is a cube and $\lambda > 0$, $\lambda.I$ denotes the cube with a center the center of I and $l(\lambda.I) = \lambda l(I)$. Obviously every $I \in \Lambda$ is an union of q^d equal dyadic cubes. Let D be the family of all dyadic cubes. If $\Gamma = \{Q_1, \ldots, Q_n\}$, $Q_i \in D$, then for any $Q \in \Gamma$ we let $\mathbf{B}_Q$ denote the collection of all children of Q with respect to Γ. The definition of child is before Lemma 2. To establish Theorem 2, we need two combinatorial lemmas.

Lemma 3 *For arbitrary collection of dyadic cubes $\Gamma = \{P_1, \ldots, P_n\}$, there exists a second collection of dyadic cubes $\tilde{\Gamma} = \{Q_1, \ldots, Q_m\}$ with the the following properties*

$$\Gamma \subset \tilde{\Gamma}, \qquad m \leq cn.$$

If $Q \in \tilde{\Gamma}$, exactly one of the following cases holds

$$i) \quad \#\mathbf{B}_Q = 0$$

$$(31) \quad ii) \quad \#\mathbf{B}_Q = 1$$

$$iii) \quad \#\mathbf{B}_Q = 2^d \text{ and } \mathbf{B}_Q \text{ consists of } 2^d \text{ equal dyadic cubes with an union } Q$$

Here $\mathbf{B}_Q$ is taken with respect to $\tilde{\Gamma}$.

An analogous assertion is proved in [5]. Similar arguments imply Lemma 3.

Lemma 4 *Let $t > 0$; $x, h \in \mathbf{R}^d$, $\|h\| \leq t$ and let $F = \{Q_0, ..., Q_r\}$ be a collection of $r+1$, not necessarily different, dyadic cubes, r is the same as in (1), with*

$$(32) \quad \begin{aligned} &\operatorname{int}(Q_i) \cap \operatorname{int}(Q_j) = \emptyset, \qquad \text{when } Q_i \neq Q_j, \\ &x + ih \in Q_i \qquad \text{for } i = 0, \ldots, r, \end{aligned}$$

Then there exists a constant c_0 such that: If $l(Q_i) \geq c_0 t$, $i = 0, \ldots, r$, then the cubes Q_i have a common point y for which $|y - x - ih| \leq c_0 t$, $i = 0, \ldots, r$.

Proof. Let $l(Q_j) = \min_{0 \leq i \leq r} l(Q_i)$. For each Q_i there is a dyadic cube $Q_i^* \subset Q_i$ with $l(Q_i^*) = l(Q_i)$ and $x + ih \in Q_i^*$. The cubes Q_i^* have equal side length and satisfy the assumptions of Lemma 4. Thus, it is sufficient to prove Lemma 4 only for cubes which have equal side lengths. Let $P_0, \ldots, P_r$ be arbitrary dyadic cubes with $l(P_i) = 1$, $i = 0, \ldots, r$. We consider the following function ψ :

$$\psi(P_0, \ldots, P_r) := \inf \frac{1}{r} \sum_{i=1}^{r} | y_i - y_{i-1} |,$$

where inf is taken over all y_i, $y_i \in P_i$, $i = 0, \ldots, r$.

Evidently, there are only finite number of values of ψ which are less than 1. So, there exists a constant c_1 such that the values of ψ does not belong to the interval $(0, c_1)$. Let now $P_0, \ldots, P_r$ be a collection of dyadic cubes with side length 1 satisfying (32) with $\|h\| \leq c_1/2$. Then $\psi(P_0, \ldots, P_r) \leq c_1/2 < c_1$ and the above arguments imply $\psi(P_0, \ldots, P_r) = 0$. Therefore $P_0, \ldots, P_r$ have a common point y and obviously $| y - x - ih | \leq d^{1/2}$. By changing the scale, we obtain Lemma 4 with $c_0 = 2c_1^{-1} d^{1/2}$. ∎

We need also one technical result (Markov type inequality) concerning polynomials.

Lemma 5 *Let φ be a polynomial of total degree not greater than k, P and Q are any dyadic cubes with $P \subset Q$, $P \neq Q$. Denote $Q' := Q \backslash P$, $D^s\varphi(x) := \sum_{|\beta|=s} \mid D^\beta(x) \mid$, here β is multiindex.*

Then the following inequalities hold

$$\| D^s\varphi \|_{C(Q')} \leq \| D^s\varphi \|_{C(Q)} \leq cl(Q)^{-s} \| \varphi \|_{C(Q')}$$

with $c = c(k,d)$; $s = 0, \ldots, k$.

Proof. The first inequality is trivial because of $Q' \subset Q$. To prove the second estimate we use the following well known inequality (see e.g. [7]).

$$\| D^s\varphi \|_{C(\lambda.I)} \leq cl(I)^{-s} \| \varphi \|_{C(I)}, \qquad s = 0, \ldots, k. \tag{33}$$

with $c = c(\lambda, k, d)$, $\lambda > 0$ and I is an arbitrary cube. Now, since Q and P are dyadic cubes, $P \subset Q$ and $P \neq Q$, there is a dyadic cube I satisfying the condition $I \subset Q' \subset Q \subset 3.I$. By the last inclusions and (33), we have

$$\| D^s\varphi \|_{C(Q)} \leq \| D^s\varphi \|_{C(3.I)} \leq cl(I)^{-s} \| \varphi \|_{C(I)} \leq cl(Q)^{-s} \| \varphi \|_{C(Q')}$$

with $s = 0, \ldots, k$. ∎

Prof of Theorem 2. Let s be a spline of freedom n . Then

$$s = \sum_{I \in M} \lambda_I N_I, \tag{34}$$

with $\lambda_I \in \mathbf{R}$, $M \subset \Lambda$ and $\#M \leq n$. Since supp N_I is an union of r^d dyadic cubes and N_I is a polynomial on each of them, s can be represented in the following way

$$s = \sum_{Q \in \Gamma} T_Q(x)\chi_Q(x), \tag{35}$$

where T_Q is a polynomial of coordinate degree $r-1$, χ_Q is the characteristic function of Q, $\Gamma \subset D$ and $\#\Gamma \leq cn$. According to Lemma 3, we can extend Γ to $\tilde{\Gamma} \supset \Gamma$, $\#\tilde{\Gamma} \leq cn$, which satisfies (31). Then

$$s = \sum_{Q \in \tilde{\Gamma}} T_Q\chi_Q$$

with T_Q the same as in (35) if $Q \in \Gamma$, and $T_Q \equiv 0$ if $Q \in \tilde{\Gamma} \backslash \Gamma$. Denote $Q' := Q \backslash \cup \{P : P \in \mathbf{B}_Q\}$ for any $Q \in \tilde{\Gamma}$. Here $\mathbf{B}_Q$ is taken with respect to $\tilde{\Gamma}$. Obviously $Q' \cap P' = \emptyset$ whenever $Q, P \in \tilde{\Gamma}$, $Q \neq P$. In view of (31) exactly one of the following possibilities holds

$$\begin{aligned} &i) \quad Q' = Q \qquad \text{if } \mathbf{B}_Q = \emptyset \\ &ii) \quad Q' = Q \backslash P \quad \text{with } P \in \tilde{\Gamma}, \text{ if } \#\mathbf{B}_Q = 1 \\ &iii) \quad Q' = \emptyset \qquad \text{if } \#\mathbf{B}_Q = 2^d \end{aligned} \tag{36}$$

Clearly, s can be represented as follows

$$s = \sum_{Q \in \tilde{\Gamma}} L_Q(x)\chi_{Q'}(x), \qquad L_Q = \sum_{P \in \tilde{\Gamma},\ P \supset Q} T_P. \tag{37}$$

To prove Theorem 2 we need an estimate for

$$\omega_r(s,t) = \sup_{|h|\le t} \left(\int_{\Omega(rh)} |\Delta_h^r s(x)|^\sigma \, dx \right)^{1/\sigma},$$

where $\Omega(rh) = \{x : x, x+rh \in \Omega\}$.

Let t and h, $|h| \le t$ be fixed. Without loss of generality we can assume that $\Omega \in \tilde{\Gamma}$. For $x \in \Omega(rh)$ let Q_x be the unique cube $Q \in \tilde{\Gamma}$ for which $x \in Q'$. We divide each Q', $Q \in \tilde{\Gamma}$, into three sets Q_1, Q_2 and Q_3 as follows

$$(38)\qquad \begin{cases} Q_1 := \begin{cases} \emptyset & \text{if } l(Q) < c_0 t \\ \{x : x+ih \in Q', i = 0,\ldots,r\} & \text{if } l(Q) \ge c_0 t \end{cases} \\ Q_2 := \{x : x \in Q'\backslash Q_1,\ l(Q_{x+ih}) \ge c_0 t,\ i = 0,\ldots,r\}, \\ Q_3 := Q'\backslash(Q_1 \cup Q_2), \end{cases}$$

where the constant c_0 is from Lemma 4. We have

$$\int_{\Omega(rh)} |\Delta_h^r s(x)|^\sigma \, dx = \sum_{Q\in\tilde{\Gamma}} \left(\int_{Q_1} + \int_{Q_2} + \int_{Q_3} \right) = A_1 + A_2 + A_3$$

a) Estimate for A_1. For $x \in Q_1$, $Q \in \tilde{\Gamma}$, $Q_1 \ne \emptyset$, we have

$$(39)\qquad |\Delta_h^r s(x)|^\sigma \le c\left(\| D^r s \|_{C(Q')} t^r \right)^\sigma \le c \| s \|_C^\sigma \left(\frac{t}{l(Q)} \right)^{r\sigma}$$

In (39) we used that s is a polynomial of total degree not greater than $d(r-1)$ on Q' and also Lemma 5. By (39) and (38), we get

$$(40)\qquad \begin{aligned} A_1 &\le \sum_{Q\in\tilde{\Gamma},\ l(Q)\ge c_0 t} \|s\|_C^\sigma \left(\frac{t}{l(Q)} \right)^{r\sigma} mes\, Q_1 \\ &\le c \| s \|_C \sum_{Q\in\tilde{\Gamma},\ l(Q)\ge c_0 t} \left(\frac{t}{l(Q)} \right)^{r\sigma} l(Q)^d \end{aligned}$$

b) Estimate for A_2. Let $Q \in \tilde{\Gamma}$, $Q_2 \ne \emptyset$ and $x \in Q_2$. We denote briefly $Q_i := Q_{x+ih}$, $i = 0,1,\ldots,r$. From (36) and (38) it follows that $Q_i' = Q_i \backslash Q_i^*$, with $Q_i^* \in \tilde{\Gamma}$, $l(Q_i^*) \le l(Q_i)/2$ or $Q_i^* = \emptyset$. In what follows we consider the empty set as dyadic cube with side length 0 i. e. $l(A) = 0$, if $A = \emptyset$. Let us define for $i = 1,\ldots,r$

$$(41)\qquad P_i := \begin{cases} Q_i, & \text{in case } l(Q_i^*) < c_0 t \\ P \text{ is the unique dyadic cube, } P \subset Q_i',\ l(P) = l(Q_i^*), \\ \text{which contains } x+ih, & \text{in case } l(Q_i^*) \ge c_0 t \end{cases}$$

Obviously $P_i \subset Q_i$, $l(P_i) \ge c_0 t$ and $x + ih \in P_i$. By Lemma 5 and (37), we get for $i = 1,\ldots,r$

$$(42)\qquad \begin{aligned} \|D^{r-1} L_{Q_i}\|_{C(P_i)} &\le \| D^{r-1} L_{Q_i} \|_{C(Q_i)} \\ &\le c l(Q_i)^{-r+1} \| L_{Q_i} \|_{C(Q_i')} \le c l(Q_i)^{-r+1} \| s \|_C . \end{aligned}$$

We define

$$s^+ := \sum_{I \in M,\ l(I) \geq c_0 rt} \lambda_I N_I, \tag{43}$$

where λ_I and M are the same as in (34). Using (41) and (43) we obtain for $i = 0, \dots, r$

$$\begin{aligned} s^+(x+ih) &= s(x+ih), \\ s^+(z) &= L_{Q_i}(z), \qquad \text{whenever } z \in P_i. \end{aligned} \tag{44}$$

Since $l(P_i) \geq c_0 t$, $x + ih \in P_i$, by applying Lemma 4 we conclude that P_i have a common point y for which $|y - x - ih| \leq c_0 t$, $i = 0, \dots, r$. Let $T_y(z)$ be the Taylor polynomial of total degree $r-2$ for s^+ taken at the point y. The definition of T_y is correct because $s^+ \in C^{r-2}$. By (44) and (42), we obtain for $i = 0, \dots, r$

$$|s(x+ih) - T_y(x+ih)| = | s^+(x+ih) - T_y(x+ih) | \tag{45}$$

$$\leq c\|D^{r-1}s^+\|_{C(P_i)}|y - x - ih|^{r-1} \leq c\|D^{r-1}L_{Q_i}\|_{C(P_i)} t^{r-1} \leq c\|s\|_C \left(\frac{t}{l(Q_i)}\right)^{r-1}.$$

Now, we are ready to estimate A_2. By (45), (38) and the fact that $\Delta_h^r T_y(x) = 0$, we get

$$\begin{aligned} A_2 &= \sum_{Q \in \bar{\Gamma}} \int_{Q_2} | \Delta_h^r s(x) |^\sigma \, dx = \sum_{Q \in \bar{\Gamma}} \int_{Q_2} | \Delta_h^r (s(x) - T_y(x)) |^\sigma \, dx \\ &\leq \sum_{Q \in \bar{\Gamma}} \sum_{i=0}^{r} \int_{Q_2} |s(x+ih) - T_y(x+ih)|^\sigma dx \\ &\leq c\|s\|_C^\sigma \sum_{Q \in \bar{\Gamma}} \sum_{i=0}^{r} \int_{Q_2} \left(\frac{t}{l(Q_{x+ih})}\right)^{\sigma(r-1)} dx \\ &\leq c\|s\|_C^\sigma \sum_{Q \in \bar{\Gamma},\ l(Q) \geq c_0 t} \int_{x:\ dist\,(x, \partial Q) < rt} \left(\frac{t}{l(Q)}\right)^{\sigma(r-1)} dx \\ &\leq c\|s\|_C^\sigma \sum_{Q \in \bar{\Gamma},\ l(Q) \geq c_0 t} t\, l(Q)^{d-1} \left(\frac{t}{l(Q)}\right)^{\sigma(r-1)} \end{aligned} \tag{46}$$

The last inequality in (46) holds because of $\text{mes}\{x :\ \text{dist}(x, \partial Q) < rt\} \leq ctl(Q)^{d-1}$.

c) Estimate of A_3

$$A_3 = \sum_{Q \in \bar{\Gamma}} \int_{Q_3} | \Delta_h^r s(x) |^\sigma \, dx \leq c\|s\|_C^\sigma \sum_{Q \in \bar{\Gamma}} \text{mes}\, Q_3$$

In view of (38), if $x \in Q_3$ then there exists i, $0 \leq i \leq r$, for which $l(Q_{x+ih}) < c_0 t$. Hence

$$\sum_{Q \in \bar{\Gamma}} \text{mes}\, Q_3 \leq c \sum_{Q \in \bar{\Gamma}\ l(Q) < c_0 t} l(Q)^d.$$

Therefore

$$A_3 \leq c\|s\|_C^\sigma \sum_{Q \in \bar{\Gamma}\ l(Q) < c_0 t} l(Q)^d. \tag{47}$$

Putting together the estimates (40), (46) and (47) and using the fact that $\alpha < r$, $\sigma = d/\alpha$, we get

$$
\begin{aligned}
| s |_{B^\alpha_\sigma}^\sigma &= \int_0^1 \leq \int_0^\infty \left(t^{-\alpha}\omega_r(s,t)_\sigma\right)^\sigma \frac{dt}{t} = \int_0^\infty t^{-d-1}\omega_r(s,t)_\sigma^\sigma \, dt \\
&\leq c\|s\|_C^\sigma \int_0^\infty \left(\sum_{Q\in\tilde{\Gamma},\ l(Q)<c_0 t} t^{-d-1} l(Q)^d \right. \\
&\quad \left. + \sum_{Q\in\tilde{\Gamma},\ l(Q)\geq c_0 t} t^{-d-1} \left(l(Q)^d \left(\frac{t}{l(Q)}\right)^{r\sigma} + t l(Q)^{d-1} \left(\frac{t}{l(Q)}\right)^{\sigma(r-1)} \right) \right) dt \\
&\leq c\|s\|_C^\sigma \sum_{Q\in\tilde{\Gamma}} \left(\int_{l(Q)/c_0}^\infty \frac{1}{t}\left(\frac{t}{l(Q)}\right)^{-d} dt \right. \\
&\quad \left. + \int_0^{l(Q)/c_0} \left(\frac{1}{t}\left(\frac{t}{l(Q)}\right)^{r\sigma-d} + \frac{1}{t}\left(\frac{t}{l(Q)}\right)^{r\sigma-d+1} \right) dt \right).
\end{aligned}
$$

Now, one can see that under the assumptions of Theorem 2 the above integrals are convergent and can be estimated by a constant depending only on r and d. This observation and the fact that $\#\tilde{\Gamma} \leq cn$ imply

$$|s|_{B^\alpha_\sigma}^\sigma \leq cn\|s\|_C^\sigma,$$

which proves Theorem 2. ■

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