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On rational approximation of meromorphic functions

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Introduction

Let E be a compact subset of the complex plane. We define the Leja extremal function Φ_E of the set E as follows:

$$\Phi_E(z) = \sup\left\{|p(z)|^{1/n}:\ p\in\mathcal{P}_n,\ n\in\mathbb{N},\ \|p\|_E \le 1\right\},$$

where $\mathcal{P}_n$ denotes the space of all polynomials of degree not greater than n and $\|\phi\|_K = \sup\{|\phi(z)|:\ z\in K\}$ for a given set K and a function ϕ defined on K. The set E is said to be *regular* if the function Φ_E is continuous. For details, see [3].

Denote by $\mathcal{R}_{n/m}$ the class of all rational functions whose numerators and denominators are polynomials of degree not greater than n and m, respectively.

Let R be greater than 1. In the sequel we shall use the following denotations:

$$E_R = \{z\in\mathbb{C}:\ \Phi_E(z) \le R\},$$
$$D_R = \{z\in\mathbb{C}:\ \Phi_E(z) < R\},$$
$$\Gamma_R = \{z\in\mathbb{C}:\ \Phi_E(z) = R\}.$$

Let E be a regular compact set. Denote by $\mathfrak{M}(R)$ the class of all functions meromorphic in D_R and by $\mathfrak{M}_m(R)$ the class of those functions of $\mathfrak{M}(R)$ that have not more than m poles in D_R.

The following theorem is well-known ([2], [4], [5]): Let f be a function defined on E. Then f can be prolonged to a function of the class $\mathfrak{M}_m(R)$ if and only if there exists a sequence of rational functions $(r_n)_{n\in\mathbb{N}}$ such that $r_n\in\mathcal{R}_{n/m}$ and

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$$\limsup_{n\to\infty} \|f - r_n\|_E^{1/n} \le 1/R.$$

The aim of the present paper is to prove an analogous result, where the class $\mathfrak{M}_m(R)$ is replaced by $\mathfrak{M}(R)$. Then, of course, the approximants cannot be restricted to rational functions whose denominators are polynomials of degree not greater than m. The idea of this paper is to consider a special sequence of degrees of denominators.

Basic result

Consider a sequence of integers (m_n) that satisfies the following conditions:

(1) $$m_n \le m_{n+1} \quad \text{for every } n;$$

(2) $$\lim_{n\to\infty} \frac{m_n \log n}{n} = 0.$$

The equality (2) implies that there exists a sequence of numbers (α_n) such that $\lim_{n\to\infty} \alpha_n = 0$ and $m_n = n\alpha_n/\log n$.

We shall prove the following

Theorem. Let (m_n) be a sequence of integers that satisfies conditions (1)-(2). Let E be a regular compact set and f be a function defined on E. Then the following conditions are equvalent:

(A) f can be extended to a function of the class $\mathfrak{M}(R)$;

(B) there exists a sequence of rational functions $r_n \in \mathcal{R}_{n/m_n}$, $n \in \mathbb{N}$, such that

(3) $$\operatorname{lin\,sup}_{n\to\infty} \|f - r_n\|_E^{1/n} \le 1/R,$$

and

(*) for every compact set $K \subset D_R$ there exists a number N_K such that for every n the function r_n has not more than N_K poles in K.

Before we prove the theorem, we shall state the following

Lemma. Let r be greater than 1 and K be a compact subset of $B(0,r)$. Let P be a polynomial of degree $\nu > 0$. Then for every $\varepsilon \in (0,1)$ there exists a compact set E^ε that can be covered by a family of discs whose sum of diameters does not exceed $4e\varepsilon$, such that for every $z \in B(0,r) \setminus E^\varepsilon$,

$$\frac{\|P\|_K}{|P(z)|} \le (3r\varepsilon^{-1})^\nu.$$

Proof of lemma. Let $\zeta_1, \ldots, \zeta_\nu$ be the zeroes of P (not necessarily distinct). Assume that $|\zeta_1|\le\ldots\le|\zeta_\mu|\le 2r<|\zeta_{\mu+1}|\le\ldots\le|\zeta_\nu|$. We may assume P to be monic. Put

$$P_1(z) = \prod_{j=1}^{\mu}(z-\zeta_j)$$

and

$$P_2(z) = \prod_{j=\mu+1}^{\nu}(z-\zeta_j).$$

Then for $|z|<r$ we have:

$$\frac{\|P\|_K}{|P(z)|} \le \frac{\|P_1\|_K}{|P_1(z)|}\cdot\frac{\|P_2\|_K}{|P_2(z)|} \le \frac{(3r)^\mu}{|P_1(z)|}\cdot\frac{\|P_2\|_K}{|P_2(z)|}.$$

Let $z_0\in K$ be such a point that $|P_2(z_0)| = \|P_2\|_K$. Then

$$\frac{\|P_2\|_K}{|P_2(z)|} = \prod_{j=\mu+1}^{\nu}\frac{|z_0-\zeta_j|}{|z-\zeta_j|} \le \prod_{j=\mu+1}^{\nu}\frac{|z_0|+|\zeta_j|}{|\zeta_j|-|z|} \le \prod_{j=\mu+1}^{\nu}\frac{|\zeta_j|+r}{|\zeta_j|-r} \le 3^{\nu-\mu}.$$

Hence,

$$\frac{\|P\|_K}{|P(z)|} \le (3r)^\nu.\varepsilon^{-\mu} \le (3r\varepsilon^{-1})^\nu$$

if $|P_1(z)| \ge \varepsilon^\mu$. But due to the Cartan lemma [1] the set $\{z: |P_1(z)|\le\varepsilon^\mu\}$ can be covered by a family of discs whose sum of diameters does not exceed $4e\varepsilon$. This completes the proof of the lemma.

Proof of the theorem.

(A)⇒(B)

Consider an increasing sequence of numbers (R_k) such that $\lim_{k\to\infty} R_k = R$. Then there exists a sequence of functions (ϕ_k) and a sequence of polynomials (Q_k) such that ϕ_k is analytic in D_{R_k}, the zeros of Q_k are the poles of f in D_{R_k} and

$$f(z) = \frac{\phi_k(z)}{Q_k(z)},$$

for $z\in D_{R_k}$. Let $(P_{k,n})_{n\in\mathbb{N}}$ be a sequence of polynomials of best approximation to ϕ_k with respect to the set E. Then

$$\limsup_{n\to\infty}\left\| f - \frac{P_{k,n}}{Q_k}\right\|_E^{1/n} \le 1/R_k$$

Hence, we can easily choose such a sequence

$$r_n = \frac{P_{k_n,n}}{Q_{k_n}},$$

that $k_n \le m_n$ and $\limsup_{k\to\infty} \|f - r_n\|_E^{1/n} \le 1/R$.

(B)$\Rightarrow$(A)

Choose a number $\rho\in(1,R)$. It is enough to show that $f\in\mathbb{M}(\rho)$. In order to prove this, let us pick two numbers r and σ, such that $\rho < r < \sigma < R$. It follows from (3) that there exists an integer N_0 such that

$$(4) \qquad \|r_{n+1} - r_n\|_E^{1/n} < 1/\sigma$$

for $n \ge N_0$. Put $r_n = p_n/q_n$, where p_n and q_n are elements of $\mathcal{P}_n$ and $\mathcal{P}_{m_n}\setminus\{0\}$, respectively. Then we get for $z\in E_r$:

$$(5) \qquad \begin{aligned} |r_{n+1}(z) - r_n(z)| &\le \frac{\|p_{n+1}q_n - p_n q_{n+1}\|_E \cdot r^{n+1+m_{n+1}}}{|q_n(z)q_{n+1}(z)|} \le \\ &\le \|r_{n+1} - r_n\|_E \cdot \frac{\|q_n q_{n+1}\|_E}{|q_{n+1}(z)q_n(z)|} \cdot r^{n+m_{n+1}+1} \end{aligned}$$

Let d be such a number that $E_\sigma \subset \{z: |z|<d\}$. Then, due to the lemma,

$$(6) \qquad \frac{\|q_n q_{n+1}\|_E}{|q_n(z).q_{n+1}(z)|} \le (3dn^2)^{m_n+m_{n+1}}$$

for $z\in E_\sigma$ except for a set F^n which can be covered by a family of discs, whose sum of diameters does not exceed $4en^{-2}$. Then we get from (4), (5) and (6) that if $z\in E_r\setminus F^n$, then

$$(7) \qquad |r_{n+1}(z) - r_n(z)| \le \left(\frac{r}{\sigma}\right)^n . (3dn^2 r)^{m_n+m_{n+1}}$$

Put $M = \sup\left\{|\Phi_E'(z)|: z\in E_r\setminus D_\rho\right\}$. Then it follows from the mean value theorem that for every α, β such that $\rho \le \alpha < \beta < r$,

$$\operatorname{dist}(\Gamma_\alpha, \Gamma_\beta) \ge M^{-1} \cdot |\beta - \alpha|.$$

Let $N_1 \ge N_0$ be such an integer that $\sum_{n=N_1}^{\infty} 4eMn^{-2} < r - \rho$. Then there exists $s\in(\rho, r)$ such that Γ_s does not intersect the set $\bigcup_{n=N}^{\infty} F^n$. Hence, the estimation (7) is valid for every $n\ge N_1$ and for every $z\in\Gamma_s$. This means that the sequence (r_n) converges uniformly on Γ_s.

Define ω_n as a polynomial:

$$\omega_n(z) = \prod_{\substack{\zeta\in E_\sigma \\ q_n(\zeta)=0}} (z-\zeta)$$

Then, according to the assumption (B), there exists a number N_3 such that for every n. deg $\omega \le N_3$ and, consequently,

$$\|\omega_n\|_{\Gamma_s} \le (2d)^{N_3}.$$

Hence, there exists a sequence of integers (n_k) such that the sequence of polynomials $(\omega_{n_k})_{k\in\mathbb{N}}$ tends to some polynomial ω and the sequence of functions $(\omega_{n_k}\cdot r_{n_k})_{k\in\mathbb{N}}$ converges uniformly on Γ_s. But the functions $\omega_n\cdot r_n$ are holomorphic in D_s, so there exists a limit function ϕ holomorphic in D_s such that

$$\lim_{k\to\infty} r_{n_k}(z)\,.\,\omega_{n_k}(z) = \phi(z)$$

for $z\in D_s$. On the other hand, the sequence $r_n(z)$ tends to $f(z)$ for

$z\in E$. Hence, $f=\phi/\omega$ in E. This means that the function ϕ/ω is a meromorphic continuation of f over the set D_s. This completes the proof.

Remark. The following example shows that in the theorem it is not possible to remove condition (*): Let

$$r_n(z) = \sum_{k=1}^{n} (z^k+1)/(k!z^{[\log k]}).$$

It is easy to see that r_n is a member of $\mathcal{R}_{n/[\log n]}$ and

$$\limsup_{n\to\infty} \|r_n - r_{n-1}\|_E^{1/n} = 0,$$

for every compact set E which does not contain the origin, but condition (*) does not hold, and the series

$$\sum_{k=1}^{\infty} (z^k+1)/(k!z^{[\log k]})$$

has an essential singularity at zero.

Corollary. If $N := \sup\{N_K : K\subset D_R\} < \infty$, then (B) implies that $f\in\mathcal{M}_N(R)$.

R e f e r e n c e s

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