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Necessary Conditions for Fourier–Hankel Multipliers.

H.P. Heinig,
Dept. of Math. and Stats.
McMaster University,
Hamilton, L8S 4K1
Canada

R. Johnson
Department of Math.,
University of Maryland,
College Park, M.D.
20742 U.S.A.

Abstract

Necessary conditions are given for the Fourier–Hankel multipliers in an index range which extends those given by G. Gasper and W. Trebels in [5] [6]. The proofs are based on weighted norm estimates for the Hankel transform.

1. **Introduction.** Let $\alpha \geq -1/2$ and $L^p_\alpha(0, \infty)$, $1 \leq p \leq \infty$, be the space of Lebesgue measurable functions f, such that

$$\|f\|_{p,\alpha} = \left\{ \int_0^\infty |f(t)|^p t^{2\alpha+1}\, dt \right\}^{1/p} < \infty,$$

and $\mathscr{H}_\alpha$ the modified Hankel transform ([9]) of order α defined formally by

$$\mathscr{H}_\alpha(f)(\rho) = \rho^{-\alpha} \int_0^\infty t^{\alpha+1} J_\alpha(\rho t)\, f(t) dt,$$

where J_α is the Bessel function of the first kind. Note that if the Fourier transform of f is defined by

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\xi x} f(x)dx, \qquad \xi \in \mathbb{R}^n$$

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and f is radial, that is, (with an abuse of notation) $f(t) = f(x)$, $|x| = t$, $x \in \mathbb{R}^n$, then $\hat{f}$ is radial, $\hat{f}(\xi) = \hat{f}(\rho)$, $\rho = |\xi|$ and $\hat{f}(\rho) = \mathscr{H}_{(n-2)/2}(f)(\rho)$.

In [5] and [6, Thm. 1], G. Gasper and W. Trebels derived an appropriate modified Hausdorff–Young inequality for the (fractional) derivative of $\mathscr{H}_\alpha(f)$ and utilized it to prove necessary conditions for Fourier–Hankel multipliers. The basis for the Hausdorff–Young inequality are the endpoint estimates

$$\sup_{\rho>0} |\rho^{\alpha+1/2} \mathscr{H}_\alpha(f)^{(\gamma)}(\rho)| \leq C \int_0^\infty |t^{\alpha+\gamma+1/2} f(t)|\,dt \tag{1.1}$$

and

$$\int_0^\infty |\rho^{\alpha+1/2} \mathscr{H}_\alpha(f)^{(\gamma)}(\rho)|^2 d\rho \leq C \int_0^\infty |t^{\alpha+\gamma+1/2} f(t)|^2 dt\,. \tag{1.2}$$

The object of this note is to obtain from (1.1) and (1.2) a weighted inequality for the operator $\mathscr{H}_\alpha(f)^{(\gamma)}$ which extends the corresponding Hausdorff–Young inequality and then just as in [5] and [6] apply it to give necessary conditions for the Fourier–Hankel multipliers operator in an extended index range.

Before giving the results we require some definition and notation. As in [6] we define $\hat{S_0}(k)$, $k \in \mathbb{N}$ to be the space of rapidly decreasing continuous functions on $[0, \infty)$ which are $C^\infty(\mathbb{R}^+\backslash\{0\})$ and have the property that $\mathscr{H}_\alpha(f)$ has compact support in $[0, \infty)$ for $\alpha \in (-1/2, k]$. Here and in the sequel $\mathbb{R}^+ = [0, \infty)$.

We say a (quasi) linear operator T, defined on suitable function spaces (e.g. simple or $\hat{S_0}(k)$) is bounded from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$, and we write $T \in [L^p(\mathbb{R}^+), L^q(\mathbb{R}^+)]$, if there is a constant $C > 0$ such that $\|Tf\|_q \leq C\|f\|_p$. Constants throughout are denoted by C, although they may be different at different occurrences.

If g is defined on $\mathbb{R}^+$ then the equimeasurable decreasing rearrangement of $|g|$ is defined by $g^*(t) = \inf\{\lambda > 0: m(\{x \in \mathbb{R}^+: |g(x)| > \lambda\}) \leq t\}$ ([3]). Of course if g is decreasing on $\mathbb{R}^+$ then $g = g^*$. Finally the conjugate index of $p > 0$ is defined by

$p' = p/(p-1)$, with $p' = \infty$ if $p = 1$.

2. **Results.** We require the following known result:

Theorem 2.1. Let T be a (quasi) linear operator defined on the space of rapidly decreasing functions, such that $T \in [L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+)] \cap [L^2(\mathbb{R}^+), L^2(\mathbb{R}^+)]$ holds. If u and v are non–negative measurable functions satisfying

$$\sup_{s>0}\left[\int_0^{1/s} u^*(t)dt\right]^{1/q}\left[\int_0^s (1/v)^*(t)^{p'-1}dt\right]^{1/p'} < \infty, \tag{2.1}$$

where $1 \le p \le q < \infty$, then there exists a constant $C > 0$, such that

$$\left\{\int_0^\infty |(Tf)(x)|^q u(x)dx\right\}^{1/q} \le C\left\{\int_0^\infty |f(x)|^p v(x)dx\right\}^{1/p}. \tag{2.2}$$

Moreover, (2.2) extends to all f for which the right side of (2.2) is finite.

This result was proved for simple functions in [7, Cor. 2.5, and Prop. 2.6] with n = 1 provided p and q are not both equal to 2 and for general p and q in [2, Thm. 1.1] and the remark following the proof. The proof however carries over provided f permits the decomposition $f = f_1 + f_2$, where $f_1 \in L^1$ and f_2 satisfies $\int_0^\infty f_2^*(t)\, t^{-1/2}dt < \infty$. (If this is the case, then the Calderón estimate ([3])

$$(Tf)^*(x) \le C\left[\int_0^{1/x} f^*(t)dt + x^{-1/2}\int_{1/x}^\infty t^{-1/2}f^*(t)dt\right]$$

holds). But clearly rapidly decreasing functions allow such a decompositon and hence the result follows.

Although we shall not require it in the sequel, we note here that (2.2) holds also in the index range $0 < q < p < \infty$, $p > 1$ provided (2.1) is replaced by the two conditions

$$\left\{\int_0^\infty \left[\left[\int_0^{1/x} u^*(t)dt\right]^{1/q}\left[\int_0^x (1/v)^*(t)^{p'-1}dt\right]^{1/q'}\right]^r (1/v)^*(x)\ ^{p'-1}dx\right\}^{1/r} < \infty$$

$$\left\{\int_0^\infty \left[\left[\int_{1/x}^\infty t^{-q/2}u^*(t)dt\right]^{1/q}\left[\int_x^\infty t^{-p'/2}(1/v)^*(t)^{p'-1}dt\right]^{1/q'}\right]^r (1/v)^*(x)^{p'-1}x^{-p'/2}dx\right\}^{1/r} < \infty$$

where $1/r = 1/q - 1/p$.

The next result yields [6, Thm 1] if $u = v \equiv 1$ and $q = p'$.

Theorem 2.2. If $1 < p \leq q < \infty$ and u,v satisfy (2.1), then for all $f \in \hat{S_0}(k)$, $k = [2\alpha + 4]$ and $0 < \gamma < \alpha + 3/2$ the inequality

$$\text{(2.3)} \quad \left[\int_0^\infty |\rho^{\alpha+1/2}\ \mathscr{H}_\alpha(f)^{(\gamma)}(\rho)|^q\ u(\rho)d\rho\right]^{1/q} \leq C\left[\int_0^\infty |t^{\alpha+\gamma+1/2}f(t)|^p v(t)dt\right]^{1/p}$$

holds.

Since $\hat{S_0}(k)$ is dense in weighted spaces (cf. [6, Lemma 1]) (2.3) holds for all f for which the right side of (2.3) is finite.

Proof. As indicated above (1.1) and (1.2) was shown to hold in [5] and [6] whenever $f \in \hat{S_0}(k)$. Writing $g(t) = t^{\alpha+\gamma+1/2}f(t)$ and

$(Tg)(\rho) = \rho^{1/2+\alpha}\ \mathscr{H}_\alpha\left[|.|^{-\alpha-\gamma-1/2}g(.)\right]^{(\gamma)}(\rho) = \rho^{\alpha+1/2}\ \mathscr{H}_\alpha(f)^{(\gamma)}(\rho)$, then T satisfies the conditions of Theorem 2.1 and hence the result follows.

Of course (2.3) holds also for $0 < q < p < \infty$, $p > 1$, provided (2.1) is replaced by the two weight conditions mentioned above.

To derive necessary conditions for the Fourier–Hankel multiplier we only require the following special case of Theorem 2.2:

Corollary 2.3. Let $1 < p \le q < \infty$ and $f \in \hat{S}_0(k)$, $k = [2\alpha+4]$. If

$$(2.4) \quad \frac{2\alpha+2}{\alpha+\gamma+3/2} < p \le \frac{2\alpha+1}{\alpha+\gamma+1/2}; \quad 0 < (2\alpha+2)\left[\frac{1}{2}-\frac{1}{p}\right] + \gamma + 1/2 \le 1/q$$

holds, then there exists a constant $C > 0$, such that

$$(2.5) \qquad \left\{\int_0^\infty |\rho^{(2\alpha+2)/p'+\gamma}\, \mathscr{H}_\alpha(f)^{(\gamma)}(\rho)|^q \frac{d\rho}{\rho}\right\}^{1/q} \le C \left\{\int_0^\infty t^{2\alpha+1} |f(t)|^p dt\right\}^{1/p}$$

Moreover, since $\hat{S}_0(k)$ is dense in L^p_α ([6, Lemma 1]) the inequality extends to all $f \in L^p_\alpha$, $\alpha > -1/2$.

Proof. Let $v(t) = t^{-p(\alpha+\gamma+1/2)+2\alpha+1}$, where $p \le (2\alpha+1)/(\alpha+\gamma+1/2)$, then v is non-decreasing. Also $(1/v)^{*p'-1} = (1/v)^{p'-1} = t^{[-p(\alpha+\gamma+1/2)+2\alpha+1](-p'/p)} = t^{p'(\alpha+\gamma+1/2)-(2\alpha+1)p'/p}$ is integrable on (0,s), if $\frac{2\alpha+2}{\alpha+\gamma+3/2} < p$. Similarly, writing $u(\rho) = \rho^{-q(\alpha+1/2)+(2\alpha+2)q/p'+\gamma q-1}$, then u is non-increasing and integrable on (0, 1/s) provided the inequalities (2.4) on the right hold. With these weight functions condition (2.1) is satisfied and hence the result follows from Theorem 2.2.

Now let $\mathscr{F}f = \hat{f}$ be the Fourier transform of f and $\mathscr{F}^{-1}$ its inverse. A tempered distribution m is called a Fourier multiplier of type (p, p), $1 \le p < \infty$, we write $m \in M^p_p$, if $\|\mathscr{F}^{-1}[m\hat{f}]\|_p \le C\|f\|_p$. The smallest C for which this inequality holds is $\|m\|_{M^p_p}$, the multiplier norm. If f is radial, then $\hat{f}(\rho) = \mathscr{H}_{(n-2)/2}(f)(\rho)$ and the radial Fourier multiplier is defined in the same way. Writing $\hat{f}(\rho) = \mathscr{H}_\alpha(f)(\rho)$, $\alpha > -1/2$, the Fourier Hankel multiplier is defined similarly, only now L^p is replaced by L^p_α.

The necessary conditions for the Fourier–Hankel multiplier, alluded to above, are now the following:

Theorem 2.4 Let $m \in C^\infty(0, \infty) \cap M^p_p$ (radial) and $\alpha > -1/2$. If $1 < p \le q < \infty$ and (2.4) holds, then there is a C > 0, such that,

$$\|m\|_\infty + \sup_j \left[\int_{2^j}^{2^{j+1}} |\rho^\gamma m^{(\gamma)}(\rho)|^q \frac{d\rho}{\rho}\right]^{1/q} \leq C\|m\|_{M_p^p}.$$

Proof. The argument is completely analogous to that of the proof of [6, Thm 2] only now instead of an interpolation argument one applies Corollary 2.3. We sketch the argument for $\gamma \in \mathbb{N}$, for the convenience of the reader only.

Let $\chi \in C^\infty$, such that,

$$\chi(\rho) = \begin{cases} 1 & \text{if } 1 \leq \rho \leq 2 \\ 0 & \text{if } \rho \leq 1/2 \text{ or } \rho \geq 4. \end{cases}$$

Then $\mathscr{H}_\alpha^{-1}(\chi) = \mathscr{H}_\alpha(\chi)$ and $\mathscr{H}_\alpha^{-1}(\chi) \in L_\alpha^p$, $p \geq 1$. Writing $\chi_j(\rho) = \chi(2^j\rho)$, then $\mathscr{H}_\alpha^{-1}(\chi_j)(t) = 2^{j(2\alpha+2)}\ \mathscr{H}_\alpha^{-1}(\chi)(2^jt)$ and

$$\|\mathscr{H}_\alpha^{-1}(\chi_j)\|_{p,\alpha} = 2^{j(2\alpha+2)/p'}\|\mathscr{H}_\alpha^{-1}(\chi)\|_{p,\alpha} = C\ 2^{j(2\alpha+2)/p'}.$$

Hence by Corollary 2.3

$$\left[\int_{2j}^{2j+1} |\rho^\gamma m^{(\gamma)}(\rho)|^q \frac{d\rho}{\rho}\right]^{1/q} = \left[\int_{2^j}^{2^{j+1}} \rho^{-(2\alpha+2)q/p'} |\rho^{(2\alpha+2)/q+\gamma} m^{(\gamma)}(\rho)|^q \frac{d\rho}{\rho}\right]^{1/q}$$

$$\leq 2^{-j(2\alpha+2)/p'}\left[\int_0^\infty |\rho^{(2\alpha+2)/p'+\gamma}(m\ \chi_j)^{(\gamma)}(\rho)|^q \frac{d\rho}{\rho}\right]^{1/q}$$

$$\leq 2^{-j(2\alpha+2)/p'}\|\mathscr{H}_\alpha^{-1}(m\ \chi_j)\|_{p,\alpha} \leq C^{-j(2\alpha+2)/p'}\|m\|_{M_p^p}\|\mathscr{H}_\alpha^{-1}(\chi_j)\|_{p,\alpha}$$

$$= C\|m\|_{M_p^p},$$

and this implies the result.

If $q = p'$ and $p = (2\alpha + 1)/(\alpha + \gamma + 1/2)$ (i.e. $\gamma = (2\alpha + 1)(\frac{1}{p} - \frac{1}{2})$) in Theorem

2.4 we obtain [6, Thm 2]. In terms of Besov spaces we note that other generalization can be given ([10]).

Weighted estimates for the Hankel transform similar to those given in Theorem 2.2 (withy $\gamma = 0$) may be found in [4] and [8].

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