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MORE APPLICABLE ERROR ESTIMATES FOR QUADRATURE RULES

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1. Introduction. For $f \in C[-1, 1]$ let

$$E_n(f) = \inf_{q \in P_n} \|f - q\|_\infty \tag{1}$$

denote the error of minimax approximation. For $f \in C^{n+1}[-1, 1]$, S.N. Bernstein [1] showed that $E_n(f)$ satisfies the inequalities

$$\frac{1}{2^n} \cdot \min_{-1\le x\le 1} |f^{(n+1)}(x)|/(n+1)! \le E_n(f) \le \frac{1}{2^n} \cdot \max_{-1\le x\le 1} |f^{(n+1)}(x)|/(n+1)! \,. \tag{2}$$

Recently Elliott and Taylor [3] obtained the following alternative lower and upper bounds for $E_n(f)$ in terms of divided differences:

$$\frac{1}{2^n} \cdot |f[\eta_0,\dots,\eta_{n+1}]| \le E_n(f) \le \frac{1}{2^n} \cdot \max_{-1\le x\le 1} |f[x, \xi_1, \xi_2,\dots, \xi_{n+1}]| \,. \tag{3}$$

In (3) the η_j and the ξ_j are respectively the extreme points and zeros of T_{n+1}. For any points $y_0,\dots,y_{n+1}$ in $[-1, 1]$ and any $f \in C^{n+1}[-1, 1]$ there is an η in $(-1, 1)$ such that

$$f[y_0,\dots,y_{n+1}] = f^{(n+1)}(\eta)/(n+1)! \tag{4}$$

(See, for example, Phillips and Taylor [5].) Thus it follows, as Elliott and Taylor [3] observed, that for $f \in C^{n+1}[-1, 1]$ the bounds in (3) in terms of divided differences are in general sharper than those in (2) involving derivatives. Moreover, the bounds for $E_n(f)$ in (2) are valid only for $f \in C^{n+1}[-1, 1]$, whereas in (3) the lower bound is valid for $f \in C[-1, 1]$ and the upper bound is valid for $f \in C[-1, 1]$ with the additional requirement that the first derivative of f exists at all of the zeros of T_{n+1}. (The latter restriction is needed because, in the right-hand inequality of (3), the maximum may occur when x coincides with one of the Chebyshev zeros ξ_j.)

The commonly used quadrature rules have errors which are most commonly expressed in terms of the appropriate order of derivative of the integrand evaluated at some unknown point, although sometimes divided differences are used at an intermediate stage in obtaining an error term. We will show here that, by expressing quadrature errors in terms of divided differences rather than in terms of derivatives, we obtain the same two advantages which Elliott and Taylor found in estimating the minimax error, as described above. (For an alternative approach, see also the bounds obtained by Sendov and Popov [6] for the errors of quadrature rules in terms of averaged moduli of smoothness or τ-moduli.)

2. Newton-Cotes rules. From the divided difference form of the error of interpolation, we have

$$f(x) - q(x) = (x-x_0)\ldots(x-x_n) \,.\, f[x,x_0,\ldots, x_n] \,,$$

where q interpolates f at $x_0,\ldots,x_n$. Integrating over [a, b], we obtain

$$\int_a^b f(x)\,dx - R(f) = E(f)\,, \tag{5}$$

where the quadrature rule is given by

$$R(f) = \int_a^b q(x)\,dx \tag{6}$$

and the error is

$$E(f) = \int_a^b (x-x_0)\ldots(x-x_n).f[x,x_0,\ldots,x_n]\,dx\,. \tag{7}$$

One of the simplest examples is the trapezoidal rule. Let us take the interval of integration [a, b] as [0, 1], $n = 1$ and $x_0 = 0$, $x_1 = 1$. Then it follows from (7) that in this case the error is

$$E(f) = \int_0^1 x(x-1).f[x,0,1]\,dx.$$

If $f \in C[0, 1]$ then, since x(x-1) has constant sign on [0, 1], we may apply the mean value theorem of integration to give

$$E(f) = f[\xi,0,1] \int_0^1 x(x-1)dx,$$

for some $\xi \in (0, 1)$. Thus, for the trapezoidal rule we have the error bounds

$$\frac{1}{6} \min_{0\le x\le 1} |f[x,0,1]| \le |E(f)| \le \frac{1}{6} \max_{0\le x\le 1} |f[x,0,1]|, \tag{8}$$

valid for $f \in C[0, 1]$, which may be compared with the more usual bounds

$$\frac{1}{12} \min_{0\le x\le 1} |f^{(2)}(x)| \le |E(f)| \le \frac{1}{12} \max_{0\le x\le 1} |f^{(2)}(x)|, \tag{9}$$

which are valid only for $f \in C^2[-1, 1]$.

For Simpson's rule, let us take [a, b] as [-1, 1], n = 2 and the x_j as -1, 0 and 1. In this case (7) yields

$$E(f) = \int_{-1}^{1} (x+1)x(x-1).f[x,-1,0,1]\, dx. \tag{10}$$

Since the mean value theorem is not immediately applicable, we use the recurrence relation for divided differences and replace f[x,-1,0,1] in (10) by

$$f[-1,0,0,1] + x.f[x,-1,0,0,1],$$

assuming f is differentiable at x = 0. This gives, in place of (10),

$$E(f) = \int_{-1}^{1} (x+1)x^2(x-1).f[x,-1,0,0,1]\, dx. \tag{11}$$

We see that the mean value theorem may now be applied to give

$$E(f) = -\frac{4}{15}.f[\xi,-1,0,0,1],$$

for some $\xi \in (-1, 1)$, assuming that $f \in C[-1, 1]$ and is twice differentiable at the origin. For this class of functions we thus obtain for the error of Simpson's rule the bounds

$$\frac{4}{15} \min_{-1\le x\le 1} |f[x,-1,0,0,1]| \le |E(f)| \le \frac{4}{15} \max_{-1\le x\le 1} |f[x,-1,0,0,1]|, \tag{12}$$

to be compared with the usual bounds

$$\frac{1}{90} \min_{-1\le x\le 1} |f^{(4)}(x)| \le |E(f)| \le \frac{1}{90} \max_{-1\le x\le 1} |f^{(4)}(x)|, \tag{13}$$

valid only for $f \in C^4[-1, 1]$.

Similarly, when we examine the derivation of the usual error formula for any closed Newton-Cotes rule (see, for example, Isaacson and Keller [4]) we see that we can modify it, as we have done above for both the trapezoidal and Simpson rules, to obtain error terms involving divided differemces.

3. Gaussian rules.

We now consider integrals of the form

$$\int_a^b \omega(x)f(x)\, dx, \tag{14}$$

where ω is a non-negative weight function. Let $p_0, p_1,...$ denote the system of polynomials which are orthogonal on [a, b] with respect to ω and, for some choice of n, let $\xi_1, \xi_2,..., \xi_{n+1}$ denote the zeros of p_{n+1}. We will find it convenient to normalize the p_j so that each orthogonal polynomial has leading coefficient unity. If we replace f in (14) by its interpolating polynomial constructed at the zeros $\xi_1, \xi_2,..., \xi_{n+1}$, it is well known (see, for example, Davis and Rabinowitz [2]) that the resulting quadrature rule is exact for $f \in P_{2n+1}$. In particular, assuming f is differentiable at each of the zeros of p_{n+1}, the rule is exact for the polynomial $q \in P_{2n+1}$ which satisfies

$$f(\xi_j) = q(\xi_j) \quad \text{and} \quad f'(\xi_j) = q'(\xi_j), \quad j = 1,...,n+1.$$

Thus we can express the error of this Gaussian rule in the form

$$E(f) = \int_a^b \omega(x)\,(p_{n+1}(x))^2 \; f[x,\xi_1,\xi_1,...,\xi_{n+1},\xi_{n+1}]\,dx, \tag{15}$$

where each of the ξ_j occurs twice in the divided difference. In particular, for the Gauss-Chebyshev rule with weight function $(1-x^2)^{-1/2}$ we have

$$\frac{\pi}{2^{2n+1}} \min_{-1\le x\le 1} |G(x)| \;\le\; |E(f)| \;\le\; \frac{\pi}{2^{2n+1}} \max_{-1\le x\le 1} |G(x)|, \tag{16}$$

where $G(x) = f[x,\xi_1,\xi_1,...,\xi_{n+1},\xi_{n+1}]$. The bounds in (16) are valid for each $f \in C[-1, 1]$ whose derivative exists at all of the zeros of the Chebyshev polynomial T_{n+1}.

References

1. S.N. Bernstein . Lecons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réele, Gauthier-Villars, Paris, 1926.
2. Philip J. Davis and Philip Rabinowitz . Methods of Numerical Integration, Academic Press, New York, 1975.
3. David Elliott and Peter J. Taylor. 'Polynomial approximation errors for functions of low order continuity', Constructive Approximation (in the press).
4. E. Isaacson and H.B. Keller . Analysis of Numerical Methods, Wiley, New York, 1966.
5. G.M. Phillips and P.J. Taylor . Theory and Applications of Numerical Analysis, Academic Press, London, 1973.
6. Blagovest Sendov and Vasil A. Popov . The Averaged Moduli of Smoothness, Wiley, New York, 1988. (First published as Usredneni Moduli na Gladkost, Bulgarian Mathematical Monographs Volume 4, Publishing House of the Bulgarian Academy of Sciences, Sofia.)

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