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On L_p–approximants and meromorphic continuation of functions

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Abstract

Sufficient and necessary conditions for the meromorphic continuation of functions, holomorphic inside and continuous on a compact in $\mathbb{C}$ are given. The results are proved in terms of best L_p-approximants, which belong to $\mathcal{R}^D_{n,m}$-the set of all rational functions of order (n,m) with poles in $D \subset \mathbb{C}$.

Introduction

Let E be a regular compact set in $\mathbb{C}$, which means: if D is unbounded component of $\overline{\mathbb{C}} \setminus E$, then D is regular with respect to Diriclet problem. Let $g_D(z,\infty)$ denote Green's function for D with pole at infinity. Set

$$E_R := \{z \in D : g_D(z,\infty) < \ln R\} \cup \{\overline{\mathbb{C}} \setminus D\} \text{ for } R > 1 \text{ and } E_R := \overline{\mathbb{C}} \setminus D, \text{ if } R = 1.$$

For each set B in $\mathbb{C}$ denote by B^0 the set of inner points of B and by $\overline{B}$ the closure of B. Denote by $\mathbb{N}$ the set of all positive integers.

We shall suppose everywhere in this paper, that there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of E, such that

(*) $K_n \subset E^0$ and K_n is regular for every $n \in \mathbb{N}$,
(**) if $g_n(z,\infty)$ denotes Green's function for unbounded component of $\overline{\mathbb{C}} \setminus K_n$ then $g_n(z,\infty) \to g_D(z,\infty)$ as $n \to \infty$ uniformly on compact subsets of D.

Walsh [1] has noticed that under some conditions on E there always exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subset of E, satisfying (*) and (**).

Let $\mathcal{A}(E)$ denote the space of all functions which are continuous in E and holomorphic in E^0. As usual $C(E)$ is the space of all continuous functions in E. Let $L_p(E), p > 0$, be the space of all measurable (with respect to Lebeque's measure μ in E) functions $g(z)$ which satisfy $\|g\|_{L_p(E)} := (\int_E \mid g(z) \mid^p d\mu)^{1/p} < \infty$. Let $f(z) \in C(E)$ and m be a fixed non negative integer. Denote by E_{R_m} the largest set E_R in which $f(z)$ admits a continuation as a function, meromorphic in E_{R_m} and has there no more than m poles (the poles are counted with regard to their multiplicities).

We shall investigate the connections between the meromorphic continuation of $f(z)$ and the sequence of best rational $L_p(E)$ approximants $R_{n,m}(z) \in \mathcal{R}_{n,m}^{\overline{\mathbb{C}}\setminus E}$ to $f(z)$

$$\|R_{n,m} - f\|_{L_p(E)} = \inf_{r \in \mathcal{R}_{n,m}^{\overline{\mathbb{C}}\setminus E}} \|f - r\|_{L_p(E)}.$$

In the sequel the set E always will satisfy conditions which leads after results of Walsh [1] the existence of $R_{n,m}(z)$.

The results of this paper are natural generalizations of results from [2], [3], [4] and [5].

Before we formulate our main results we define as it is done in [8] the m_α-convergence, $\alpha > 0$ and convergence in capacity: Let $e \subset \mathbb{C}$, $\alpha > 0$. Set $m_\alpha(e) := \inf \sum_\nu |U_\nu|^\alpha$, where the infimum is taken over all systems of disks U_ν, such that $\cup_\nu U_\nu \supseteq e$, $|U_\nu|$ denotes the radius of U_ν . Let cap(e) denote the outher logarithmic capacity of e (if e is a compact set, then cap(e) is equal to capacity of e). Let Ω be a domain in $\mathbb{C}$ and $\varphi : \Omega \to \overline{\mathbb{C}}$. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of meromorphic in Ω functions. The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ m_α-converges to φ on compact subsets of Ω if for every compact set $K \subset \Omega$ and for every $\varepsilon > 0$ $m_\alpha\{z \in K : |(\varphi_n - \varphi)(z)| \geq \varepsilon\} \to 0$ *as* $n \to \infty$. The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in capacity on compact subsets of Ω if for every compact set $K \subset \Omega$ and for every $\varepsilon > 0$ *cap* $\{z \in K : |(\varphi_n - \varphi)(z)| \geq \varepsilon\} \to 0$ *as* $n \to \infty$.

Main results

Theorem 1. *Let* $\overline{E^o} = E$, $f(z) \in \mathcal{A}(E)$, $\sigma > 1$, $R^*_{n,m}(z) \in \mathcal{R}_{n,m}^{\overline{\mathbb{C}}\setminus E}$ *and* m *be a fixed non negative integer. If* $\limsup_{n\in N} \|f - R^*_{n,m}\|_{L_p(E)}^{1/n} \leq 1/\sigma$ *then* $E_\sigma \subseteq E_{R_m}$.

Theorem 2. *Let* $\overline{E^o} = E$, $f(z) \in \mathcal{A}(E)$, $1 < R_m < \infty$ *and* m *be a fixed non negative integer. Then* $\{R_{n,m}(z)\}_{n\in\mathbb{N}}$ *does not converge in capacity on compact subsets of* $\overline{\mathbb{C}} \setminus \overline{E}_{R_m}$.

Theorem 1 is a generalization of well known theorem of Saff-Gonchar [2]. Theorem 2 is a generalization of one Vavilov's result [3], concerning Pade approximants.

As a consequence of Theorem 1 and Theorem 2 we can prove the following corollaries:

Corollary 1. *Let* $f(z), E$ *and* $R_{n,m}(z)$ *be as in Theorem 1 and let* $R_{n,m}(z) = \frac{a_n z^{n} + \ldots}{Q_{n,m}(z)}$, *where* $Q_{n,m}(z)$ *is normalized in the following way:*

$$(1) \qquad Q_{n,m}(z) = \prod_{k=1}^{l_n}(z - \beta_{n,k}) \prod_{k=l_n+1}^{m_n} (z/\beta_{n,k} - 1),\ m_n := \deg Q_{n,m} \quad \textit{and}$$

$$|\beta_{n,1}| \leq \ldots |\beta_{n,l_n}| < A \leq |\beta_{n,l_n+1}| \leq \ldots \leq |\beta_{n,m_n}|,\ A > 2\mathrm{diam}(E).$$

If for some $R > 1$ *we have* $\limsup_{n\in\mathbb{N}} |a_n|^{1/n} \leq \frac{1}{R cap(E)}$ *then* $R < R_m$.

Corollary 2. *Let* $f(z)$ *admit a continuation as a function, which has no more than m-meromorphic in* E_R, $R > 1$ *and let* E, $R_{n,m}(z)$ *be as in Corollary 1. Then*

$$\limsup_{n\in N} |a_n|^{1/n} \leq \frac{1}{R\mathrm{cap}(E)}.$$

Corollary 3. *Let* $f(z)$, E *and* $R_{n,m}(z)$ *be as in Corollary 1. Then the folowing conditions are equivalent*

(i) $f(z)$ *cannot be continued as m-meromorphic in* $\mathbb{C} \setminus D$,

(ii) $\limsup_{n\in\mathbb{N}} |a_n|^{1/n} = 1/cap(E)$.

Corollary 1, Corollary 2 and Corollary 3 are generalizations of some results in [4].

Corollary 4. *Let* $D = \overline{\mathbb{C}} \setminus E$, $f(z) \in \mathcal{A}(E)$, $m = n$ *and* U *be domain in* $\mathbb{C}$ *such that the following conditions are valid:*

(i) $E \subset U$,

(ii) $R_{n,m}(z)$ *are holomorphic in* U,

(iii) $\tau_n(K) = o(n)$, $n \to \infty$, *where* K *is any compact subset of* U *and* $\tau_n(K)$ *denotes the number of zeros of* $R_{n,m}(z)$ *on* K.

Then $f(z)$ *is holomorphic in* U *and* $R_{n,m}(z) \to f(z)$ *as* $n \to \infty$ *in the sense of Chebyshev on compact subsets of* U.

Corollary 5 is a generalization of Theorem 1[5].

Preliminary results

Theorem 4[2]. *Let* $\|\cdot\|$ *denote the sup-norm on* E, E *be a regular compact set in* $\mathbb{C}$ *and let* $f(z) \in C(E)$. *We set* $\rho_{n,m} := \inf_{r\in\mathcal{R}^{\overline{\mathbb{C}}}_{n,m}} \|f - r\|_E$. *If* m *is a fixed non negative integer and* $\limsup_{n\in\mathbb{N}} (\rho_{n,m})^{1/n} \leq 1/\sigma < 1$ *then* $E_\sigma \subseteq E_{R_m}$.

Theorem 5[1]. *Let* $f(z)$ *be holomorphic on* C, *where* C *is a compact Jordan domain in* $\mathbb{C}$. *Let* $\pi_n(z)$ *be a best* $L_p(E)$, $p > 0$ *polynomial approximant of order* n *to* $f(z)$. *Then* $\limsup_{n\in\mathbb{N}} \|f - \pi_n\|_E^{1/n} \leq 1/R$ *for every* R *provided* $1 < R < R_0$.

Theorem 6[5]. *Let* E *be a regular compact set in* $\mathbb{C}$ *with connected complement* $\overline{\mathbb{C}} \setminus E$ *and let* $f(z) \in \mathcal{A}(E)$. *Let* $R^{\#}_{n,n}(z) \in \mathcal{R}^{\overline{\mathbb{C}}}_{n,n}$ *denotes the best Chebyshev approximant to* $f(z)$ *on* E. *Assume that there is a domain* W, $E \subset W$ *such that* $R^{\#}_{n,n}(z)$ *is holomorphic in* W *for each* $n \in N$ *and* $\tau_n(K) = o(n)$ *as* $n \to \infty$ *for each compact subset* K *of* W. *Then* $f(z)$ *is a holomorphic in* W *and* $R^{\#}_{n,n}(z)$ *converges to* $f(z)$ *as* $n \to \infty$ *uniformly on compact subsets of* W.

Lemma 1[1]. *Let* $g(z)$ *be an analytic function in bounded domain* C *and let* $g(z) \in L_p(\overline{C})$, $p > 0$. *Then for every compact subset* $K \subset C$ $\|g\|_K \leq \frac{1}{(\pi^2 \rho_K)^{1/p}} \|g\|_{L_p(\overline{C})}$, *where* ρ_K *denotes the distance between* K *and* ∂C.

Proofs

Proof of Theorem 1. Let $1 < R < R_1 \leq \sigma$ and R, R_1 be fixed. Let $\Gamma_{n,\delta} := \{z : g_n(z,\infty) = \ln \frac{R_1 \exp(\delta)}{R}\}$, where $\varepsilon < 0$ and $1 < \frac{R_1 \exp(\varepsilon)}{R} < R_1/R$. From the conditions on E we get $g_n(z,\infty) \to g_D(z,\infty)$ as $n \to \infty$ uniformly in $\overline{E}_{\rho_-} \setminus E_{\rho_+}$, where $\rho_+ = \frac{R_1 \exp(\varepsilon)}{R}$ and $\rho_- = \frac{R_1 \exp(-\varepsilon)}{R}$.

Clearly for all sufficiently large $n \in \mathbb{N}$ $E \subset K_{n,\varepsilon/2}$, where $K_{n,\varepsilon/2} := \{z : g_n(z,\infty) < \ln \frac{R_1 \exp(\varepsilon/2)}{R}\}$, and by Lemma 1 and the assumptions of the theorem we get

$$(2) \qquad \|f - R^*_{n,m}\|_{K_i} \leq \left(\frac{1}{\pi^2 \rho_{K_i}}\right)^{1/p} \|f - R^*_{n,m}\|_{L_p(E)}$$

for all sufficiently large $i \in \mathbb{N}$.

Fix an integer $i \in \mathbb{N}$ such that (2) is valid. Since $1/R_1 \geq 1/\sigma$, from (2) and from the assumptions of the theorem we get

$$(3) \qquad \limsup_{n \in \mathbb{N}} \|f - R^*_{n,m}\|_{K_i}^{1/n} \leq 1/R_1.$$

Let $\delta > 0$. Following the considerations in [2] one can see that there exsists a set Ω_δ such that $\Omega_\delta \subset K_{i,\varepsilon/2}$, $m_1(\Omega_\delta) < \delta$, all poles of $R^*_{n,m}, n \in \mathbb{N}$ belong to Ω_δ and

$$\limsup_{n \in \mathbb{N}} \|f - R^*_{n,m}\|_{K_{i,\varepsilon/2} \setminus \Omega_\delta}^{1/n} \leq 1/R.$$

From this inequality we get (remember that $E \subset K_{i,\varepsilon/2}$)

$$\limsup_{n \in \mathbb{N}} \|f - R^*_{n,m}\|_{E \setminus \Omega_\delta}^{1/n} \leq 1/R.$$

But $R^*_{n,m}(z) \in \mathcal{R}_{n,m}^{\overline{\mathbb{C}} \setminus E}$ and from the last inequality we get $\limsup_{n \in \mathbb{N}} \|f - R^*_{n,m}\|_E^{1/n} \leq 1/R$. Consequently, if $R \to \sigma$, then

$$(4) \qquad \limsup_{n \in \mathbb{N}} \|f - R^*_{n,m}\|_E^{1/n} \leq 1/\sigma.$$

Now from (4) and from Theorem 4 we get the statement of the theorem.

Proof of Theorem 2. The proof of this theorem is based on the following two lemmas.

Lemma 2. *Let $\overline{E^0} = E, f(z) \in \mathcal{A}(E)$, m be a fixed non negative integer and $R_m > 1$. If $1 < R < R_m$ then $\limsup_{n \in \mathbb{N}} \|f - R_{n,m}\|_{L_p(E)}^{1/n} \leq 1/R$.*

Proof of Lemma 2. Since $R_m > 1$, then $f(z)$ admits a continuation as a function which is no more than m-meromorphic in E_{R_m}. Denote by $\beta_1, ..., \beta_{m_1}$, $m_1 \leq m$ the poles of $f(z)$ in E_{R_m}. One can prove (using the conditions on E) as it has been done in Theorem 5 that $\limsup_{n \in \mathbb{N}} \|F - \pi_n\|_{L_p(E)}^{1/n} \leq 1/R$, where $F(z) := f(z) \prod_{i=1}^{m_1} (z - \beta_i)$ and $\pi_n(z) \in \mathcal{R}_{n,0}$ is the best $L_p(E)$ approximant of $F(z)$. Consequently $\limsup_{n \in \mathbb{N}} \|f - R_{n,m}\|_{L_p(E)}^{1/n} \leq 1/R$.

Lemma 3. *Let E be a regular compact in $\mathbb{C}$, $f(z) \in C(E)$, m be a fixed non negative integer and $1 < R_m < \infty$. Then each sequence $\{r^*_{n,m}\}_{n\in\mathbb{N}}$, $r^*_{n,m}(z) \in \mathcal{R}^{\overline{\mathbb{C}}}_{n,m}$ and $\limsup_{n\in\mathbb{N}} \|f - r^*_{n,m}\|_E^{1/n} \le 1/R_m$ does not converge in capacity on compact subsets of $\overline{\mathbb{C}} \setminus \overline{E}_{R_m}$.*

The proof of Lemma 3 is similar to the proof of one Vavilov's theorem [3]. We give only scheme of the proof. Suppose that there is a closed disc γ_1 such that $\gamma \cap \overline{E}_{R_n} = \emptyset$ and $\{r^*_{n,,m}\}_{n\in\mathbb{N}}$ converges in capacity in γ_1. One can find, that for some closed disc $\gamma \subset \gamma_1$, there is valid

(5) $\qquad |J_n(z)| \le C_1, n$ is sufficiently large, C_1 is independent on n,

where
$$J_n(z) := \prod_\nu{}_\rho (z - z_{n,\nu}) \prod_\nu{}_\rho (z - z_{n+1,\nu}) \{r^*_{n+1,m}(z) - r^*_{n,m}(z)\},$$
$$\rho := \sup_{z\in\gamma_1} \{\exp(g_D(z,\infty))\},$$
$\prod_\nu{}_\rho$ is product over these $\nu \in N$, such that $z_{n,\nu} \in \overline{E}_\rho$,

$z_{n,\nu}$ are the poles of $r^*_{n,m}(z)$ in $\mathbb{C}$.

For each $R < R_m$ one can easily find using [2] that

(6) $\quad |J_n(z)| \le C_2, z \in E_R$, n-sufficiently large, C_2 depends only on m and R.

Denote by $\zeta_{n,\nu}$ the poles of $J_n(z)$ in $\mathbb{C}$ and by μ_n the order of the pole of $J_n(z)$ at infinity. It is clear that $\mu_n \to \infty$ as $n \to \infty$ in the case when $f(z)$ does not belong to $\mathcal{R}^{\overline{\mathbb{C}}}_{n,m}$. Define for each $r < R_m$ $G_r := \overline{\mathbb{C}} \setminus \{\overline{E}_r \cup \gamma\}$ and let $g_r(z, \zeta_{n,\nu})$ be its Green's functions with pole at $\zeta_{n,\nu}$. Set

$$H_{n,r}(z) := \frac{1}{\mu_n} \ln |J_n(z)| - \ln \frac{\exp\{g_D(z,\infty)\}}{R_m} - \frac{1}{\mu_n} \sum_\nu g_r(z, \zeta_{n,\nu}).$$

If $R_0 \in (R_m, \inf_{z\in\gamma}\{\exp\{g_D(z,\infty)\}\})$ one can find, using (5), (6) and the fact that $H_{n,r}(z)$ is subharmonic in G_r, that there exists $r_0 < R_m$ such that $H_{n,r_0}(z) \le -\lambda$, $z \in \delta E_{R_0}$, $\lambda > 0$, n is sufficiently large. Using this inequality and the consideration in [2] one can find $R_1 \in (R_m, R_0)$ and $q < 1$ such that $|J_n(z)| < C_3$ for $z \in \overline{E}_{R_1}$ and all sufficiently large $n \in \mathbb{N}$, where C_3 is a constant that is not depending on $n \in \mathbb{N}$.

Following [2] one can prove that $\{r^*_{n,m}\}_{n\in\mathbb{N}}$ m_1-converges on compact subsets of E_{R_1}, that means that $f(z)$ admits continuation as m-meromorphic in E_{R_1}. But this contradicts the definition of R_m.

We now continue the proof of Theorem 2. From Lemma 2 and Theorem 1 we get

$$\limsup_{n\in\mathbb{N}} \|f - R_{n,m}\|_E^{1/n} = R_m$$

that, in view of Lemma 3, implies the statement of Theorem 2.

Proof of Corollary 1. Let $T_n(z)$ be the Chebyshev polynomial for E with zeros on E ($\deg T_n = n$). One can easily find that

$$\|T_n\|^{1/n}_{L_p(E)} \to \operatorname{cap}(E) \quad \text{as} \quad n \to \infty.$$

Now using Theorem 1 and following the proof of Theorem 1[4], where $\|\cdot\|_E$ is replaced by $\|\cdot\|_{L_p(E)}$ we get the statement of Corollary 1.

Proof of Corollary 2. From Lemma 2 and from the proof of Theorem 1 we find

$$\limsup_{n\in\mathbb{N}} \|f - R_{n,m}\|_E^{1/n} \leq 1/R.$$

Using this inequality and following the proof of Theorem 2[4] we get the statement of Corollary 2.

Proof of Corollary 3. One can easily find, that

$$\|R_{n,m}\|_{L_p(E)} \leq \|f\|_{L_p(E)} + \|f - R_{n,m}\|_{L_p(E)} \leq C(E,f),$$

where $C(E,f)$ is a constant, which depends only on E and $f(z)$. Let U be a closed disc such that $U \subset E^0$. From Lemma 1 we get $\|R_{n,m}\|_U \leq C_4/\rho_U^{2/p}$, where C_4 does not depend on $n \in \mathbb{N}$, $\rho_U^{2/p} > 0$. Following the arguments in [4] and using (1) we get $|a_n| \leq C_5 r^n \exp(n\theta)/\rho_U^{2/p}$, where $\theta > 0$ is arbitrary chosen, C_5 is independing on $n \in \mathbb{N}$ and r denotes the radius of U.

Since $\text{cap}(U)/\text{cap}(E) \to 0$ as $r \to 0$ and $\rho_U \to \rho(z_U, \delta E)$ as $r \to 0$, where z_U denotes the center of U and $\rho(z_U, \delta E)$ denotes the distance between z_U and δE, (recall that θ is arbitrary) we get $\limsup_{n\in\mathbb{N}} |a_n|^{1/n} \leq \text{cap}(E)$. From this, Corollary 1 and Corollary 2 we get the statement of Corollary 3.

Proof of Corollary 4. Let U be a closed disc such that $U \subset E^0$. Since $D = \overline{\mathbb{C}}\backslash E$, Mergelian's theorem implies $\rho_{n,0} \to 0$ as $n \to \infty$. From this and Lemma 1 we get

$$\|R_{n,n} - f\|_U \leq C(p,U)\|R_{n,n} - f\|_{L_p(E)} \leq C_6 \rho_{n,0},$$

where $C(p,U)$ and C_6 do not depend on $n \in \mathbb{N}$. Consequently $\|R_{n,n} - f\|_U \to 0$ as $n \to \infty$. Now apllying Theorem 6 to $f(z)$, $R^{\#}_{n,n}(z) := R_{n,n}(z)$ and $W := U$ we get the statement of Corollary 4.

We can make some further remarks concerning Theorem 1 and Theorem 2. From [2] and the proof of Theorem 1 one can see, that conditions of $f(z)$ and $R^*_{n,m}(z)$ in Theorem 1 are not necessary. It's sufficient to assume that $f(z)$ is meromorphic on E and $R^*_{n,m}(z) \in \mathcal{R}^{\overline{\mathbb{C}}}_{n,m}$. Theorem 2 is a valid not only for $\{R_{n,m}(z)\}_{n\in\mathbb{N}}$, but also for every sequence $\{\hat{R}_{n,m}(z)\}_{n\in\mathbb{N}}$, of functions, such that

$$\hat{R}_{n,m}(z) \in \mathcal{R}^{\overline{\mathbb{C}}\backslash E}_{n,m} \quad \text{and} \quad \limsup_{n\in\mathbb{N}} \|f - \hat{R}_{n,m}\|_{L_p(E)}^{1/n} \leq 1/R_m.$$

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