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# On discrete norm estimates related to multilevel preconditioners in the finite element method

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**Abstract.** Recently, some new preconditioners for solving elliptic finite element discretizations by conjugate gradient methods have been proposed ([Y1],[BPX]). They are based on certain multilevel splittings of the finite element spaces under consideration. In this note we apply standard function space techniques to derive sharp estimates for the condition numbers of the preconditioned linear systems. In particular, uniform bounds are found for the Bramble-Schatz-Xu preconditioner.

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## 0. Introduction

In this paper we study certain discrete norm estimates related to multilevel finite element methods for the solution of the model problem

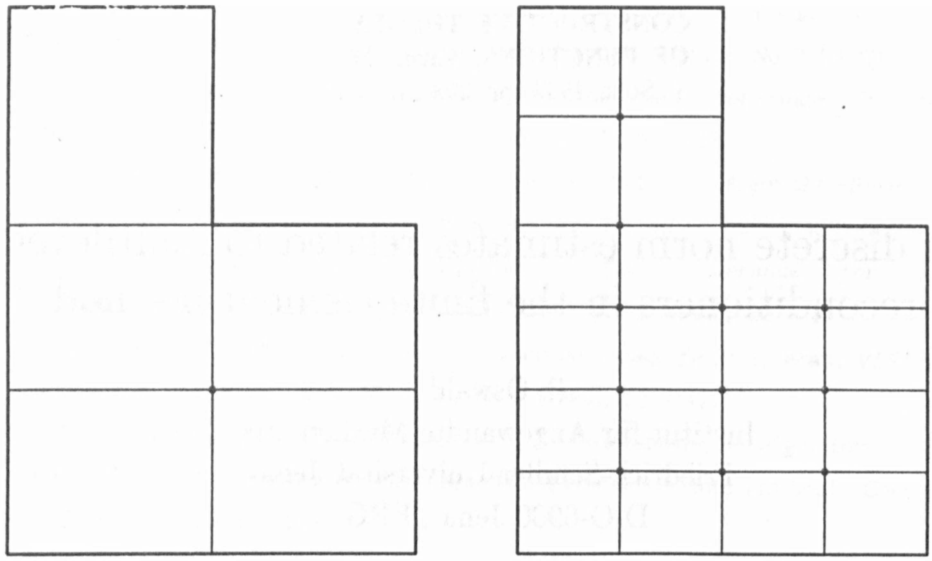
(1) $$-\nabla\cdot(p\cdot\nabla u)+q\cdot u=f,\qquad u\in H_0^1(\Omega),$$

Since we want to concentrate on some particular examples of practical importance rather than to develop the general theory, let us suppose that $\Omega\subset R^d$ is a "rectangular like" domain, $f\in L_2(\Omega)$ , and that $p\in C^1(\Omega)$ , $q\in C(\Omega)$ satisfy

$$0<\alpha_1\le p(x)\le\alpha_2,\qquad 0<\beta_1\le q(x)\le\beta_2,\qquad x\in\Omega,$$

with some fixed constants. Let $\mathcal{R}_0$ be an appropriate initial partition of $\Omega$ into a finite number of $d$-dimensional rectangles, and generate $\mathcal{R}_1,\mathcal{R}_2,\dots$ by dyadically refining the initial partition. Thus, $\mathcal{R}_k$ consists of rectangles similar to those contained in $\mathcal{R}_0$ but scaled by a factor $2^{-k}$.

Let $S_k, k=0,1,\dots$ be the sequence of rectangular linear $C^0$ finite element spaces corresponding to $\{\mathcal{R}_k\}$ equipped with zero boundary values. The set of nodal basis functions for $S_k$ is denoted by $\mathcal{N}_k=\{N_{k,j}:\ j=1,\dots,n_k\}$ where $n_k=\dim S_k$ is equal to the number of interior nodes in $\mathcal{R}_k$ (cf. Figure 1 for an illustration of the refinement process and the sets $\mathcal{V}_k$ of vertices resp. nodes, detailed definitions can be found in [C]).



$\mathcal{R}_0$ $\mathcal{R}_1$

**Figure 1.**

Clearly,

$$S_0 \subset S_1 \subset \ldots \subset S_k \subset \ldots \subset H_0^1(\Omega), \tag{2}$$

and the $S_k$ can be used within the framework of conforming finite element methods for discretizing (1). To this end, let us denote

$$a(u,v) = \int_\Omega (p \cdot \nabla u \cdot \nabla v + q \cdot u \cdot v)\, d\omega \quad , \quad \phi(v) = \int_\Omega f \cdot v\, d\omega, \quad u,v \in H_0^1(\Omega)$$

and consider the following approximate problem: Find $u_k \in S_k$ such that

$$a(u_k, v_k) = \phi(v_k) \qquad \text{for all } v_k \in S_k. \tag{3}$$

If we denote by $x = (x_1, \ldots, x_{n_k})$ the coefficient vector of the representation of $u_k$ with respect to the nodal basis $\mathcal{N}_k$, i.e.

$$u_k = \sum_{j=1}^{n_k} x_j \cdot N_{k,j}$$

then (3) reduces to a linear system

$$A \cdot x = b \tag{4}$$

with a positive definite sparse matrix $A$ consisting of the elements $a_{i,j} = a(N_{k,i}, N_{k,j})$ and $b = (\phi(N_{k,i}))$.

The design of efficient solvers for large sparse linear systems such as (4) has been a field of intensive research for many years, see [HY, AB, H, Dja] et.al. for some information in this direction. Among the iterative methods, the so-called preconditioned conjugate gradient method attracts much attention : Since the usual conjugate gradient method is fast converging for systems with well-conditioned positive definite coefficient matrix but, unfortunately, the condition number $\kappa(A)$ of the system (4) significantly grows for $k \to \infty$, i.e. if the gridsize tends to zero, it is recommended first to improve the condition of (4) by transforming the system into an equivalent one

$$(5) \qquad (C \cdot A) \cdot x = C \cdot b$$

where $\kappa(C^{1/2} \cdot A \cdot C^{1/2}) \ll \kappa(A)$. Moreover, the preconditioning matrix $C$ is supposed to be positive definite and, in some sense, simple (for the theory and proposals how to construct reasonable $C$, see [AB, H]).

One possible way to implement the above strategy is to determine a positive definite matrix $B$ such that, on the one hand, $B^{-1}$ can efficiently be computed (usually, $O(n_k)$ is the desirable bound for the number of arithmetical operations when solving a linear system with coefficient matrix $B$) and that, on the other hand, $A$ and $B$ are "almost" spectrally equivalent, i.e.

$$(6) \qquad \lambda_B \cdot (Bx, x) \le (Ax, x) \le \Lambda_B \cdot (Bx, x), \quad x \in R^{n_k}$$

with two positive constants $\lambda_B$, $\Lambda_B$ possessing a small ratio $\Lambda_B/\lambda_B$. Evidently, since $\kappa(B^{-1/2} \cdot A \cdot B^{-1/2}) \le \Lambda_B/\lambda_B$ , $C = B^{-1}$ will be a good choice.

Mention that

$$\begin{aligned} (Ax, x) &= \sum_{i,j=1}^{n_k} a(N_{k,i}, N_{k,j}) x_j x_i \\ &= a(u_k, u_k) \approx \|u_k\|_{H^1}^2 \end{aligned}$$

Thus, if $(Bx, x)$ can also be interpreted in terms of some norm of $u_k$ then (6) reduces to a comparison of different norms on $S_k$, a problem which can be dealt with by function-theoretic methods.

In this note we consider two examples of preconditioners of this type that have recently been proposed by Yserentant [Y1] (the so-called hierarchical basis preconditioner) and by Bramble, Pasciak, Xu [BPX, X], and give new proofs of the corresponding relations (6) using techniques from function space theory. In particular, for the hierarchical basis method we reproduce the estimate from [Y1]

$$(7) \qquad \Lambda_H/\lambda_H = O(k^2)\ , \quad k \to \infty, d = 2,$$

while for the BPX-method we improve the previous results quoted in [BPX, X, Y2] to the final bound

$$(8) \qquad \Lambda_X/\lambda_X = O(1)\ , \quad k \to \infty,$$

for arbitrary $d$.

Additionally, the preconditioning for the finite element discretization of (1) by nonconforming Wilson rectangles is discussed.

## 1. Preliminaries on function spaces

In this Section we outline the proof of the following constructive characterization of $H_0^1(\Omega)$ with respect to $\{S_k\}$ .

**Lemma 1.** *A function $f \in L_2(\Omega)$ belongs to $H_0^1(\Omega)$ if and only if there exists a representation*

$$u = \sum_{k=0}^{\infty} v_k , \quad v_k \in S_k , \ k = 0, 1, \ldots . \tag{9}$$

*with the property*

$$\sum_{k=0}^{\infty} 4^k \cdot \|v_k\|_{L_2}^2 < \infty. \tag{10}$$

*Moreover, one has*

$$\|u\|_{H^1}^2 \approx |||u|||^2 \equiv \inf \left\{ \sum_{k=0}^{\infty} 4^k \cdot \|v_k\|_{L_2}^2 \right\} \tag{11}$$

*(equivalence up to constants depending on $\mathcal{R}_0$, only; the infimum is taken with respect to all representations* (9)-(10)*).*

Assertions of this type are standard in the theory of function spaces, cf. [N, BIN, T], for the special case we are interested in see [DP, O1, O2]. The proof of Lemma 1 will be sketched for the sake of completeness, only.

First of all, we have (up to equivalent norms)

$$H_0^1(\Omega) \cong B_{2,2;0}^1(\Omega) \tag{12}$$

where the Besov space $B_{2,2;0}^1(\Omega)$ consists of all $u \in L_2(\Omega)$ with $u|_{\partial\Omega} = 0$ and finite norm

$$\|u\|_{B_{2,2}^1}^2 \equiv \|u\|_{L_2}^2 + \sum_{k=0}^{\infty} 4^k \cdot \omega_2(2^{-k}, u)_{L_2}^2 < \infty. \tag{13}$$

The existence of a reasonable trace follows from (13). Here, $\omega_2(\cdot, u)_{L_2}$ stands for the second order $L_2$ modulus of continuity of $u$ . (12) holds for rather general domains, cf. [T, BIN] .

Next we replace the moduli in the Besov norm by best approximations with respect to the subspaces $S_k$ . Let

$$s_k(u)_{L_2} = \inf_{v \in S_k} \|u - v\|_{L_2}, \quad k = 0, 1, \ldots,$$

then

$$\|u\|^2_{A^1_{2,2}} = \|u\|^2_{L_2} + \sum_{k=0}^{\infty} 4^k \cdot s_k(u)^2_{L_2} \tag{14}$$

defines an equivalent norm in the above introduced Besov space. The equivalence follows by the usual technique of Jackson–Bernstein inequalities for piecewise polynomial approximation. Without the assumption for the trace to the boundary, the result is e.g. contained in [DP, O1]. In [O2], for triangular elements in the plane, a detailed proof is given including several restrictions on the traces.

Finally, the equivalence of $\|\cdot\|_{A^1_{2,2}}$ given by (14) and $|||\cdot|||$ from (11) is a simple exercise : if $u_k \in S_k$, $k = 0, 1, \ldots$, denotes finite element functions best (or nearly best) approximating some given $u$ then a suitable representation (9),(10) is determined by $v_0 = u_0$, $v_k = u_k - u_{k-1}$, $k = 1, 2, \ldots$ .

Since we are more interested in norms defined on the subspaces $S_k$ , we formulate the finite–dimensional version of Lemma 1 separately.

**Lemma 2.** *For any $u_k \in S_k$ we have*

$$\|u_k\|^2_{H^1} \approx |||u_k|||^2_k = \inf \left\{ \sum_{l=0}^{k} 4^l \cdot \|v_l\|^2_{L_2} \right\} \tag{15}$$

*where the infimum is taken with respect to all representations*

$$u_k = \sum_{l=0}^{k} v_l , \quad v_l \in S_l,\ l = 0, \ldots, k. \tag{16}$$

*The constants in the equivalence relation* (15) *are independent of k.*

*Proof.* The upper estimate in (15) follows directly from (2),(11). For the lower estimate, once again using (2),(11), we find a representation

$$u_k = \sum_{l=0}^{\infty} \bar{v}_l, \bar{v}_l \in S_l, l = 0, 1, \ldots, \quad \sum_{l=0}^{\infty} 4^l \cdot \|\bar{v}_l\|^2_{L_2} \le c \cdot \|u_k\|^2_{H^1}.$$

To get a finite representation (16), put $v_l = \bar{v}_l$, $l = 0, \ldots, k-1$ and $v_k = \bar{v}_k + \sum_{l=k+1}^{\infty} \bar{v}_l = u_k - \sum_{l=0}^{k-1} v_l \in S_k$ . With this notation, we have

$$\begin{aligned}
\sum_{l=0}^{k} 4^l \cdot \|v_l\|^2_{L_2} &\le \sum_{l=0}^{k-1} 4^l \cdot \|\bar{v}_l\|^2_{L_2} + 4^k \left( \sum_{l=k}^{\infty} 4^{-(l-k)} \cdot 4^{l-k} \|\bar{v}_l\|_{L_2} \right)^2 \\
&\le \sum_{l=0}^{k-1} 4^l \cdot \|\bar{v}_l\|^2_{L_2} + 4^k \sum_{j=0}^{\infty} 4^{-j} \sum_{l=k}^{\infty} 4^{l-k} \|\bar{v}_l\|^2_{L_2} \\
&\le \frac{5}{4} \sum_{l=0}^{\infty} 4^l \cdot \|\bar{v}_l\|^2_{L_2} \le c \cdot \|u_k\|^2_{H^1}.
\end{aligned}$$

Lemma 2 will be explored below together with another simple but useful property of the spaces $S_l$:

(17) $$\|v_l\|_{L_2}^2 \approx 2^{-ld} \cdot \sum_{j=1}^{n_l} \alpha_j^2 , \quad v_l = \sum_{j=1}^{n_l} \alpha_j N_{l,j}$$

where the constants in the equivalence relation (17) depend only on $\mathcal{R}_0$ but not on $v_l \in S_l$, $l = 0, 1, \ldots$. This so-called $L_2$ stablity property of the nodal basis is obvious from the piecewise structure of the finite element functions and the dyadic refinement process of the underlying partitions.

## 2. The hierarchical basis preconditioner

Let us introduce a hierarchy in the set $\mathcal{V}_k$ of nodal points by writing

$$\mathcal{V}_k = \mathcal{V}_0 \cup (\mathcal{V}_1 \setminus \mathcal{V}_0) \cup \ldots \cup (\mathcal{V}_k \setminus \mathcal{V}_{k-1})$$

and renumber the nodal points accordingly, i.e. $P_j \in \mathcal{V}_k$ with $j = n_{l-1}+1, \ldots, n_l$ belongs to $\mathcal{V}_k \setminus \mathcal{V}_{k-1}$, $l = 0, \ldots, k$ ($\mathcal{V}_{-1} = \emptyset, n_{-1} = 0$).

To each nodal point $P_j, j = n_{l-1}+1, \ldots, n_l$ , we associate a hierarchical basis function

(18) $$H_j = 2^{l(d/2-1)} \cdot N_{l,j}$$

which is a scaled copy of the nodal basis function from $\mathcal{N}_l$ corresponding to this $P_j$.

Clearly, $\{H_j, j = 1, \ldots, n_k\}$ forms a basis in $S_k$ . The coefficients of the basis representation

$$u_k = \sum_{j=1}^{n_k} x_{H,j} \cdot H_j$$

can be determined recursively by the formulae

(19a) $$x_{H,j} = I_0 u_k(P_j), \quad j = 1, \ldots, n_0,$$

(19b) $$x_{H,j} = 2^{l(1-d/2)} \cdot (I_l u_k - I_{l-1} u_k)(P_j), \quad j = n_{l-1}+1, \ldots, n_l,\ l = 1, \ldots, k$$

where $I_l : S_k \to S_l$ is the natural interpolation projection given by

$$I_l u_k \in S_l , \quad I_l u_k(P) = u_k(P) , \quad P \in \mathcal{V}_l.$$

$l = 0, 1, \ldots, k$ ($I_{-1} u_k = 0$).

Thus, the transformations

(20) $$x = L x_H \qquad \text{resp.} \qquad x_H = L^{-1} x$$

of the hierarchical basis coefficients into the nodal basis coefficients vice versa are elementary and require only $O(n_k)$ arithmetical operations.

If we put $B_H = (L^{-1})^T \cdot L^{-1}$ then by (17),(19),(20)

$$\begin{aligned}(B_H x, x) &= (L^{-1}x, L^{-1}x) = (x_H, x_H) \\ &= \sum_{l=0}^{k} \sum_{j=n_{l-1}+1}^{n_l} 2^{l(2-d)} \cdot ((I_l u_k - I_{l-1} u_k)(P_j))^2 \\ &\approx \sum_{l=0}^{k} 4^l \cdot \|I_l u_k - I_{l-1} u_k\|_{L_2}^2 \equiv |||u_k|||_H^2\end{aligned}$$

and the efficiency of the proposed hierarchical preconditioner $C_H = B_H^{-1} = L \cdot L^T$ depends on the corresponding constants in (6) or, what is the same according to Lemma 2, in

(21) $$\bar{\lambda}_H \cdot |||u_k|||_H^2 \le |||u_k|||_k^2 \le \bar{\Lambda}_H \cdot |||u_k|||_H^2, \quad u_k \in S_k.$$

Obviously, since $v_l = I_l u_k - I_{l-1} u_k \in S_l$, $l = 0, \ldots, k$ , yields an representation (16), we have

(22) $$\bar{\Lambda}_H \le 1$$

What concerns $\bar{\lambda}_H$ , consider an arbitrary decomposition (16) of $u_k \in S_k$. Since

$$I_l u_k = \sum_{m=0}^{l} v_m + \tilde{v}_l \, ; \quad \tilde{v}_l = I_l(\sum_{m=l+1}^{k} v_m) \, ,$$

we obtain

$$\begin{aligned}|||u_k|||_H^2 &= \sum_{l=0}^{k} 4^l \cdot \|v_l + \tilde{v}_l - \tilde{v}_{l-1}\|_{L_2}^2 \\ &\le c \cdot \left( \sum_{l=0}^{k} 4^l \|v_l\|_{L_2}^2 + \sum_{l=0}^{k} 4^l \|\tilde{v}_l\|_{L_2}^2 \right)\end{aligned}$$

Due to (17) and the definition of $I_l$, we have

$$\begin{aligned}\|\tilde{v}_l\|_{L_2}^2 &\le c \cdot 2^{-ld} \sum_{P \in \mathcal{V}_l} \tilde{v}_l(P)^2 \\ &\le c \cdot 2^{-ld} \sum_{P \in \mathcal{V}_l} \left( \sum_{m=l+1}^{k} 2^{\alpha(m-l)} \cdot 2^{-\alpha(m-l)} \cdot v_m(P) \right)^2 \\ &\le c \cdot \beta_{\alpha,k-l} \sum_{m=l+1}^{k} 2^{(d-\alpha)(m-l)} \cdot \|v_m\|_{L_2}^2\end{aligned}$$

where

$$\beta_{\alpha,m} = \begin{cases} 2^{\alpha m} & : \ \alpha > 0 \\ m & : \ \alpha = 0 \\ 1 & : \ \alpha < 0 \end{cases}$$

Now, substituting into the preceeding inequality, taking $\alpha = d-2$ $(d \geq 2)$ resp. $\alpha = -1/2$ $(d=1)$ and changing the order of summation, we finally get

$$\bar{\lambda}_H^{-1} \leq c \cdot \beta^2_{(d-2)/2,k}\,, \quad k = 1,2,\ldots. \tag{23}$$

This proves

**Theorem 1.** *For the hierarchical preconditioner $C_H = L \cdot L^T$ we have*

$$\kappa(B_H^{-1/2} \cdot A \cdot B_H^{-1/2}) \leq c \cdot \beta^2_{(d-2)/2,k}\,, \quad k = 1,2,\ldots. \tag{24}$$

*with a constant c independent of the number of refinement levels k.*

*Remark 1.*For $d = 2$, this is the $O(k^2)$ bound obtained by Yserentant [Y1], the case $d = 3$ has been proved by Ong [On] (actually, these authors considered triangular linear elements but there is no substantial difference to the case of rectangular elements). Mention that the estimates (22)-(24) are asymptotically sharp (for the proof, see [Y1, O3]). As a consequence, the hierarchical preconditioner does not improve the asymptotic behaviour of the condition of (4) if $d \geq 4$.

*Remark 2.* The function space technique can be used for other conforming elements, too. E.g., we studied the hierarchical preconditioner for Lagrange elements of higher degree [O3] and for some triangular $C^1$ elements for discretizing the biharmonic equation [O4]. What concerns nonconforming elements (since (2) is violated, this case requires additional efforts) only P1 elements have successfully been treated [O5]. We mention also the somewhat different approach by Dörfler [D].

## 3. The Bramble-Pasciak-Xu preconditioner

In [BPX, X, Y2] it has been proved that the BPX–preconditioner $C_X = B_X^{-1}$ proposed in [BPX] is spectrally equivalent to $\tilde{C}_X = \tilde{B}_X^{-1}$ produced by the form

$$(\tilde{B}_X x, x) = |||u_k|||_X^2 \equiv \sum_{l=0}^{k} 4^l \cdot \|Q_l u_k - Q_{l-1} u_k\|^2_{L_2} \tag{25}$$

Here, $Q_l : S_k \to S_l\,; l = 0,\ldots,k$, denotes the $L_2$ projection, i.e. $Q_l u_k \in S_l$ is defined by the orthogonality property

$$\iint_\Omega (u_k - Q_l u_k) \cdot v_l \, d\omega = 0 \quad \text{for all } v_l \in S_l.$$

For $l = -1$, we put $Q_{-1} u_k = 0$.

Thus, by (6) and Lemma 2 we are led to the estimation of the constants in

$$\tilde{\lambda}_X \cdot |||u_k|||_X^2 \leq |||u|||_k^2 \leq \tilde{\Lambda}_X \cdot |||u_k|||_X^2\,, \; u_k \in S_k.$$

As above, we automatically have $\tilde{\Lambda}_X \leq 1$. For the lower estimate, observe that by the orthogonality properties of the projections $Q_l$ and for any representation (16) one obtains

$$\|Q_l u_k - Q_{l-1}u_k\|_{L_2}^2 \leq \|u_k - Q_{l-1}u_k\|_{L_2}^2 \leq \|\sum_{m=l}^{k} v_m\|_{L_2}^2 \leq c \cdot 4^{-l\epsilon} \sum_{m=l}^{k} 4^{m\epsilon} \cdot \|v_m\|_{L_2}^2,$$

$0 < \epsilon < 1$ , which yields

$$|||u_k|||_X^2 \leq c \cdot \sum_{m=0}^{k} \left( 4^{m\epsilon} \cdot \|v_m\|_{L_2}^2 \sum_{l=o}^{m} 4^{l(1-\epsilon)} \right) \leq c \cdot \sum_{m=0}^{k} 4^m \cdot \|v_m\|_{L_2}^2.$$

Since this inequality holds for any representation (16), after taking the infimum, we arrive at $\tilde{\lambda}_X \geq c$ with some absolute constant $c > 0$ which proves

**Theorem 2.** *For the Bramble–Pasciak–Xu preconditioner $C_X$ we have*

$$\kappa(B_X^{-1/2} \cdot A \cdot B_X^{-1/2}) \leq c \tag{26}$$

*with a constant which is independent of $k = 0, 1, \ldots$ .*

*Remark 3.* (26) improves the original estimates of [BPX] where, under additional regularity assumptions, an $O(k)$ bound was obtained.

*Remark 4.* The difference between the results in Theorems 1 and 2 may be explained from the properties of the underlying projections $I_l$ resp. $Q_l$ . While $\{Q_l\}$ is uniformly bounded as operator sequence acting from $L_2$ to $L_2$, the interpolation projections $I_l$ are not well–defined even on $H_0^1(\Omega)$, $d \geq 2$ . It is an intriguing question whether there exist other reasonable splittings of $S_k$ (say, produced by projections of quasi–interpolant type).

*Remark 5.* The results of Theorems 1 and 2 can be extended to other $H^s$ resp. Besov space norms, e.g.

$$\|u_k\|_{H^s}^2 \approx \sum_{l=0}^{k} 4^{ls} \cdot \|Q_l u_k - Q_{l-1}u_k\|_{L_2}^2 , \quad ,0 < s < 3/2, \; d \geq 1. \tag{27}$$

Especially the case $s = 1/2$ might be of interest in connection with domain decomposition techniques.

## 4. Preconditioning for the nonconforming Wilson element.

We finish the paper by some simple observations how to derive analogous estimates for the nonconforming Wilson rectangle. The corresponding subspaces $\hat{S}_k$ are defined as follows. With each rectangle $R = \prod_{i=1}^{d}(a_i, b_i) \in \mathcal{R}_k$ we associate the quadratic bubble functions

$$\Psi_{R,i}(\xi) = \begin{cases} (b_i - \xi_i)(\xi_i - a_i)(b_i - a_i)^{-2} & , \quad \xi \in R \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$i = 1, \ldots, d$. Denoting $\bar{\mathcal{N}}_k = \{\Psi_{R,i}, R \in \mathcal{R}_k, i = 1, \ldots, d\}$ and $\bar{S}_k = \mathrm{span}\bar{\mathcal{N}}_k$ we introduce

$$\hat{S}_k = S_k \oplus \bar{S}_k \tag{28}$$

Since $\hat{S}_k \not\subset H_0^1(\Omega)$, a modified bilinear form on $\hat{S}_k \times \hat{S}_k$ is defined by

$$a_k(\hat{u}_k, \hat{v}_k) = \sum_{R \in \mathcal{R}_k} \iint_R (p \nabla \hat{u}_k \nabla \hat{v}_k + q \hat{u}_k \hat{v}_k)\, d\omega. \tag{29}$$

The discretization of (1) with respect to $\hat{S}_k$ looks now as follows : Find $\hat{u}_k \in \hat{S}_k$ such that

$$a_k(\hat{u}_k, \hat{v}_k) = \phi(\hat{v}_k) \quad \text{for all} \quad \hat{v}_k \in \hat{S}_k. \tag{30}$$

In [C], a detailed investigation of (30) is given for $d = 3$. E.g., the additional degrees of freedom lead to generally better approximation rates since polynomials of total degree 2 can be reproduced.

We add the following observation concerning preconditioners for the linear system arising from (30) when the nodal basis $\hat{\mathcal{N}}_k = \mathcal{N}_k \cup \bar{\mathcal{N}}_k$ in $\hat{S}_k$ is used. To this end, let $\hat{u}_k \in \hat{S}_k$ be splitted according to (28), i.e. $\hat{u}_k = u_k + \bar{u}_k$ where

$$u_k = I_k \hat{u}_k = \sum_{j=1}^{n_k} x_j N_{k,j} \in S_k$$

and

$$\bar{u}_k = \sum_{R \in \mathcal{R}_k} \sum_{i=1}^{d} \bar{x}_{R,i} \Psi_{R,i} \in \bar{S}_k.$$

**Lemma 3.** *The discrete norm*

$$|||\hat{u}_k|||_{\hat{S}_k}^2 = |||u_k|||_k^2 + 4^k \cdot \|\bar{u}_k\|_{L_2}^2$$

*satisfies the equivalence relation*

$$|||\hat{u}_k|||_{\hat{S}_k}^2 \approx a_k(\hat{u}_k, \hat{u}_k) \quad , \quad \hat{u}_k \in \hat{S}_k. \tag{31}$$

The elementary proof of (31) is left to the reader. From (31), the following is obvious : If $B$ produces a suitable preconditioner $C = B^{-1}$ for the linear system (4) (i.e. the nodal basis discretization of (1)) then

$$\hat{B} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \quad \text{with} \quad \hat{C} = \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix}$$

are appropriate for preconditioning the linear system

$$\hat{A} \cdot \hat{x} = \hat{b} \quad , \quad \hat{x} = (x, \bar{x}) \tag{32}$$

which represents the nodal basis discretization of (30).

Indeed,

$$(\hat{A}\hat{x},\hat{x}) = a_k(\hat{u}_k,\hat{u}_k) \approx |||u_k|||_k^2 + 4^k \cdot \|\bar{u}_k\|_{L_2}^2$$

by Lemma 3, and

$$(\hat{B}\hat{x},\hat{x}) = (Bx,x) + (\bar{x},\bar{x}) \approx (Bx,x) + 4^k \cdot \|\bar{u}_k\|_{L_2}^2$$

by the definition of the new nodal basis functions.

Thus, if (6) holds true for $B$ then

$$\lambda_{\hat{B}} \cdot (\hat{B}\hat{x},\hat{x}) \le (\hat{A}\hat{x},\hat{x}) \le \Lambda_{\hat{B}} \cdot (\hat{B}\hat{x},\hat{x}) \tag{33}$$

with

$$\Lambda_{\hat{B}}/\lambda_{\hat{B}} \le c \cdot (\max\{\Lambda_B, 1\})/(\min\{\lambda_B, 1\}) \tag{34}$$

**Theorem 3.** *The hierarchical basis preconditioner resp. the Bramble–Pasciak–Xu preconditioner can directly be carried over to the discretization of (1) by Wilson rectangles. The condition number estimates remain unchanged.*

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