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Characterization of Functions in Terms of Rate of Convergence of a Quadrature Process III

Q.I. RAHMAN AND G. SCHMEISSER

1 Introduction

In recent papers (see [10], [13–15]) we studied the problem of characterizing regularity properties of a function in terms of rate of convergence of a quadrature process. Since there exist continuous, but not absolutely continuous, functions for which the remainders of the trapezoidal rule all vanish (a striking non-trivial example is quoted in the introduction in [13]), the problem has to be formulated appropriately in order to make sense. We followed two different concepts.

Concept 1. For a quadrature process

$$\int_I w(x)f(x)\,dx = \sum_{\nu=1}^{n} A_{n\nu} f(x_{n\nu}) + R_n[f]$$

over an interval I with a weight function w, let $\mathcal{T} = (T_\alpha)_{\alpha\in A}$, with $0 \in A$, be a family of transformations $T_\alpha : C(I) \to C(I)$ such that

(i) $T_0(f) = f$,

(ii) $\displaystyle\int_I w(x)\,T_\alpha(f)(x)\,dx = \int_I w(x)\,f(x)\,dx \quad$ for all $\alpha \in A$,

(iii) T_α preserves the regularity property to be characterized.

Defining

$$R_n^*[f] := \sup_{\alpha\in A} |\,R_n[T_\alpha(f)]\,|\,,$$

it is reasonable to ask if the regularity property in question implies a certain speed of convergence for the sequence $\left(R_n^*[f]\right)_{n\in\mathbb{N}}$ and vice versa, provided the family $\mathcal{T}$ is sufficiently rich.

Note that $\mathcal{T}$ should destroy certain properties of f, like symmetry, which favour the numerical quadrature process while disregarding the regularity or lack thereof. For the integration of periodic functions over a period with $w(x) \equiv 1$ the translations $T_\alpha(f)(x) := f(x+\alpha)$ form an appropriate family $\mathcal{T}$, cf. [14]; for non-periodic functions and $w(x) = (1-x^2)^{-1/2}$ see [15].

Concept 2. If for a 2π-periodic function f we assume certain regularity conditions stronger than continuity (absolute continuity will be sufficient), then, as a consequence of the

Möbius inversion formula, there is a one to one correspondence between the Fourier coefficients of f and the remainders of the trapezoidal rule on $[0, 2\pi]$. This may be used for the problem in question if the quadrature formula is the trapezoidal rule or is deducable from it.

The Möbius inversion formula has been (explicitly or implicitly) used for various related problems in the theory of numerical quadrature (see [3–7], [9] and [11–12]), among them the closest to our investigations are those in [4] and [3].

In both concepts the quadrature process has to be chosen appropriately. The three cases, namely, the of quadrature of

(a) periodic functions over a period,

(b) non-periodic functions over a compact interval,

(c) functions over the whole real line,

require separate treatment. In Case (b) the following quadrature process

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx =: Q_n[f] + \mathcal{R}_n[f], \tag{1}$$

where

$$Q_n[f] := \begin{cases} \dfrac{\pi}{n}\left(f(1) + 2\displaystyle\sum_{\nu=1}^{k-1} f\left(\cos\frac{2\nu\pi}{n}\right) + f(-1)\right) & \text{if } n = 2k, \\[2ex] \dfrac{\pi}{n}\left(f(1) + 2\displaystyle\sum_{\nu=1}^{k} f\left(\cos\frac{2\nu\pi}{n}\right)\right) & \text{if } n = 2k+1, \end{cases} \tag{2}$$

is suitable. It uses alternatingly a Lobatto-Tschebyscheff and a Radau-Tschebyscheff formula [8, pp. 103–104].

The table given below contains references to the various situations studied so far. Here we will treat Case (b) following Concept 2.

Case	Integration problem	Quadrature process	Results Concept 1	Results Concept 2
(a)	$\int_0^b f(x)\,dx$	trapezoidal rule	[14]	[13]
(b)	$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx$	formula (1)–(2)	[15]	this paper
(c)	$\int_{-\infty}^{\infty} f(x)\,dx$	trapezoidal rule	[10]	unpublished

2 Statement of the Results

The following results are completely analogous to those in [15]. Here, however, we have the stronger assumption of *absolute continuity,* instead of usual continuity, which entails stronger and perhaps more elegant conclusions.

Hereafter $\mathcal{R}_n[f]$ will stand for the remainder of the quadrature formula (1). By $\mathcal{P}_n$ we denote the space of all polynomials of degree at most n.

Theorem 1. *Let f be an absolutely continuous function on $[-1,1]$. Then $f \in \mathcal{P}_k$ if and only if $\mathcal{R}_n[f] = 0$ for all $n > k$.*

In the next two theorems entire functions are characterized. For the notion of *order* and *type* we refer to [2].

Theorem 2. *An absolutely continuous function f on $[-1,1]$ is the restriction of an entire function of order ρ if and only if*

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/\,|\,\mathcal{R}_n[f]\,|)} = \rho. \tag{3}$$

Theorem 3. *An absolutely continuous function f on $[-1,1]$ is the restriction of an entire function of finite positive order ρ and type τ if and only if*

$$\limsup_{n\to\infty} \frac{n}{e\rho}\left(2^n\,|\,\mathcal{R}_n[f]\,|\right)^{\rho/n} = \tau. \tag{4}$$

In the next theorem we shall characterize a space of holomorphic functions whose precise description is given in the following

Definition 1. For $r > 1$ let $\mathcal{B}_r^2$ be the normed linear space of all functions f holomorphic inside the elliptic region bounded by

$$\mathcal{E}_r := \left\{ z \in \mathbf{C} : \, z = \frac{1}{2}\left(r + r^{-1}\right)\cos\theta + \frac{i}{2}\left(r - r^{-1}\right)\sin\theta, \;\; 0 \le \theta \le 2\pi \right\}$$

and satisfying

$$\|f\| := \left(\sup_{1<t<r} \int_{\mathcal{E}_t} \frac{|\,f(\zeta)\,|^2}{\sqrt{1-\zeta^2}}\, d\zeta \right)^{\frac{1}{2}} < \infty.$$

(A branch of the root can be chosen such that the integral becomes non-negative.)

By ℓ^2 we denote the Hilbert space of sequences $(a_n)_{n\in\mathbf{N}}$ for which $\sum_{n\in\mathbf{N}} |\,a_n\,|^2 < \infty$.

Theorem 4. *For an absolutely continuous function f on $[-1,1]$ the following statements are equivalent:*

(i) *f is the restriction to $[-1,1]$ of a function belonging to $\mathcal{B}_r^2$,*

(ii) *$\mathcal{R}_n[f] = b_n r^{-n}$ for all $n \in \mathbf{N}$, where $(b_n)_{n\in\mathbf{N}} \in \ell^2$.*

For the characterization of functions of relatively low regularity we introduce a weighted Sobolev space which can also be described in several other ways (see [15, Remark 1]).

Definition 2. For $k \in \mathbf{N}$ let $W_2^k(-1,1)$ be the space of functions $f : (-1,1) \to \mathbf{R}$ for which $f^{(k-1)}$ exits and is absolutely continuous with $\|f^{(k)}\|_k < \infty$, where

$$\|g\|_k := \left(\int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} \, |g(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Theorem 5. *For an absolutely continuous function f on $[-1,1]$ and $k \geq 2$ the following statements are equivalent:*

(i) $f \in W_2^k(-1,1)$,

(ii) $\mathcal{R}_n[f] = b_n n^{-k}$ *for all* $n \in \mathbf{N}$, *where* $(b_n)_{n\in\mathbf{N}} \in \ell^2$.

Remark. Theorem 5 does not extend to $k = 1$. There exists a function $f \in W_2^1(-1,1)$ for which (ii) is not true (see [15, Theorem 6]).

3 Lemmas

As usual, let $T_n(x) := \cos(n \arccos x)$ be the n^{th} Tschebyscheff polynomial of the first kind. We shall make decisive use of the following representation for $\mathcal{R}_n[f]$.

Lemma 1. *If f is an absolutely continuous function on $[-1,1]$, then*

$$\mathcal{R}_n[f] = -\pi \sum_{j=1}^{\infty} c_{nj}[f], \qquad n = 1,2,\ldots, \tag{5}$$

where

$$c_n[f] := \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\, T_n(x)}{\sqrt{1-x^2}} \, dx, \qquad n = 0,1,\ldots. \tag{6}$$

Proof. Note that $f(\cos\theta)$ is an even absolutely continuous 2π-periodic function of θ and

$$c_\nu[f] = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos\theta) \cos\nu\theta \; d\theta, \qquad \nu = 0,1,\ldots.$$

Hence [1, p. 116]

$$f(\cos\theta) = \frac{1}{2} c_0[f] + \sum_{\nu=1}^{\infty} c_\nu[f] \cos\nu\theta, \tag{7}$$

where the series converges uniformly. As such

$$f(x) = \frac{1}{2} c_0[f] + \sum_{\nu=1}^{\infty} c_\nu[f]\, T_\nu(x) \quad \text{for } x \in [-1,1], \tag{8}$$

the series on the right being uniformly convergent. Applying the functional $\mathcal{R}_n$ to the two sides, we get

$$\mathcal{R}_n[f] = \frac{1}{2}c_0[f]\,\mathcal{R}_n[T_0] + \sum_{\nu=1}^{\infty} c_\nu[f]\,\mathcal{R}_n[T_\nu]. \tag{9}$$

Using the formula

$$T_\nu(\cos\varphi) = T_\nu\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right) = \frac{1}{2}\left(e^{i\nu\varphi}+e^{-i\nu\varphi}\right),$$

it is not difficult to verify that for $n \in \mathbf{N}_0$

$$\mathcal{R}_n[T_\nu] = \begin{cases} -\pi & \text{if } \nu \text{ is a multiple of } n,\\ 0 & \text{otherwise}\end{cases}$$

whereby (9) reduces to (5). □

The equations (5) can be inverted. For this purpose we need to recall the Möbius function $\mu : \mathbf{N} \to \{-1,0,1\}$ defined by

$$\mu(n) := \begin{cases} 1 & \text{if } n = 1,\\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes},\\ 0 & \text{otherwise, i.e. } n \text{ is divisible by a square of a prime.}\end{cases}$$

Lemma 2. *If f is an absolutely continuous function on $[-1,1]$, then in the notation of Lemma 1*

$$c_n[f] = -\frac{1}{\pi}\sum_{j=1}^{\infty}\mu(j)\,\mathcal{R}_{nj}[f] \qquad n = 1,2,\ldots.$$

Proof. Since $f(\cos\theta)$ is an absolutely continuous 2π-periodic function for which (7) holds and $\mathcal{R}_n[f]$ has the representation (5), the desired result follows immediately from Lemmas 1 and 2 in [13]. □

We shall also employ the following lemmas proved in [15].

Lemma 3 [15, Lemma 4]. *If one of the two functions*

$$f(z) = \sum_{n=0}^{\infty} c_n T_n(z) \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} c_n z^n$$

is entire, so is the other one and both are of the same order ρ. Moreover, if $\rho \in (0,+\infty)$ and g is of type τ, then f is of type $2^\rho\tau$.

Lemma 4 [15, Lemma 3]. *A function $f \in C[-1,1]$ is the restriction to $[-1,1]$ of a function belonging to $\mathcal{B}^2_\tau$ if and only if for all $n \in \mathbf{N}$ the coefficients in (6) can be represented as $c_n[f] = a_n r^{-n}$ with $(a_n)_{n\in\mathbf{N}} \in \ell^2$.*

Lemma 5 [15, Lemma 5]. *A function $f \in C[-1,1]$ belongs to $W_2^k(-1,1)$ for some $k \in \mathbf{N}$ if and only if for all $n \in \mathbf{N}$ the coefficients in (6) can be represented as $c_n[f] = a_n n^{-k}$ with $(a_n)_{n\in\mathbf{N}} \in \ell^2$.*

4 Proofs of the Theorems

Proof of Theorem 1. Clearly, $f \in \mathcal{P}_k$ if and only if $c_n[f] = 0$ for all $n > k$. By Lemmas 1 and 2 this holds if and only if $\mathcal{R}_n[f] = 0$ for all $n > k$. □

Proof of Theorem 2. Being absolutely continuous, f may according to (6) and (8) be written as $f(z) = \sum_{n\in\mathbf{N}_0} c_n T_n(z)$, where $c_0 := c_0[f]/2$ and $c_n := c_n[f]$ for $n \in \mathbf{N}$.

If f is an entire function of order $\rho < +\infty$, then by Lemma 3 the power series $\sum_{n\in\mathbf{N}_0} c_n z^n$ also represents an entire function of order ρ and therefore [2, Theorem 2.2.2]

$$\limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)} = \rho.$$

Consequently, given $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbf{N}$ such that

$$|c_n| \le n^{-n/(\rho+\varepsilon)} \qquad \text{for } n \ge n_\varepsilon.$$

Using Lemma 1 we obtain for all sufficiently large n

$$\begin{aligned} |\mathcal{R}_n[f]| &\le \pi \sum_{j=1}^{\infty} |c_{nj}| \le \pi \sum_{j=1}^{\infty} (nj)^{-nj/(\rho+\varepsilon)} \\ &< \pi \sum_{j=1}^{\infty} \left(n^{-n/(\rho+\varepsilon)}\right)^j < 2\pi\, n^{-n/(\rho+\varepsilon)}. \end{aligned}$$

Hence

(10) $$\lambda := \limsup_{n\to\infty} \frac{n\log n}{\log(1/|\mathcal{R}_n[f]|)} \le \rho,$$

which obviously holds for $\rho = +\infty$ as well.

Conversely, let λ be defined as in (10) and assume $\lambda < +\infty$. Then, given $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbf{N}$ such that

$$|\mathcal{R}_n[f]| \le n^{-n/(\lambda+\varepsilon)} \qquad \text{for } n \ge n_\varepsilon.$$

Using Lemma 2 we conclude that for all sufficiently large n

$$|c_n| \le \frac{1}{\pi} \sum_{j=1}^{\infty} (nj)^{-nj/(\lambda+\varepsilon)} < \frac{2}{\pi} n^{-n/(\lambda+\varepsilon)}$$

and so

$$\limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)} \le \lambda,$$

which holds for $\lambda = +\infty$ as well. Hence the function $g(z) := \sum_{n\in\mathbb{N}_0} c_n z^n$ is entire and of order at most λ. By Lemma 3, the same is true about the function f. With this the proof of Theorem 2 is complete. □

Proof of Theorem 3. By Lemma 3 the function $f(z) = \sum_{n\in\mathbb{N}_0}^{\infty} c_n T_n(z)$ is of order $\rho \in (0, +\infty)$ and type τ if and only if $g(z) := \sum_{n\in\mathbb{N}_0}^{\infty} c_n z^n$ is also of order ρ and of type $2^{-\rho}\tau$, which is the case [2, Theorem 2.2.10] if and only if

$$\tau = \frac{2^\rho}{e\rho} \limsup_{n\to\infty} n \mid c_n \mid^{\rho/n}.$$

Proceeding analogously to the previous proof, we obtain

$$\limsup_{n\to\infty} n \mid c_n \mid^{\rho/n} = \limsup_{n\to\infty} n \mid \mathcal{R}_n[f] \mid^{\rho/n}$$

and conclude that Theorem 3 holds. □

Proof of Theorem 4. In view of Lemma 4 it is enough to show that for $r > 1$

(11) $$\left(r^n \mathcal{R}_n[f]\right)_{n\in\mathbb{N}} \in \ell^2 \iff \left(r^n c_n[f]\right)_{n\in\mathbb{N}} \in \ell^2.$$

The equivalence in (11) follows from Lemmas 1 and 2 if we verify that

$$b_n := \sum_{j=1}^{\infty} a_{nj} r^{-n(j-1)} \qquad (n \in \mathbb{N})$$

transforms a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2$ into a sequence $(b_n)_{n\in\mathbb{N}} \in \ell^2$. This is readily seen to be true. Indeed, if $(a_n)_{n\in\mathbb{N}} \in \ell^2$, then by the Cauchy-Schwarz inequality

$$\begin{aligned} |b_n|^2 &\le \left(\sum_{j=1}^{\infty} \left(r^{-n}\right)^{j-1}\right)\left(\sum_{j=1}^{\infty} |a_{nj}|^2 \left(r^{-n}\right)^{j-1}\right) \\ &\le \frac{1}{1-r^{-n}}\left(|a_n|^2 + r^{-n}\sum_{j=2}^{\infty} |a_{nj}|^2\right) \\ &\le \frac{1}{1-r^{-1}}\left(|a_n|^2 + r^{-n}\sum_{\nu=1}^{\infty} |a_\nu|^2\right), \end{aligned}$$

which shows that $(b_n)_{n\in\mathbb{N}} \in \ell^2$. □

Proof of Theorem 5. In view of Lemma 5 it is enough to show that for $k \ge 2$

(12) $$\left(n^k \mathcal{R}_n[f]\right)_{n\in\mathbb{N}} \in \ell^2 \iff \left(n^k c_n[f]\right)_{n\in\mathbb{N}} \in \ell^2.$$

The equivalence in (12) follows from Lemmas 1 and 2 because [13, Lemma 4].

$$b_n := \sum_{j=1}^{\infty} \frac{a_{nj}}{j^k} \qquad (n \in \mathbb{N})$$

transforms a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2$ into a sequence $(b_n)_{n\in\mathbb{N}} \in \ell^2$ if $k > 1$. □

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Qazi Ibadur Rahman
Dépt. de Math. et de Statistique
Université de Montréal
C.P. 6128, succursale A
Montréal, P.Q., H3C 3J7
CANADA

Gerhard Schmeisser
Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstrasse $1\frac{1}{2}$
D-8520 Erlangen
GERMANY