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A GAMLEN-GAUDET TYPE THEOREM IN THE UNCONDITIONAL PART OF $L_\infty(0,1)$ WITH RESPECT TO THE HAAR SYSTEM

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We consider the space U_∞, the unconditional part of the space $L_\infty(0,1)$ with respect to the Haar system $(h_i)_{i=1}^{\infty}$ and we classify the isomorphy types of the closed subspaces of U_∞ spanned by the subsequences of the Haar system. More precisely we show that there are only ten such distinct subspaces. This result is motivated by the previous work of Gamlen-Gaudet [3] and P. F. X. Müller [7].

§0 – Introduction

Let $k=0,1,\ldots;\ 0\leq m\leq 2^k-1$. Then we define the dyadic interval $(km)=$ $=(2^{-k}m,2^{-k}(m+1)]$ and the Haar function h_{km} which is 1 on the left half of $(2^{-k}m,2^{-k}(m+1)]$, -1 on the right half and zero elsewhere.

We use also the notation h_n instead of h_{km}, where $n=2^k+m$ and $h_0(x)=1$ for $x\in[0,1]$.

1973 j. Gamlen and R. Gaudet characterized the closed subspaces of $L_p(0,1)$, $(1<p<\infty)$, spanned by subsequences of the Haar system $(h_i)_{i=0}^{\infty}$. They proved that there are only two distinct (up to an isomorphism) classes of such subspaces, namely l_p and $L_p(0,1)$ itself.

It seems to be natural to ask ourselves what can be said about the subspaces $[h_{k_i}]_{i=1}^{\infty}$ in the extremal cases $p=1$ or $p=\infty$.

It is clear that the unconditionality of the Haar system in the space $L_p(0,1)$, $(1<p<\infty)$, plays an important role in the proof of the Gamlen-Gaudet theorem, so since the Haar system is a conditional basis in $L_1(0,1)$

an moreover $L_\infty(0,1)$ is a nonseparable space, we can not expect ourselves to a straightforward extension of the Gamlen-Gaudet theorem in these extremal cases.

1987 P. F. X. Müller found the right answer to this problem for p=1. More precisely he characterized the closed subspaces $[h_{km}]_{(k,m)\in B}$, where $B\subseteq \mathbb{N}$ is an infinite subset, of the dyadic Hardy space H_1.

We recall the definition (cf. [7]) of H_1: for $f=\sum_{(km)} a_{km}h_{km}\in L_1(0,1)$, put $S(f)=(\sum_{(km)} a_{km}^2 h_{km}^2)^{1/2}$, $||f||_{H_1}:=\int S(f)$ and $H_1:=\{f\in L_1; ||f||_{H_1}<\infty\}$.

But it was remarked in [2] that H_1 is the unconditional part of $L_1(0,1)$ with respect to the Haar system $(h_i)_{i=0}^\infty$, i.e. $H_1=\{f\in L_1;\ \int f dt=0$, s.t. $f=\sum_{i=1}^\infty a_i h_i$ the series being unconditionally convergent in $L_1(0,1)\}$ and $||f||_{H_1}$ is equivalent to $\sup_{\varepsilon_i=\pm 1} ||\sum_{i=1}^\infty a_i\varepsilon_i h_i||_{L_1}$.

So instead of $L_1(0,1)$ Müller considered H_1, the unconditional part of L_1, and he was able to find that there are only three distinct subspaces $[h_{k_i}]_i$ of H_1, namely l_1, H_1 and $(\sum_{n=1}^\infty H_1(n))_{l_1}$, where $H_1(n)$ is the subspace in H_1 spanned by the first 2^n-1 Haar functions.

Motivated by this last result we intend to characterize the closed subspaces $[h_{k_i}]_{i=1}^\infty$ of the unconditional part of the space $L_\infty(0,1)$ with respect to the Haar system and we get ten such subspaces.

In order to describe them we first consider the subspace a_0, the unconditonal part of the space c_0 (of all null-convergent sequences) with respect to a system $(c_i)_{i=1}^\infty$ similar to the Haar system of functions and we classify the subspaces $[c_{k_i}]_{i=1}^\infty$ of a_0.

Let us mention that the unexplained terminology used in this paper follows [5].

§1 - The space a_0 and Gamlen-Gaudet theorem for a_0

In order to classify the subspaces $[h_{k_i}]_{i=1}^\infty$ of U_∞ it is useful to study the discrete variant of the space U_∞ and first of all we will introduce the system $(c_i)_{i=1}^\infty$ of sequences, system which is similar to the classical

Haar system of functions.

We describe now the so-called discrete Haar system indexed w.r. to $\mathbb{N}^2$ $(c_{ij})_{i=0,j=0}^{\infty}$ as follows:

$c_{i,0}=\sum_{i=0}^{2^i-1} e_j-\sum_{j=2^i}^{2^{i+1}-1} e_j$ and $c_{i,k}=U_{2^{i+1}k}(c_{i,0})$ where $i=0,1,2,\dots$ and $U_n(x):=$ $:=(x_n,x_{n+1},\dots)$ for $x:=(x_0,x_1,\dots)$. Here $(e_n)_{n=1}^{\infty}$ stands for the standard unit basis in any space of sequences.

Sometimes it is useful to rearrange the system $(c_{ij})_{i=0,j=0}^{\infty}$ w.r. to $\mathbb{N}^*$ proceeding as follows:

$c_{2^i}:=c_{i,0}$ for $i=0,1,2,\dots$

$c_{2^0+2^i}:=c_{i-1,2^0}$ for $i=1,2,\dots$

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$c_{\sum_{j=0}^{k-1}\varepsilon_j 2^j+2^k+2^i}:=c_{i-k-1,\ \sum_{j=0}^{k-1}\varepsilon_j 2^j+2^k}$, where $\varepsilon_j=0$ or 1, $1\leqslant k<i$ and $i=2,3,\dots$

In what follows we call the discrete Haar system anyone of these two representations: $(c_i)_{i=1}^{\infty}$ or $(c_{ij})_{i=0,j=0}^{\infty}$.

The following definition is essentially contained in [5]. Let E be a real Banach space and $(x_n)_{n=1}^{\infty}$ a basic sequence in E. We call the unconditional part of E w.r. to $(x_n)_{n=1}^{\infty}$ and we denote by $Unc(E,x_n)$ the space $Unc(E,x_n):=\{x\in E$ s.t. $x=\sum_{n=1}^{\infty} a_n x_n$ unconditionally converges in E$\}$ endowed with the norm $||x||:=\sup_{\varepsilon_j=\pm1} ||\sum_{i=1}^{\infty}\varepsilon_i a_i x_i||_E$ for $x=\sum_{n=1}^{\infty} a_n x_n$. It is well-known that $Unc(E,x_n)$ is a Banach space. Recall that a Schauder basis $(x_n)_{n=1}^{\infty}$ unconditionally convereges in E iff $\sup_{\varepsilon_i=\pm1;||\Sigma a_i x_i||\leqslant 1} ||\sum_{i=1}^{\infty} a_i\varepsilon_i x_i||_E<\infty$.

Now we study the properties of the discrete Haar system $(c_i)_{i=1}^{\infty}$ in the space c_0.

Let $f=(f_i)_{i=1}^{\infty}\in c_0$ and denote by $a_j(f):=|\text{supp } c_j|^{-1}\sum_{i\in \text{supp } c_j} f_i c_j(i)$.

Using the classical Cèsaro theorem we have:

Lemma 1.1 The Fourier-Haar series $\sum_{j=1}^{\infty} a_j(f)c_j$ coordinatewise converges to $f\epsilon c_0$.

But more is true, namely:

Theorem 1.2 The discrete Haar system $(c_i)_{i=1}^{\infty}$ is a Schauder basis of c_0.

Proof By Lemma 1.2 it follows that the set $\{\sum_{i=1}^{k} a_i c_i,\ a_i\epsilon\mathbb{R},\ k\epsilon\mathbb{N}\}$ is dense in c_0. We show that $\|\sum_{i=1}^{n} a_i c_i\|_{c_0} \leq 2\|\sum_{i=1}^{m} a_i c_i\|_{c_0}$ for all $a_i\epsilon\mathbb{R}$ and all $n\leq m$. Then Nikolskii's criterion ends the proof.

Case I If $n=2^k<n\leq 2^{k+1}-1$, the above inequality follows remarking that $|a|\leq|a+b|\vee|a-b|$ for all $a,b\epsilon\mathbb{R}$.

Case II For $n=2^k-1<m=2^{k+l}$ for $l\geq 1$, we have $\sum_{i=1}^{2^k-1} a_i c_i = E(f|\sigma')-E(f|\sigma)$, where $f=\sum_{i=1}^{2^{k+l}} a_i c_i$,σ is the finite algebra generated by the following subsets $\{0,1,\dots,2^k-1\},\{2^k,\dots,2^{k+1}-1\},\{2^{k+1},\dots,2^{k+2}-1\},\dots,$ $\{2^{k+l-1},\dots,2^{k+l}-1\}$, σ' is the algebra generated by $\{0\},\{1\},\dots,\{2^k-1\},$ $\{2^k,\dots,2^{k+1}-1\},\dots,\{2^{k+l-1},\dots,2^{k+l}-1\}$ and $E(f|\sigma)$ is the conditional expectation of f with respect to σ. Hence $\|\sum_{i=1}^{n} a_i c_i\|\leq 2\|\sum_{i=1}^{m} a_i c_i\|$ for all $a_i\epsilon\mathbb{R}$.

The third case, the general one, follows easy from the cases I and II. ∎

It is clear that $(c_i)_{i=1}^{\infty}$ is not an unconditional basis in c_0 and in what follows we denote by $a_0:=\mathrm{Unc}(c_0,c_i)=\{f\epsilon c_0;\ f=\sum_{i=1}^{\infty} a_i c_i,$ the series being unconditionally convergent in $c_0\}$, endowed with the norm

$$|||f|||=\sup_{\epsilon_j=\pm 1}\|\sum_{j=1}^{\infty} a_j\epsilon_j c_j\|_{c_0}=\sup_{i\epsilon\mathbb{N}}\sum_{\{j;\,i\epsilon\,\mathrm{supp}\,c_j\}}|a_j|. \qquad (I.1)$$

It is clear that $a_0=\{f\epsilon c_0;\ \text{s.t. } |||f|||<\infty\}$.

Now we classify the subspaces of a_0 spanned by the subsequences of the basis $(c_i)_{i=1}^{\infty}$.

In the sequel B is an infinite subset of $\mathbb{N}^2$.

We have the following obvious types of subspaces a_B of a_0 generated by $(c_{ij})_{(i,j)\varepsilon B}$.

1. $B=\{(j_k,0);\ 1\leq j_1<j_2<\ldots\}\cup A$, where A is a finite family of indices of the form (j,k). It is easy to show, using (1.1), that $a_B\simeq l_1$.

2. If there is a constant M>0 such that for each $i\varepsilon\mathbb{N}$, $|\{j;i\varepsilon \text{supp } c_{jk},\ (j,k)\varepsilon B\}|\leq M$, then $a_B\simeq c_0$.

3. If $B=B_1\cup B_2$, where B_1 (resp. B_2) is the family from case 1. (resp. from case 2), then $a_B\simeq l_1\oplus c_0$.

4. Let $A_n:=\{(j,k);\ 0\leq j\leq n-1,\ 2^{n-j-1}\leq k\leq 2^{n-j}-1\}$ and $B_n=\{(j,2^{n-j-1});\ 0\leq j\leq n-1$, where n=1,2,... It is obvious that, putting $B=\bigcup_{n=1}^{\infty} B_n$, we get $a_B\simeq(\sum_{n=1}^{\infty} l_1(n))_{c_0}$.

5. If $B=B_1\cup B_4$, where B_1 (resp. B_4) is the family from the case 1. (resp. from the case 4.) then $a_B\simeq l_1\oplus(\sum_{n=1}^{\infty} l_1(n))_{c_0}$.

6. If $B=\bigcup_{n=1}^{\infty} A_n$, we denote by X_0 the space a_B.

7. Let $B=\mathbb{N}^2$. Then a_B coincides with a_0.

In what follows we will show that only these seven isomorphy types of the subspaces a_B of a_0 are possible. This is the Gamlen-Gaudet theorem for a_0.

It is known (cf. [1]) that the first five of these seven subspaces are pairwise nonisomorphic.

<u>Proposition 1.3 The space</u> X_0 <u>is not isomorphic to anyone of the first five spaces above.</u>

<u>Proof</u> Since l_1 and c_0 have an unique unconditional basis it is clear that X_0 is not isomorphic neither to l_1 nor to c_0.

If X_0 would be isomorphic to $l_1\oplus c_0$, then $(\sum_n l_1(n))_{c_0}$ should be complemented in $l_1\oplus c_0$. This is impossible since by a theorem of Edelstein and Wojtaszczyk (cf. [5]- Remark 2 after Theorem 2.c.13) every complemented subspace of $l_1\oplus c_0$ should be isomorphic to one of the spaces

l_1, c_0 or $l_1 \oplus c_0$ and by [1] $(\sum_{n=1}^{\infty} l_1(n))_{c_0}$ is not isomorphic to anyone of these three spaces.

We will show now that $X_0 \not\approx (\sum_{n=1}^{\infty} l_1(n))_{c_0}$.

Assume the contrary is true and then by [1] it would follow that the basis $(c_{ij})_{(i,j)\in \cup_n A_n}$ of X_0 would be equivalent to the standard basis of $(\sum_{n=1}^{\infty} l_1(n))_{c_0}$.

This in turn imply that there is a constant $M>0$ indepent of n and a partition $(\tau_s)_{s=1}^{r}$ of the set A_n into pairwise disjoint subsets such that:

$$M^{-1} \max_{1\leq s\leq r} \sum_{\tau_s} |a_{ki}| \leq \left\|\left| \sum_{A_n} a_{ki} c_{ki} \right\|\right| \leq M \max_{1\leq s\leq r} \sum_{\tau_s} |a_{ki}| \qquad (1.2)$$

for all $a_{ki} \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Let us put $l=(j_0, j_1, \ldots, j_{n-1})$ such that $\bigcap_{i=0}^{n-1} \operatorname{supp} c_{ij_i} \neq \emptyset$ and let us denote $B_{nl} := \{(i,j_i); 0\leq i\leq n-1,\ l=(j_0,\ldots,j_{n-1})\}$. We have 2^n such distinct subsets B_{nl} of A_n. Fix B_{nl} and let $\tau_{i_1}, \ldots, \tau_{i_k}$ a partition of B_{nl} such that $\tau_i \cap B_{nl} \neq \emptyset$ for all $1\leq m\leq k$. Then, by (1.1.) and (1.2), we get the relation:

$$k \leq M. \qquad (1.3)$$

Moreover, for each $1\leq s\leq r$ and $0\leq i\leq n-1$ put $C_{i,s} := \{j;\ \text{s.t. } (i,j)\in \tau_s$ and $\operatorname{supp} c_{ij}$ be pairwise disjoint$\}$.

Then we have, by (1.1) and (1.2):

$$|C_{i,s}| \leq M \qquad (1.4)$$

for $1\leq s\leq r$, $0\leq i\leq n-1$.

Next we show that the relations (1.2), (1.3) and (1.4) lead to a contradiction.

Take $n=Mk$ for $k\in \mathbb{N}$ which will be fixed later and take M in (1.2) sufficiently large such that

$$2^{M-1} \geq M(M+1). \qquad (1.5)$$

Let $B_{nl}^{M} = \{(i,j_i)\in B_{nl};\ M(k-1)\leq i\leq Mk-1\}$ for all $B_{nl} \subset A_n$ and let $M_1 \leq M$

the number of all subsets τ_{i_j}, $1\leq j\leq M_1$, such that $\tau_{i_j}\cap B^M_{nl}\neq\emptyset$.

If $M_1=M$ for all B_{nl}, by (1.3) it follows that all the subsets $B_{nl}\setminus B^M_{nl}$ are covered by the subsets $(\tau_{i_j})^M_{j=1}$, hence at least one of the subsets $(\tau_{i_j})^M_{j=1}$, let us say τ_{i_1}, has the property that

$$|\tau_{i_1}|\geq M^{-1}(2^{Mk}-2^M) \ . \tag{1.6}$$

Taking $k\in\mathbb{N}$ sufficiently large we have that:

$$|\tau_{i_1}|\geq 2M^2k \tag{1.7}$$

and putting $a_{ij}=1$, for $(i,j)\in\tau_{i_1}$ and 0 otherwise, we get the false relation $2Mk\leq Mk$.

Consequently there are $2^{M(k-1)}$ subsets $B_{nl}\subset A_n$ such that $1\leq M_1<M$ and moreover there is an unique $j_k\in\mathbb{N}$ with the property that $(M(k-1),j_k)$ belongs to these $2^{M(k-1)}$ subsets. Take one of these subsets B_{nl} and denote by B^{2M}_{nl} the set $\{(i,j_i)\in B_{nl};\ M(l-2)\leq i\leq M(k-1)-1\}$.

Each of these B^{2M}_{nl} is covered by $0\leq M_2<M$ sets $(\tau_{i_j})^{M_1+M_2}_{j=M_1+1}$. If $M_2=0$ for all these B^{2M}_{nl}, then all B^{2M}_{nl} are covered by $\tau_{i_1},\dots,\tau_{i_{M_1}}$ and since $|\{(M(k-2),j_1)\in\bigcup_1 B^{2M}_{nl}\}|=2^{M-1}$, it follows that at least one of the sets $\tau_{i_1},\dots,\tau_{i_{M_1}}$ has $M_12^{M-1}\geq M+1$ elements $(M(k-2),j_1)$, which clearly contradicts (1.4). Consequently there are $2^{M(k-2)}$ sets $B_{nl}\subsetneq A_n$ such that B^{2M}_{nl} is covered by M_2 sets $(\tau_{i_j})^{M_1+M_2}_{j=M_1+1}$, where $1\leq M_2<M$. Assuming $M_1+M_2<M$, we proceed as above finding the sets $\tau_{i_{M_1+M_2+1}},\dots,\tau_{i_{M_1+M_2+M_3}}$ for $1\leq M_3<M$ and after at most M steps we find $2^{M(k-M)}$ sets $B'_{nl}\subsetneq A_n$ such that $|\{(0,j_1)\in\bigcup_1 B'_{nl}\}|=2^{Mk-M^2}$ and such that B'_{nl} be covered by at most M sets $\tau_{i_1},\dots,\tau_{i_M}$. Consequently there is a set, say τ_1, such that

$|\{(0,j_1)\in\tau_1\}|\geq M^{-1}2^{Mk-M^2}$. Take k verifying (1.7) and

$$M^{-1}2^{Mk-M^2}\geq M+1 . \qquad (1.8)$$

Then it follows that (1.4) is violated.

So, (1.2) does not hold for any n, thus $X_0 \not\approx (\sum_{n=1}^{\infty} l_1(n))_{c_0}$, and by Proposition 4.1 - [1], X_0 is not a complemented subspace of $(\sum_{n=1}^{\infty} l_1)_{c_0}$.

If $X_0\approx l_1\oplus(\sum_{n=1}^{\infty} l_1(n))_{c_0}$, then X_0 would be complemented into $(\sum_{n=1}^{\infty} l_1)_{c_0}$, which we have seen that is false. ■

Proposition 1.4 All seven spaces a_B are pairwise nonisomorphic.

Proof Obviously

$$X_0\oplus l_1\approx a_0. \qquad (1.9)$$

By Proposition 1.3, relation (1.9) and by theorem of Edelstein and Wojtaszczyk it follows that a_0 is not isomorphic to the first five spaces of the list of a_B. It remains to prove that

$$X_0\not\approx a_0. \qquad (1.10)$$

But using the fact that X_0 is an order continuous Banach lattice with respect to the order given by the basis $(c_j)_{j=1}^{\infty}$ we get X_0 contains l_1 (cf. [6]), so (1.10) is proved. ■

Theorem 1.5 The seven spaces listed above are all the subspaces of a_0 of the type a_B.

Proof Let $B\subset N^2$ an infinite subset. Then $B=B_1\cup B_2$, where $B_1\subseteq \bigcup_{n=1}^{\infty} A_n$ and $B_2\subset\{(i,0);i\in N\}$. By (1.9) it suffices to show that (up to an isomorphism) c_0, $(\sum_{n=1}^{\infty} l_1(n))_{c_0}$ and X_0 are all the subspaces a_B of X_0.

Since $X_0=(\sum_{n=1}^{\infty} a_{A_n})_{c_0}$ it is clear that one of the following situations occurs.

1. There is a subsequence $(n_i)_{i\in N}\subseteq N$ such that the set $\{c_{kj};(k,j)\in B\cap A_{n_i}\}$ contains the elements $c_{k(l),j}$, (with $0\leq k(l)\leq n_i$ and $1\leq l\leq r_i$ such that $\lim_{i\to\infty} r_i=\infty$), such that there is j so that the support of $c_{k(r_i),j}$ intersects the supports of at least two elements $c_{k(r_i-1),s}$, each of these

elemnts having supports which intersect the support of at least two elemts $c_{k(r_i-2),s}$, etc.

The nonexistence of such a sequence $(n_i)_i$ implies the existence of a n_0 such that for each $n \geq n_0$ the sets $\{i;(i,j) \in B \cap A_n\}$ be in one of the following situations:

2. There is a natural number K independent of n, such that $|\{i;(i,j) \in B \cap A_n\}| \leq K$ for all $n \geq n_0$.

3. $\sup_n |\{i,(i,j) \in B \cap A_n\}| = \infty$, but all the supports of elements c_{ij}, with $(i,j) \in B \cap A_n$ for $n \geq n_0$, are pairwise disjoint.

4. Denoting by $k(l) := |\{i,(i,j) \in B \cap A_n\}|$ for $1 \leq l \leq r$, we have $\lim_{n\to\infty} k(r_n) = \infty$ and moreover $B \cap A_n = \bigcup_{k=1}^{M} C_k$, where M does not depend on n, such that the support of each $c_{k(r_n),j}$, with $(k(r_n),j) \in C_k$, intersects only one support of an element $c_{k(r_n-1),s}$ with $(k(r_n-1),s) \in C_k$ for $k=1,2,\dots,M$; etc.

In cases 2. and 3., obviously $a_B \simeq c_0$ and in the case 4. it follows that $a_B \simeq (\sum_{n=1}^{\infty} l_1(n))_{c_0}$.

In case 1. $Y := a_B$ contains a complemented subspace isomorphic to X_0, namely the subspace $Z := (\sum_{n=1}^{\infty} Z_n)_{c_0}$, where $Z_n := [c_{k(r_n),j}; c_{k(r_n-1),s_1}; c_{k(r_n-1),s_2}; c_{k(r_n-2),s_{11}}; c_{k(r_n-2),s_{12}}; \dots]$, where $\operatorname{supp} c_{k(r_n-1),s_i} \cap \operatorname{supp} c_{k(r_n),j} \neq \emptyset$ for $i=1,2$ and similar relations for $c_{k(r_n-2),s_{1i}}$ with $i=1,2$; etc.

Since

$$X_0 \simeq (\sum_{n=1}^{\infty} X_0)_{c_0} \qquad (1.11)$$

the decomposition method of Pelczyński gives us that $Y \simeq X_0$, which ends the proof of Theorem 1.5. ∎

Let us remark that the Haar system $(h_{nk})_{n=0,k=0}^{\infty\ \ 2^k-1}$ is a basic sequence in $L_\infty(0,1)$, so we can consider the Banach space $U_\infty := Unc(L_\infty(0,1),h_{nk})$. The norm in U_∞ is given by:

$$|||x||| = \sup_{n\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{k\leq n;\, t\in \operatorname{supp} h_{kj}} |a_{kj}| \tag{2.1}$$

Let $B\subset\mathbb{N}^2$ an infinite subset. We intend to classify the subspaces $U_B := [h_{nk}]_{(n,k)\in B}$ of U_∞.

First we list the obvious subspaces $U_B\subseteq U_\infty$.

<u>Proposition 2.1</u> <u>Besides the seven subspaces a_B of a_0 there are three other subspaces U_B, namely $(\sum_{n=1}^{\infty} l_1)_{c_0}$, $(\sum_{n=1}^{\infty} l_1)_{c_0}\oplus X_0$ and U_∞. All these ten spaces are pairwise nonisomorphic.</u>

<u>Proof</u> All these spaces have a natural representation as subspaces of the type U_B. Let us mention for instance that $U_B\simeq U_\infty$ if $B=\{(n_i,k_{ij});\ n_1<n_2<n_3<\ldots,\ 1\leq j\leq 2^{i-1},\ \operatorname{supp} h_{n_ik_{ij}}\cap \operatorname{supp} h_{n_ik_{il}}=\emptyset$ for $j\neq l$ and $\operatorname{supp} h_{n_ik_{ij}}\cap \operatorname{supp} h_{n_{i+1}k_{i+1,l}}\neq\emptyset,\ l=2j-1$ or $2j\}$. Moreover, for $B=\{(n_i,k_i);\ n_1<n_2<\ldots,\ \operatorname{supp} h_{n_ik_i}\cap \operatorname{supp} h_{n_{i+1}k_{i+1}}\neq\emptyset,\ i\in\mathbb{N}\}$, then $U_B\simeq l_1$ and for $B=\bigcup_{i=1}^{\infty} B_i$, where B_i are the above sets such that $\operatorname{supp} h_{nk}\cap \operatorname{supp} h_{nl}=\emptyset$ for all $(n,k)\in B_i$, $(m,l)\in B_j$ for $i\neq j$, then $U_B\simeq(\sum_{n=1}^{\infty} l_1)_{c_0}$. Finally for $B=\bigcup_{n=1}^{\infty} A_n$, where $A_n=\{(m_i,k_{ij});\ i\leq n;\ m_1<m_2<\ldots<m_n;\ 1\leq j\leq 2^{i-1},\ \operatorname{supp} h_{m_ik_{ij}}\cap \operatorname{supp} h_{m_ik_{il}}=\emptyset$ for $j\neq l$ and $\operatorname{supp} h_{m_ik_{ij}}\cap \operatorname{supp} h_{m_{i+1}k_{i+1,l}}=\emptyset$ for $l=2j-1$ or $l=2j\}$ and moreover $\operatorname{supp} h_{mk}\cap \operatorname{supp} h_{lj}=\emptyset$ for all $(m,k)\in A_n$ and $(l,j)\in A_p$ for $n\neq p$, then $U_B\simeq X_0$.

It remains to show that the last three spaces of the above list are pairwise nonisomorphic and moreover nonisomorphic to any first seven spaces of the list. It is known (cf. [1]) that $(\sum_{n=1}^{\infty} l_1)_{c_0}$ is nonisomorphic to the first five spaces. Since X_0 is not isomorphic to a complemented subspace of $(\sum_{n=1}^{\infty} l_1)_{c_0}$, it follows that $(\sum_n l_1)_{c_0} \not\approx X_0$ and $(\sum_n l_1)_{c_0} \not\approx l_1 \oplus X_0$.

Similarly we may show that $(\sum_n l_1)_{c_0} \oplus X_0$ is not isomorphic to anyone of the first eight spaces of the list except perhaps $l_1 \oplus X_0$ (this case will be studied later).

Simple arguments show that U_∞ is not isomorphic to all the spaces which precede it in this list except $l_1 \oplus X_0$ and $(\sum_n l_1)_{c_0} \oplus X_0$.

It remains to show that $(\sum_n l_1)_{c_0} \oplus X_0$ is not isomorphic neither to $l_1 \oplus X_0$ not to U_∞ and U_∞ is not isomorphic to $l_1 \oplus X_0$.

In order to prove these three nonisomorphisms we will use Theorem 1.1 - [1]. First let us show that $U_\infty \not\approx l_1 \oplus X_0$. Assume the contrary is true. Applying Theorem 1.1 - [1] for m=2, for l_1 and X_0 instead of the spaces $(X_i)_{i=1}^2$ from the statement of this theorem and for $z_n = T(h_n)$, $T: U_\infty \to l_1 \oplus X_0$ being an isomorphism, we get a partition of $\mathbb{N}$, $(A_i)_{i=1}^2$ such that $(h_n)_{n\in A_1}$ be equivalent to $(P_1 Th_n \otimes r_n)_{n\in A_1} \subseteq L_2([0,1],X_0)$ and $(h_n)_{n\in A_2}$ be equivalent to $(P_2 Th_n \otimes r_n)_{n\in A_2} \subseteq L_1([0,1],l_1)$, P_1 and P_2 being the canonical projections of $X_0 \oplus l_1$ onto its components.

Since $l_1 \not\subset X_0$ it follows that $(h_i \otimes h_j)_{i,j}$ is a shrinking basis in $L_2([0,1],X_0)$ and consequently $l_1 \not\subset [h_n]_{n\in A_1} \subset L_2([0,1],X_0)$. Denoting by $\sigma_i := \{t\in[0,1]$, t belongs to an infinity of the supports of h_n, $n\in A_i\}, i=1,2$, it follows that $\sigma_1 = \emptyset$ and thus $|\sigma_2| = \infty$. Hence $(\sum_n l_1)_{c_0} \subseteq [h_n]_{n\in A_2}$ (cf. Corollary 2.3 and the proof which follows after it). This inclusion is impossible, otherwise $c_0 \subset L_1([0,1],l_1)$ (cf. [9]).

Completely similar one shows that $(\sum_n l_1)_{c_0} \oplus X_0 \not\approx l_1 \oplus X_0$.

Now we show that $U_\infty \not\approx (\sum_n l_1)_{c_0} \oplus X_0$. otherwise it would exist an isomorphism $T: U_\infty \to (\sum_n l_1)_{c_0} \oplus X_0$, consequently it would exist a decomposition of $\mathbb{N}$ into $A_1 \cup A_2$ such that $[h_n]_{n\in A_2} \subset L_2([0,1],X_0)$ and $(h_n)_{n\in A_1}$ would be

M-equivalent to $(u_n\otimes r_n)_{n\epsilon A_1}$ into $L_\infty([0,1],(\sum_n l_1)_{c_0})$. Since $l_1\not\subset[h_n]_{n\epsilon A_2}$ it follows that $U_\infty\subset[h_n]_{n\epsilon A_1}$ (see the proof of Proposition 2.2).

By the proof of Theorem 1.1 - [1] it follows the existence of an M non depending on $k\epsilon N$ such that, denoting by $B_k=A_1\cap\{1,2,\dots,k\}$, we have that $(h_n)_{n\epsilon B_k}$ is M-equivalent to $(\hat{x}_n)_{n\epsilon B_k}\epsilon[\sum_1^{2^k}(\sum_{n=1}^{\infty} l_1)_{c_0}]_\infty$ and moreover $[h_n]_{n\epsilon B_k}$ is M-complemented in this last space for all $k\epsilon N$. By $\hat{x}_n$ we denote $(\epsilon_n^1 u_n,\epsilon_n^2 u_n,\dots,\epsilon_n^{2^k} u_n)$, where $\epsilon_n^i=\pm1$ for $1\leqslant i\leqslant 2^k$ and $u_n=P_1Th_n$ for $n\epsilon B_k$.

But there is the isometry $U_k:[\sum_1^{2^k}(\sum_{n=1}^{\infty} l_1)_{c_0}]_\infty\to(\sum_n l_1)_{c_0}$ for $k\epsilon N$. put now $v_{nk}=U_k\hat{x}_n$ for $n\epsilon B_k$ and it follows that $(h_n)_{n\epsilon B_k}$ is M-equivalent to $(v_{nk})_{n\epsilon B_k}$ for all $k\epsilon N$. Moreover, let Q_k be the projection of $[\sum_1^{2^k}(\sum_{n=1}^{\infty} l_1)_{c_0}]_\infty$ onto $[\hat{x}_n]_{n\epsilon B_k}$. Then $R_k:=U_kQ_kU_k^{-1}$ is the projection of $(\sum_{n=1}^{\infty} l_1)_{c_0}$ onto $[v_{nk}]_{n\epsilon B_k}$ and $||R_k||\leqslant||Q_k||\leqslant M$ for all $k\epsilon N$.

Hence $(h_n)_{n\epsilon B_k}$ is M-equivalent to the M-unconditional and M-complemented sequence $(v_{nk})_{n\epsilon B_k}$ in $(\sum_n l_1)_{c_0}$ for all $k\epsilon N$.

By Proposition 4.1 - [1] it follows that there is $K>0$ non depending on $k\epsilon N$, such that for all $k\epsilon N$ there is a partition of B_k into the sets τ_s, $1\leqslant s\leqslant r_k$, such that

$$K^{-1}\max_s\sum_{n\epsilon\tau_s}|a_n|\leqslant|||\sum_{n\epsilon B_k}a_nh_n|||\leqslant K.\max_s\sum_{n\epsilon\tau_s}|a_n|$$

for all $a_n\epsilon R$.

Since $X_0\subseteq U_\infty\subsetneq[h_n]_{n\epsilon A_1}$, it follows that for a fixed $l\epsilon N$, there is $k(l)\epsilon N$ such that for the set $A_l:=\{(j,k);0\leqslant j\leqslant n-1,\ 2^{n-j-1}\leqslant k\leqslant 2^{n-j}-1\}$, there is a set $A_l'\subseteq B_{k(l)}$ so that $[h_{jk}]_{(j,k)\epsilon A_l}$ is isometric to $[h_n]_{n\epsilon A_l'}$.

Taking $\tau_s'=\tau_s\cap A_l'$ we have:

$$K^{-1}\max_s\sum_{n\epsilon\tau_s'}|a_n|\leqslant|||\sum_{n\epsilon A_l'}a_nh_n|||\leqslant K.\max_s\sum_{n\epsilon\tau_s'}|a_n|$$

for all $a_n \in R$ and all $l \in N$, which is impossible by the proof of Proposition 1.3. ∎

Denote by σ the set $\{t \in [0,1];$ t belongs to an infinity of the supports of h_{nk}; $(n,k) \in B\}$.

As in the proof of Theorem 1.5 we can show that $\sigma = \emptyset$ implies that U_B is isomorphic to one of the following spaces: c_0, $(\sum_{n=1}^{\infty} l_1(n))_{c_0}$ and X_0.

If $|\sigma| < \infty$, then U_B is obviously isomorphic to one of the following spaces: l_1, $c_0 \oplus l_1$, $l_1 \oplus (\sum_{n=1}^{\infty} l_1(n))_{c_0}$, $l_1 \oplus X_0$. It remains to show that $|\sigma| = \infty$ implies that U_B must be isomorphic to $(\sum_{n=1}^{\infty} l_1)_{c_0}$, $(\sum_{n=1}^{\infty} l_1)_{c_0} \oplus X_0$ or U_∞ itself.

Now we denote by σ' the set of accumulations points of σ.

<u>Proposition 2.2</u> <u>Let $\sigma \subset \sigma'$. Then $U_B \approx U_\infty$.</u>

<u>Proof</u> Let $t_0 \in \sigma$. There is a dyadic interval $I_{k_0 i_0}$ such that supp $h_{k_0 i_0} = I_{k_0 i_0}$ and $t_0 \in \overset{\circ}{I}_{k_0 i_0}$. Since $t_0 \in \sigma'$ there are $t_1, t_2 \in \sigma$ with $t_1 \neq t_2$ such that $t_1, t_2 \in I_{k_0 i_0}$. Hence there is $I_{k_1 i_1}, I_{k_1 i_2} \subset I_{k_0 i_0}$ such that $I_{k_1 i_1} \cap I_{k_1 i_2} = \emptyset$, $t_1 \in I_{k_1 i_1}$, $t_2 \in I_{k_1 i_2}$. Since $t_1, t_2 \in \sigma'$, there are $t_{11}, t_{12} \in \sigma \cap I_{k_1 i_1}$ and t_{21}, $t_{22} \in \sigma \cap I_{k_1 i_2}$. t_{11}, t_{12} belonging to σ it follows that there are $I_{k_2 i_3}$, $I_{k_2 i_4} \subset I_{k_1 i_1}$ so that $I_{k_2 i_3} \cap I_{k_2 i_4} = \emptyset$ and $t_{11} \in I_{k_2 i_3}$, $t_{12} \in I_{k_2 i_4}$. Similarly there are disjoint intervals $I_{k_2 i_5}, I_{k_2 i_6}$ included into $I_{k_1 i_2}$ such that $t_{21} \in I_{k_2 i_5}$, $t_{22} \in I_{k_2 i_6}$ and so on.

It follows easily that $[h_{k_n i_j}]_{n=1, 2^n-1}^{\infty\ 2^{n+1}-2} \approx U_\infty$ and the decomposition method of Pelczynski shows that $U_B \approx U_\infty$. ∎

<u>Corollara 2.3</u> <u>If σ is an uncountable set, then $U_B \approx U_\infty$.</u>

<u>Proof</u> By a well-known (cf. [8]) theorem it follows that $\sigma = \sigma_1 \cup \sigma_2$, where σ_2 is a perfect set (i.e. $\sigma_2 = \sigma_2'$). Consequently U_B contains a complemented subspace isomorphic to U_∞, hence U_B itself is isomorphic to U_∞. ∎

Now it remains the case that σ is a countable set so that $\sigma\not\subset\sigma'$.

Put then $\tau_0:=\sigma\setminus\sigma'\neq\emptyset$ and $\tau_1=\sigma'\cap\sigma$.

If $\sigma_1=\emptyset$, then it is easy to show that U_B is isomorphic either to $(\sum_n l_1)_{c_0}$ or to $(\sum_n l_1)_{c_0}\oplus X_0$.

If $\sigma_1\neq\emptyset$ and $\sigma_1\subset\sigma_1'$ by Proposition 2.2 it follows that $U_B\simeq U_\infty$. Hence let us assume that $\sigma_1\not\subset\sigma_1'$ put $\sigma_2=\sigma_1\cap\sigma_1'$ and proceeding as above we can define for each ordinal number of the second class $\alpha<\Omega$ (cf. [8]) the set σ_α. More precisely if α is a limit ordinal, that is, there is not a β so that $\beta+1=\alpha$, we take $\sigma^{(\alpha)}=\bigcap_{\beta<\alpha}\bar{\sigma}_\beta$ and $\sigma_\alpha=\sigma\cap\sigma^{(\alpha)}$.

It is known that there is an ordinal $\alpha_0<\Omega$ so that $\sigma_{\alpha_0}=\sigma_\beta$ for all $\beta>\alpha_0$ and we may assume $\sigma_{\alpha_0}=\emptyset$. So we get the sequence of pairwise disjoint sets $\tau_0=\sigma\setminus\sigma'$, $\tau_1=\sigma_1\setminus\sigma_1',\dots,\tau_\beta=\sigma_\beta\setminus(\sigma_\beta)'$ for all $\beta\leq\alpha_0$. Then $\sigma=\bigcup_{0\leq\beta\leq\alpha_0}\tau_\beta$.

Now let $\tau_0=\{t_{0i}\}_{i=1}^{\infty}$, where τ_{0i} are the isolated points of σ and let us denote by $B_0\subset B$ the set of indices such that $B_0=\bigcup_{i=1}^{\infty}B_{0i}$, where $t_{0i}\in\bigcap_{(k,j)\in B_{0i}}\operatorname{supp}h_{kj}$, $i=1,2,\dots$; and $\operatorname{supp}h_{kj}\cap\operatorname{supp}h_{ml}=\emptyset$ for $(k,j)\in B_{0i}$ and $(m,l)\in B_{0n}$ with $i\neq n$. Then $[h_{kj}]_{(k,j)\in B_0}$ is isometric to $(\sum_n l_1)_{c_0}$ by the map T_0 which carries the l^{th} function h_{kj} with $(k,j)\in B_{0i}$ onto e_{li} the l^{th} element of the i^{th} copy of l_1 into $(\sum_{n=1}^{\infty}l_1)_{c_0}$.

Put $\tau_1=\{t_{1i}\}_{i=1}^{\infty}$ and let us consider the set of indices $B_1\subset B\setminus B_0$, disjoint from B_0 such that $B_1=\bigcup_{i=1}^{\infty}B_{1i}$ with $t_{1i}\in\bigcap_{(k,j)\in B_{1i}}\operatorname{supp}h_{kj}$, $i=1,2,\dots$ and $\operatorname{supp}h_{kj}\cap\operatorname{supp}h_{ml}=\emptyset$ for $(k,j)\in B_{1i}$ and $(m,l)\in B_{1n}$ with $i\neq n$.

Next we show that there is an isometry $T_1:[h_i]_{i\in B_0\cup B_1}\to(\sum_n l_1)_{c_0}$ which extends T_0.

Then $|||\sum_{i=1}^{\infty}\sum_{(j,k)\in B_{1i}}a^i_{jk}h_{jk}+\sum_{i=1}^{\infty}\sum_{(j,k)\in B_{0i}}b^i_{kj}h_{kj}|||=$

$$=|||\sum_{i\in N_1}\sum_{(k,j)\in B_{0i}} b^i_{kj}h_{kj}|||V|||\sum_i\sum_{(k,j)\in B_{1i}} a^i_{kj}h_{kj}+\sum_{i\in N\setminus N_1}\sum_{(k,j)\in B_{0i}} b^i_{kj}h_{kj}|||,$$

where $N_1\subset N$ so that $\operatorname{supp} h_{kj}\cap\operatorname{supp} h_{ml}=\emptyset$ for $(k,j)\in\bigcup_{i\in N_1} B_{0i}$ and $(m,l)\in B_1$.

Hence we may assume that for all $(k,j)\in B_{0i}$, $\operatorname{supp} h_{kj}\subset\operatorname{supp} h_{ml}$, where $(m,l)\in B_1$ and it follows that: $|||\sum_i\sum_{(k,j)\in B_{1i}} a^i_{kj}h_{kj}+\sum_i\sum_{(k,j)\in B_{0i}} b^i_{kj}h_{kj}|||=$

$$=\bigvee_{i=1}^{\infty}|||\sum_{(k,j)\in B_{1i}} a^i_{kj}h_{kj}+\sum_{l=1}^{\infty}\sum_{B_{0l}} b^i_{kj}h_{kj}|||.$$

Fix $i\in N$. We have $|||\sum_{(k,j)\in B_{1i}} a^i_{kj}h_{kj}+\sum_{l=1}^{\infty}\sum_{(k,j)\in B_{0i}} b^l_{kj}h_{kj}|||=$(since $\operatorname{supp} h_{kj}\subset\operatorname{supp} h_{k_1m_1}$ for all $(k,j)\in\bigcup_{l\in N_1} B_{0l}$ and $(k_1,m_1)\in B_{1i}$ $)=|a^i_{k_1m_1}|+$

$$+\sup(|||\sum_{(k_2,j)\in B_{1i}} a^i_{k_2j}h_{k_2j}+\sum_{l\notin N_1}\sum_{(k,j)\in B_{0l}} b^l_{kj}h_{kj}|||,|||\sum_{l\in N_1}\sum_{B_{0l}} b^l_{kj}h_{kj}|||)\leqslant$$

$\leqslant$(using a set N_2 whose definition is like those of N_1)$<$

$$<|||\sum_{l\in N_1}\sum_{(k,j)\in B_{0l}} b^l_{kj}h_{kj}|||V|||\sum_{l\in N_2}\sum_{(k,j)\in B_{0l}} b^l_{kj}h_{kj}|||V(|a^i_{k_1m_1}|+|a^i_{k_2m_2}|+$$

$$+|||\sum_{(k_3,j)\in B_{1i}} a^i_{k_3j}h_{k_3j}+\sum_{l\notin N_1\cup N_2}\sum_{(k,j)\in B_{0l}} b^l_{kj}h_{kj}|||)\leqslant\ldots$$

$$\leqslant\bigvee_{m=1}^{\infty}|||\sum_{l\in N_m}\sum_{(k,j)\in B_{0l}} b^l_{kj}h_{kj}|||V(\sum_{j=1}^{\infty}|a^i_{k_jm_j}|).$$

It follows that $|||\sum_i\sum_{(k,j)\in B_{1i}} a^i_{kj}h_{kj}+\sum_i\sum_{(k,j)\in B_{0i}} b^i_{kj}h_{kj}|||\leqslant$

$$\leqslant\bigvee_{m=1}^{\infty}|||\sum_{l\in N_m}\sum_{B_{0l}} b^l_{kj}h_{kj}|||V\bigvee_{i=1}^{\infty}(\sum_{j=1}^{\infty}|a^i_{k_jm_j}|).$$

The reverse inequality is obvious $(h_i)_i$ being an unconditional basis. This gives us the isometry T_1 which extends T_0.

Consequently, for all finite ordinals n we may define isometries $T_n:[h_{kj}]_{(k,j)\in\bigcup_{1\leqslant i\leqslant n} B_i}\to(\sum_{n=1}^{\infty} l_1)_{c_0}$ which map the natural bases one onto another. If $\alpha\leqslant\alpha_0$ is a limit ordinal, the above procedure works unchanged and consequently using the transfinite induction we get an isometry

$$T_{\alpha_0}:[h_{kj}]_{(k,j)\in\bigcup_{\alpha\leqslant\alpha_0} B_\alpha}\to(\sum_{n=1}^{\infty} l_1)_{c_0}.$$

Let $B':=B\setminus\bigcup_{\alpha\leqslant\alpha_0} B_\alpha$ and remark that $U_{B'}$ is either of finite dimension or isomorphic to one of the following spaces: $(\sum_{n=1}^{\infty} l_1(n))_{c_0}$ or X_0.

Consequently U_B is isomorphic either to $(\sum_n l_1)_{c_0}$ or to $(\sum_n l_1)_{c_0} \oplus X_0$.

This ends the proof of Gamlen-Gaudet theorem for U_∞:

Theorem 2.4 All the isomorphy types of subspaces U_B of U_∞ are those from the statement of Proposition 2.1 .

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