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HERMITE–FEJER INTERPOLATION FOR WEIGHTS ON THE WHOLE REAL LINE

by

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Abstract

We discuss recent results on weighted L_1 and L_∞ convergence of Hermite–Fejér interpolation at the zeros of orthogonal polynomials associated with weights on the whole real line. We also prove the following new analogue of a 1985 result of Nevai and Vertesi: Let w be a bounded weight on $(-\infty,\infty)$, and assume that the coefficient $\gamma_{n-1}(w)/\gamma_n(w)$ in the recurrence relation for its orthogonal polynomials satisfies for some $\epsilon > 0$,

$$\gamma_{n-1}(w)/\gamma_n(w) = O(n^{1/2-\epsilon}),\ n \to \infty.$$

Then if $H_n(f,x)$ denotes the nth Hermite–Fejér interpolation polynomial to f at the zeros of the orthogonal polynomial of degree n for w, we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |H_n(P,x) - P(x)|\ w(x)\,dx = 0,$$

for each polynomial P. Stronger results are given under more regularity conditions on w.

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§1. INTRODUCTION

In this paper, we shall discuss the convergence of *Hermite–Fejér interpolation.* The interpolation takes place at the zeros $\{x_{kn}\}_{k=1}^{n} = \{x_{kn}(w)\}_{k=1}^{n}$ of the nth orthonormal polynomial for a *weight* w,

$$p_n(x) := p_n(w,x) := \gamma_n x^n + ..., \gamma_n = \gamma_n(w) > 0, \tag{1.1}$$

which is defined by the condition

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, w(x)\, dx = \delta_{mn}. \tag{1.2}$$

The zeros are ordered so that

$$-\infty < x_{nn} < x_{n-1,n} < x_{n-2,n} < ... < x_{1n} < \infty. \tag{1.3}$$

Here, by a *weight* w, we mean a function w: $\mathbb{R} \to [0,\infty)$ satisfying

$$0 < \int_{-\infty}^{\infty} |t|^{\ell}\, w(t)\, dt < \infty, \ell = 0, 1, 2, ... \tag{1.4}$$

Recall that the nth *Hermite–Fejér interpolation polynomial* to a function f at the zeros $\{x_{jn}\}_{j=1}^{n}$ of $p_n(w,x)$, is a polynomial $H_n(f,x)$, of degree at most 2n–1, defined by

$$\left.\begin{array}{l} H_n(f,x_{jn}) = f(x_{jn}) \\ H_n'(f,x_{jn}) = 0, \end{array}\right\} , 1 \leq j \leq n. \tag{1.5}$$

As motivation for our study, we briefly review some of the history of the Hermite–Fejér operator, apparently first defined by Fejér in 1916. In 1930 Fejér showed that if w(x) is the Chebyshev weight $1/\sqrt{1 - x^2}$ on (–1,1), so that interpolation takes place at the zeros of the Chebyshev polynomial $T_n(x)$, then

$$\lim_{n\to\infty} \| f(\cdot) - H_n(f,\cdot) \|_{L_\infty[-1,1]} = 0, \tag{1.6}$$

for each function f continuous on [–1,1]. This new proof of the Weiestrass approximation theorem was especially surprising, since ordinary Lagrange or Hermite interpolation does not in general converge.

Quite naturally, Fejér's result inspired much further research. Generalizations have included:

(i) Interpolation at zeros of $p_n(w,x)$ for weights w on (–1,1) other than the Chebyshev weight;

(ii) Convergence in L_p or weighted L_p norms;

(iii) Higher–order Hermite–Fejér interpolation, involving higher–order derivatives of the interpolating polynomial.

We cannot hope to review the contributions of the many authors here, other than to recall such names as Turan, Szegö, Shohat, Egervary, Freud, Grunwald, Maté, Nevai, Szabados, Totik, Knoop, Vertesi, Varma, Prasad, Schonhage, Totik, Xu. We shall only recall three results for which we present analogues for weights on ℝ.

The following result of Nevai and Vertesi, proved in 1985, is the only known result for general weights on (–1,1):

Theorem 1.1 [9,p.46] Let w be a weight on (–1,1). Then for every polynomial P,

$$\lim_{n\to\infty} \int_{-1}^{1} |H_n(P,x) - P(x)|\ w(x)\,dx = 0. \tag{1.7}$$

This theorem is the closest relative in Hermite–Fejér interpolation to the classical (1937) Erdos–Turan theorem on L_2 convergence of Lagrange interpolation. Thus if L_2 is the natural setting for Lagrange interpolation at the zeros of orthogonal polynomials, so L_1 is the natural setting for Hermite–Fejér interpolation.

If however, we want convergence on functions other than polynomials, we have to assume far more about the weight. The following 1985 result of Nevai and Vertesi is fairly typical:

Theorem 1.2 [9,p.47] Let $w(x) := g(x)\,(1-x)^a\,(1+x)^b$, where $a > -3/4$, $b > -3/4$, and g is a positive function with $g \in \mathrm{Lip}\ 1$. Then for each bounded and Riemann integrable function f on [–1,1],

$$\lim_{n\to\infty} \int_{-1}^{1} |H_n(f,x) - f(x)|\ w(x)\,dx = 0. \tag{1.8}$$

Note the restrictions on a and b, which are known to be necessary. Surprisingly in sup–norm convergence, we can dispense with these, if we require uniform convergence only in proper subintervals of $(-1,1)$:

Theorem 1.3 [10,Theorem 2.1] Let $w(x) := g(x)\,(1-x)^a\,(1+x)^b$, where $a > -1$, $b > -1$, and g is a positive function with $g' \in \text{Lip } 1$. Then for every continuous function f on $[-1,1]$, and for all $r \in (0,1)$,

$$\lim_{n\to\infty} \|f(\cdot) - H_n(f,\cdot)\|_{L_\infty[-r,r]} = 0. \tag{1.9}$$

The problem of pointwise convergence at ± 1 is quite enigmatic [10]. In Section 2 of this paper, we shall state analogues of Theorems 1.1 to 1.3 for weights on $\mathbb{R}$, and in Section 3, we shall prove one of them.

§2. HERMITE–FEJER INTERPOLATION FOR WEIGHTS ON $\mathbb{R}$

In recent years, there has been a real push to develop the theory of orthogonal polynomials, weighted approximation, and potential theory, for weights on $\mathbb{R}$ [6,7]. One feature of these weights that complicates matters, from the point of view of Hermite–Fejér interpolation, is the unboundedness of the recurrence coefficients $\gamma_{n-1}(w)/\gamma_n(w)$. The necessity to bound this, and to apply the Cauchy–Schwartz inequality to estimate quantities about which little is known, explains the restrictions in the following new result. It is the closest analogue we can derive to the Nevai–Vertesi result on polynomials (Theorem 1.1):

Theorem 2.1 Let w be a weight on $\mathbb{R}$, and assume either that

(I) There exist $\epsilon > 0$ and $\beta \geq 0$ such that

$$\sup_{t\in\mathbb{R}} w(t)/(1+|t|)^\beta < \infty; \tag{2.1}$$

and

$$\gamma_{n-1}(w)/\gamma_n(w) = O(n^{1/2-\epsilon}),\ n \to \infty, \tag{2.2}$$

or

(ii) For $\ell = 0, 1, 2, \ldots,$

$$\sup_{t\in\mathbb{R}} w(t)\, |t|^{\ell} < \infty; \tag{2.3}$$

and

$$\gamma_{n-1}(w)/\gamma_n(w) = o(n^{1/2}),\ n \to \infty. \tag{2.4}$$

Then, for each polynomial P,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |H_n(P,x) - P(x)|\ w(x)\, dx = 0. \tag{2.5}$$

We shall prove this result in Section 3. Its distinguishing feature is the relatively few regularity assumptions on w. We may replace $\gamma_{n-1}(w)/\gamma_n(w)$ in (2.2) or (2.4) by $x_{1n} = x_{1n}(w)$, and we note that $\exp(-|x|^{\alpha})$, $\alpha > 2$, satisfies these two conditions. However under additional regularity conditions on w, that are valid for $w(x) = \exp(-|x|^{\alpha})$, $\alpha > 3/2$, we may say much more. To avoid technical complications, we shall quote only a special case of the result in [5], which may be viewed as a real line analogue of Theorem 1.2.

Note that $\exp_k$ denotes exp(exp(exp ...)), the kth iterated exponential.

__Theorem 2.2__ Let $f:\mathbb{R} \to \mathbb{R}$ be bounded and Riemann integrable in each finite interval.

(I) If $w(x) := \exp(-2|x|^{\beta})$, $\beta > 3/2$, let for some $\epsilon > 0$,

$$\sup_{x\in\mathbb{R}} |f(x)|\ w(x)\, |x|^{2\beta-1+\epsilon} < \infty. \tag{2.6}$$

(II) If $w(x) := \exp(-2\exp_k(|x|^{\beta}))$, $\beta \geq 1$, $k \geq 1$, let for some $\epsilon > 0$,

$$\sup_{x\in\mathbb{R}} |f(x)|\ w(x)\, (\exp_k(|x|^{\beta}))^{2+\epsilon} < \infty. \tag{2.7}$$

Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |H_n(f,x) - f(x)|\ w(x)\, dx = 0. \tag{2.8}$$

Essentially for both cases, the growth restriction on f has the form

$$\sup_{x\in\mathbb{R}} |f(x)|\; w(x)\; |\log w(x)|^{A} < \infty,$$

for a suitable A. The damping factor $|\log w|^{A}$ decays relatively slowly to w, and this extends to general weights $w := \exp(-2Q)$, under mild and explicit conditions on Q' and Q". Moreover [5], similar results hold for the Hermite–interpolation operator, and related operators.

What about sup–norm convergence? It turns out that here we have to assume a bound on the orthogonal polynomials for w. To state this bound, we assume $w := \exp(-2Q)$, where Q is even and convex, and define the nth *Mhaskar–Rahmanov–Saff* number a_n [6], the root of

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\, dt. \tag{2.9}$$

If $Q^{[-1]}$ denotes the inverse function of Q, then a_n grows roughly like $Q^{[-1]}(n)$. The bound we assume is: There exist C, $\sigma > 0$, such that

$$\sup_{x\in\mathbb{R}} |p_n(w,x)|\; w(x)^{1/2}\, [1 + |Q'(x)|]^{-\sigma} \le C / \sqrt{a_n},\; n \ge 1. \tag{2.10}$$

Such a bound is known to be available for $w(x) := \exp(-|x|^{\beta})$, $\beta > 1$, or $w(x) := \exp(-\exp_k(|x|^{\alpha}))$, $k \ge 1$, $\beta \ge 1$ [3,11]. In fact for the former weight with β a positive even integer, results of Bonan and Clark [1] ensure that we may choose $\sigma \ge \beta/(6(\beta-1))$ ($\le 1/3$). For simplicity, we state the L_∞ result from [4] only for special weights:

<u>Theorem 2.2</u> Let $w := \exp(-2Q)$ satisfy (2.10) for some $\sigma > 0$. Let f: $\mathbb{R} \to \mathbb{R}$ be continuous and assume that for some $\epsilon > 0$,

$$\sup_{x\in\mathbb{R}} |f(x)|\; w(x)\, [1 + |Q'(x)|]^{2\sigma+2+\epsilon}\, [1 + |x|]^{2} < \infty. \tag{2.11}$$

Let

$\kappa > 2\sigma + 1$, if $w(x) = \exp(-2|x|^{\beta})$, some $\beta > 1$;

$\kappa > \max\{2\sigma+1, 4\sigma\}$ if $w(x) = \exp(-2\exp_k(|x|^{\beta}))$, $\beta \ge 1$.

Then

$$\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} |H_n(f,x) - f(x)|\; w(x)\, [1+|Q'(x)|]^{-\kappa}\, [1+|x|]^{-1}\right) = 0. \tag{2.12}$$

The proof of this result, given in far greater generality in [4], involves the usual tricks of the Hermite–Fejér trade, and of weights on $\mathbb{R}$, as well as a new variant of the Posse–Markov–Stieltjes inequality.

§3. PROOF OF THEOREM 2.1

We shall need the Christoffel functions

$$\lambda_n(w,x) := \inf_{P\in\mathscr{P}_{n-1}} \int_{-\infty}^{\infty} P^2(t)\, w(t)dt \,/\, P^2(x),\ n \geq 1,\ x \in \mathbb{R}, \tag{3.1}$$

and the Christoffel numbers

$$\lambda_{jn} := \lambda_n(w,x_{jn}),\ 1 \leq j \leq n,\ n \geq 1. \tag{3.2}$$

Throughout $C, C_1, C_2, \ldots$ denote constants independent of n, x and polynomials P of degree $\leq$ n. We shall denote by $v(x) = 1/\sqrt{1 - x^2}$ the Chebyshev weight on $(-1,1)$, and let

$$K_m(s,t) := \sum_{j=0}^{m-1} p_j(v,s)\, p_j(v,t) \tag{3.3}$$

be the corresponding kernel function. K_m [7] admits the estimates

$$|K_m(s,t)| \leq C/[\tfrac{1}{m} + |s-t|],\ s, t \in [-1,1],\ m \geq 1, \tag{3.4}$$

and

$$K_m(s,s) \geq C_1\, m\,,\ s \in [-\tfrac{1}{2},\tfrac{1}{2}],\ m \geq 1. \tag{3.5}$$

We proceed to prove a simple finite–infinite range inequality:

<u>Lemma 3.1</u> Let w be a weight on $\mathbb{R}$. Let $r > 1$, and $P \in \mathscr{P}_{n-1}$. Then

$$\int_{-\infty}^{\infty} P^2(x)\, w(x)\, dx \leq \frac{r^2}{r^2-1} \int_{-rx_{1,n+1}}^{rx_{1,n+1}} P^2(x)\, w(x)\, dx. \tag{3.6}$$

<u>Proof</u> Now

$$\int_{|x|\geq rx_{1,n+1}} P^2(x)\, w(x)\, dx \leq \int_{|x|\geq rx_{1,n+1}} \left(\frac{x}{rx_{1,n+1}}\right)^2 P^2(x)\, w(x)\, dx$$

$$\leq \int_{-\infty}^{\infty} \left(\frac{x}{rx_{1,n+1}}\right)^2 P^2(x)\, w(x)\, dx$$

$$= \sum_{j=1}^{n+1} \lambda_{j,n+1} \left(\frac{x_{j,n+1}}{rx_{1,n+1}}\right)^2 P^2(x_{j,n+1})$$

$$\leq r^{-2} \sum_{j=1}^{n+1} \lambda_{j,n+1} P^2(x_{j,n+1})$$

$$= r^{-2} \int_{-\infty}^{\infty} P^2(x)\, w(x)\, dx = r^{-2} \left[\int_{-rx_{1,n+1}}^{rx_{1,n+1}} + \int_{|x| \geq rx_{1,n+1}}\right] P^2(x)\, w(x)\, dx,$$

by two applications of the Gauss–Jacobi quadrature formula. Hence

$$(1 - r^{-2}) \int_{|x| \geq rx_{1,n+1}} P^2(x)\, w(x)\, dx \leq r^{-2} \int_{-rx_{1,n+1}}^{rx_{1,n+1}} P^2(x)\, w(x)\, dx,$$

and the result follows. □

Next, we estimate Christoffel numbers:

__Lemma 3.2__ Let w be a weight on ℝ. Let $k \geq 1$. If (2.1) holds, then, given $\epsilon > 0$, there exists C such that for $n \geq 1$,

$$\max_{1 \leq j \leq n} \lambda_{jn} (1 + x_{jn}^2)^{2k} \leq C\, (x_{1,n+1}/n)^{1-\epsilon}. \tag{3.7}$$

If (2.3) holds, then there exists C such that for $n \geq 1$,

$$\max_{1 \leq j \leq n} \lambda_{jn} (1 + x_{jn}^2)^{2k} \leq C\, x_{1,n+1}/n. \tag{3.8}$$

__Proof__ Suppose first that (2.1) holds. For $n \geq 2k$, let $m := m(n) := n-1-2k$. Then

$$P(t) := (1 + t^2)^k\, K_m\left(\frac{t}{2x_{1,n+1}}, \frac{x_{jn}}{2x_{1,n+1}}\right)$$

is a polynomial of degree $\leq n-1$, and so by the definition of the Christoffel numbers, and by Lemma 3.1,

$$\lambda_{jn} \leq \frac{4}{3} \int_{-2x_{1,n+1}}^{2x_{1,n+1}} P^2(t)\, w(t)\, dt / P^2(x_{jn}).$$

Let $p^{-1} + q^{-1} = 1$. Using (3.4) and (3.5) and then Hölder's inequality yields

$$\lambda_{jn}(1+x_{jn}^2)^{2k} \le C_2 \int_{-2x_{1,n+1}}^{2x_{1,n+1}} (1+t^2)^{2k}\left[1+\frac{m|t-x_{jn}|}{2x_{1,n+1}}\right]^{-2} w(t)\,dt \tag{3.9}$$

$$\le C_3\left\{\int_{-\infty}^{\infty} (1+t^2)^{2kp}(1+|t|)^{\beta p/q}\, w(t)\,dt\right\}^{1/p}$$

$$\times \left\{\int_{-\infty}^{\infty} \left[1+\frac{m|t-x_{jn}|}{2x_{1,n+1}}\right]^{-2q} w(t)(1+|t|)^{-\beta}\,dt\right\}^{1/q}$$

$$\le C_4\left\{\int_{-\infty}^{\infty} \left[1+\frac{m|t-x_{jn}|}{2x_{1,n+1}}\right]^{-2q} dt\right\}^{1/q} = C_4\left(\frac{2x_{1,n+1}}{m}\right)^{1/q}\left\{\int_{-\infty}^{\infty}[1+|s|]^{-2q}\,ds\right\}^{1/q},$$

by (2.1), and the convergence of the moments of w. Since we may choose $1/q \ge 1-\epsilon$, and since $m \ge n/2$, n large enough, the proof follows in this case. If instead of (2.1), we assume (2.3), then using (2.3) in (3.9) yields

$$\lambda_{jn}(1+x_{jn}^2)^{2k} \le C_5 \int_{-2x_{1,n+1}}^{2x_{1,n+1}} \left[1+\frac{m|t-x_{jn}|}{2x_{1,n+1}}\right]^{-2} dt \le C_6\,\frac{2x_{1,n+1}}{m}. \quad \square$$

Lemma 3.3

$$x_{1n} \le 2 \max_{1\le k\le n-1} \frac{\gamma_{k-1}}{\gamma_k}. \tag{3.10}$$

Proof See, for example, Freud [2,p.18,Thm.4]. □

Proof of Theorem 2.1

Suppose first that (2.1) and (2.2) hold. Let P be a polynomial. Careful use of the Cauchy–Schwarz inequality yields (see [9,p.46,(51)])

$$\int_{-\infty}^{\infty} |P(x)-H_n(P,x)|\,w(x)\,dx \le \frac{\gamma_{n-1}}{\gamma_n}\left[\sum_{j=1}^{n} P'(x_{jn})^2\, p_{n-1}(w,x_{jn})^2\,\lambda_{jn}^3\right]^{1/2}$$

$$\le \frac{\gamma_{n-1}}{\gamma_n} \max_{1\le j\le n} |\lambda_{jn}P'(x_{jn})| \left[\sum_{j=1}^{n} \lambda_{jn}\, p_{n-1}(w,x_{jn})^2\right]^{1/2}$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \max_{1\le j\le n} |\lambda_{jn}P'(x_{jn})|,$$

by the Gauss–quadrature formula, and orthonormality. If $\delta > 0$ and $2k \ge$ degree of P, then we obtain from (3.7) of Lemma 3.2, for some $A = A(P)$,

$$\int_{-\infty}^{\infty} |P(x) - H_n(P,x)|\, w(x)\, dx \le A \frac{\gamma_{n-1}}{\gamma_n} \max_{1\le j\le n} \lambda_{jn} (1 + x_{jn}^2)^k \tag{3.11}$$

$$\le A\, C \frac{\gamma_{n-1}}{\gamma_n} \left(\frac{x_{1,n+1}}{n}\right)^{1-\delta} = o(1),$$

by (2.2) and Lemma 3.3, if we choose δ small enough.

If (2.3) and (2.4) hold, (3.8) of Lemma 3.2 and (3.11) yield

$$\int_{-\infty}^{\infty} |P(x) - H_n(P,x)|\, w(x)\, dx \le A\, C \frac{\gamma_{n-1}}{\gamma_n} \frac{x_{1,n+1}}{n} = o(1),$$

by (2.4) and Lemma 3.3. □

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