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Sequences in the Walsh Table for $|x|^\alpha$

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Abstract

We investigate the convergence behavior of best uniform rational approximants r^*_{mn} with numerator degree m and denominator degree n to the function $|x|^\alpha$, $\alpha > 0$, on $[-1,1]$ for ray sequences in the lower triangle of the Walsh table, i.e. for sequences $\{(m,n)\}$ of indices with

$$\frac{m}{n} \to c \in [1,\infty) \text{ as } m+n \to \infty.$$

The results will be compared with those for diagonal sequences $(m = n)$ and sequences of best polynomial approximants. Sketches of the proofs will be given.

§1. Introduction

Our aim is to investigate the convergence behavior of ray sequences in the Walsh table of the function

$$f(x) = |x|^\alpha, \quad x \in [-1,1], \quad 0 < \alpha. \tag{1}$$

Let Π_n denote the collection of all real polynomials p of degree at most n and let $\mathcal{R}_{mn} := \{p/q \mid p \in \Pi_m,\ q \in \Pi_n,\ q \not\equiv 0\}$. By $r^*_{mn} = r^*_{mn}(f,[-1,1];\cdot) \in \mathcal{R}_{mn}$ we denote the *best uniform rational approximant* to f on the interval $[-1,1]$, i.e.

$$E_{mn}(f,[-1,1]) := \|f - r^*_{mn}\|_{[-1,1]} = \inf_{r\in\mathcal{R}_{mn}} \|f - r\|_{[-1,1]}, \tag{2}$$

where $\|\cdot\|_{[-1,1]}$ is the sup–norm on $[-1,1]$.

We know that for each pair $m,n \in \mathbb{N}$ the best rational approximant r^*_{mn} exists and is unique (see [Me], §9.1 and §9.2, or [Ri], §5.1). The doubly infinite array of

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all r^*_{mn}, $m, n \in \mathbb{N}$, is called the *Walsh table* of the function f with respect to approximation on $[-1, 1]$.

An infinite sequence $N = N_c \subseteq \mathbb{N}^2$ of indices (m, n), as well as the sequence $\{r^*_{mn}\}_{(m,n)\in N_c}$ of approximants, is called a *ray sequence* with associated *numerator-denominator ratio* c if

$$\frac{m}{n} \to c \in [0, \infty] \quad \text{as} \quad m + n \to \infty, \quad (m, n) \in N_c. \tag{3}$$

Since the sequence $\{r^*_{nn}\}_{n=0}^{\infty}$ fills the diagonal of the Walsh table, it is called the *diagonal* sequence, and any sequence with $c = 1$ is called *near diagonal*. Best polynomial approximants r^*_{m0} and best reciprocal polynomial approximants r^*_{0n} fill the first column and the first row of the table, and they have c-values $c = \infty$ and $c = 0$, respectively. It will turn out that the asymptotic behavior of the approximants $\{r^*_{mn}\}_{(m,n)\in N_c}$ essentially depends on the parameter c. We shall investigate ray sequences in the upper triangle of the Walsh table, for which we have $1 \leq c \leq \infty$.

Since f is an even function, it is an immediate consequence of the uniqueness of best approximants that the same is true for r^*_{mn}.. Thus we have

$$r^*_{2m+i,2n+j}\left(|x|^\alpha, [-1, 1]; \cdot\right) = r^*_{2m,2n}\left(|x|^\alpha, [-1, 1]; \cdot\right) \quad \text{for } m, n \in \mathbb{N} \text{ and } i, j \in \{0, 1\}. \tag{4}$$

Replacing x^2 by x in the function and the approximant gives the identity

$$r^*_{2m,2n}\left(|x|^{2\alpha}, [-1, 1]; \cdot\right) = r^*_{mn}\left(x^\alpha, [0, 1]; \cdot\right) \tag{5}$$

for all $m, n \in \mathbb{N}$ and $0 \leq \alpha$. Hence, the investigation of the Walsh table of $|x|^\alpha$ with respect to the interval $[-1, 1]$ is equivalent to an investigation of the Walsh table of $x^{\alpha/2}$ with respect to $[0, 1]$.

As a prototype of the approximation problem of $|x|^\alpha$ on $[-1, 1]$ or $x^{\alpha/2}$ on $[0, 1]$ one can consider the approximation of $|x|$ on $[-1, 1]$. Much attention has been given to this problem in the literature. After the pioneering result by Newman [Ne], who showed in 1964 that

$$\frac{1}{2} e^{-9\sqrt{n}} \leq E_{n,n}\left(|x|, [-1, 1]\right) \leq 3e^{-\sqrt{n}} \quad \text{for } n = 4, 5, \ldots, \tag{6}$$

a series of results has been published about rational approximants of $|x|^\alpha$ and/or

x^α. We have taken the following compilation from [Vj1].

$$E_{nn}(x^\alpha,[0,1]) \le e^{-c(\alpha)\sqrt[3]{n}}, \quad \alpha\in\mathbb{R}_+, \text{ in [FrSz], 1967,}$$

$$E_{nn}(x^{1/3},[0,1]) \le e^{-c\sqrt{n}}, \quad \text{in [Bu2], 1968,}$$

$$E_{nn}(x^\alpha,[0,1]) \le e^{-c(\alpha)\sqrt{n}}, \quad \alpha\in\mathbb{R}_+, \text{ in [Go1], 1967,}$$

$$\frac{1}{3}e^{-\pi\sqrt{2n}} \le E_{nn}(x^{1/2},[0,1]) \le e^{-\pi\sqrt{2n}(1-\mathcal{O}(n^{-1/4}))}, \quad \text{in [Bu1], 1968,}$$

$$e^{-c(\alpha)\sqrt{n}} \le E_{nn}(x^\alpha,[0,1]), \quad \alpha\in Q_+\setminus\mathbb{N}, \text{ in [Go2], 1972,}$$

$$e^{-4\pi\sqrt{\alpha n}(1+\epsilon)} \le E_{nn}(x^\alpha,[0,1]) \le e^{-\pi\sqrt{\alpha n}(1-\epsilon)}, \ \alpha\in\mathbb{R}_+\setminus\mathbb{N}, \epsilon>0, n\ge n(\epsilon,\alpha), \text{ in [Go3], 1974,}$$

$$E_{nn}(x^{1/2},[0,1]) \le cne^{-\pi\sqrt{2n}}, \quad \text{in [Vj3], 1974,}$$

$$\frac{1}{3}e^{-\pi\sqrt{2n}} \le E_{nn}(x^{1/2},[0,1]) \le ce^{-\pi\sqrt{2n}}, \quad \text{in [Vj2], 1975,}$$

$$e^{-c_1(s)\sqrt{n}} \le E_{nn}(\sqrt[s]{x},[0,1]) \le e^{-c_2(s)\sqrt{n}}, \quad s\in\mathbb{N}, \text{ in [Tz], 1976.}$$

Here, c, $c(\alpha), \ldots$ denote constants. In [Ga], 1979, and [Vj1], 1980, T. Ganelius and N.S. Vjacheslavov have proved independently that

$$c_1(\alpha)e^{-2\pi\sqrt{\alpha n}} \le E_{nn}(x^\alpha,[0,1]) \le c_2(\alpha)e^{-2\pi\sqrt{\alpha n}}, \ n=1,2,\ldots,$$

where, for the proof of the upper bound, it was necessary to assume that $\alpha\in Q_+$, and it could not be proved that $c_2(\alpha)$ depends continuously on α. The lower bound holds for all $\alpha\in\mathbb{R}_+\setminus\mathbb{N}$. Without the restriction that α is rational, T. Ganelius [Ga] was able to prove the somewhat weaker result that

$$c_1(\alpha)e^{-2\pi\sqrt{\alpha n}} \le E_{nn}(x^\alpha,[0,1]) \le c_2(\alpha)e^{-2\pi\sqrt{\alpha n}+c_3(\alpha)\sqrt[4]{n}}, \ n=1,2,\ldots \tag{7}$$

for all $\alpha\in\mathbb{R}_+\setminus\mathbb{N}$, where $c_3(\alpha)>0$ is a third constant.

Based on high precision calculations, Varga, Ruttan, and Carpenter [VRC] recently made the conjecture that the limit

$$\lim_{n\to\infty} e^{\pi\sqrt{n}}E_{nn}(|x|,[-1,1]) = 8 \tag{8}$$

holds true, and this conjecture has subsequently been proved in [St].

If we consider best polynomial instead of best rational approximants, then we have in case of the approximation of the function $f(x)=|x|^\alpha$ the classical result by Bernstein [Be1], [Be2] that the limit

$$\lim_{m\to\infty} m^\alpha E_{m,0}(|x|^\alpha,[-1,1]) = \beta(\alpha) \tag{9}$$

exists for each $\alpha>0$. For $\alpha=1$, the limit (8) is the analogue of (9) in the rational case.

A comparison of (7) and (9) shows that there is an essential difference in the rate of convergence in the polynomial and diagonal rational case. Since ray sequences are intermediate between both types of approximants, it is interesting to know how the convergence behavior and especially the rate of convergence changes with the value of the numerator–denominator ratio c in the ray sequence N_c. The next theorem shows that for all $c \in (0, \infty)$ the rate is more similar to the diagonal case than to the polynomial one.

Theorem 1. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and let $N_c \subseteq \mathbb{N}^2$ be a ray sequence with numerator–denominator ratio $c \in (0, \infty)$. If we choose constants*

$$\underline{c} < \min(1, \sqrt{c}), \quad \overline{c} > \max(1, \sqrt{c}), \tag{10}$$

then

$$e^{-\pi \overline{c} \sqrt{\alpha n}} \leq E_{mn}(|x|^\alpha, [-1, 1]) \leq e^{-\pi \underline{c} \sqrt{\alpha n}} \tag{11}$$

for all $(m, n) \in N_c$ with $m + n$ sufficiently large.

The proof of Theorem 1 follows immediately from Ganelius' result stated in (7) together with identity (5) and the observation that $\mathcal{R}_{ll} \subseteq \mathcal{R}_{mn} \subseteq \mathcal{R}_{kk}$ with $l = \min(m, n)$ and $k = \max(m, n)$.

If $c \neq 1$, then Theorem 1 does not contain the precise coefficient in the exponent of the error estimate. However, it turns out that the estimate (11) is good enough for the determination of the asymptotic distribution of zeros and poles of the approximants and for determining the exact region of convergence, which will be described in the theorems below.

We conjecture that for every ray sequence N_c, $c \in (0, \infty)$, the limit

$$\lim_{\substack{m+n \to \infty \\ (m,n) \in N_c}} \frac{-1}{\pi \sqrt{\alpha n}} \log(E_{mn}(|x|^\alpha, [-1, 1])) \tag{12}$$

exists. It is not clear whether it depends on α. By Theorem 1 we only know that its value has to lie between $\min(1, \sqrt{c})$ and $\max(1, \sqrt{c})$.

For the special case of the function $f(x) = |x|$ it has been proved by Blatt, Iserles, and Saff in [BIS] that the sequence $\{r^*_{nn}\}$ converges not only on the interval $[-1, 1]$ but also in the two half–planes $H_+ := \{z : \text{Re}(z) > 0\}$ and $H_- := \{z : \text{Re}(z) < 0\}$. We have

$$\lim_{n \to \infty} r^*_{nn}(z) = \begin{cases} z & \text{for } z \in H_+ \\ -z & \text{for } z \in H_- . \end{cases} \tag{13}$$

Further, it has been shown that all the poles and zeros lie on the imaginary axis, and that they interlace on each halfaxis.

The convergence behavior is completely different in the polynomial case. There no over-convergence exists; outside of the interval $[-1,1]$ we have

$$\lim_{m\to\infty} r^*_{m0}(z) = \infty \tag{14}$$

uniformly on every compact subset of $\mathbb{C}\setminus[-1,1]$. Thus, the question arises; what happens in the intermediate case of ray sequence with $c \in (1,\infty)$?

Theorem 2. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and $r^*_{mn} = r^*_{mn}(|x|^\alpha, [-1,1]; \cdot)$. For any ray sequence $N_c \subseteq \mathbb{N}^2$ with $c \in (1,\infty]$ we have*

$$\lim_{\substack{m+n\to\infty \\ (m,n)\in N_c}} |r^*_{mn}(z)|^{1/(m+n)} = \exp\left[\frac{c-1}{c+1} g_{\overline{\mathbb{C}}\setminus[-1,1]}(z,\infty)\right] \tag{15}$$

uniformly on every compact subset of $\overline{\mathbb{C}}\setminus[-1,1]$, where

$$g_{\overline{\mathbb{C}}\setminus[-1,1]}(z,\infty) = \log\left|z+\sqrt{z^2-1}\right| \tag{16}$$

is the Green function for $\overline{\mathbb{C}}\setminus[-1,1]$ with pole at ∞.

Corollary 3. *If $c \in (1,\infty]$, then $(c-1)/(c+1) > 0$, and therefore we have*

$$\lim_{\substack{m+n\to\infty \\ (m,n)\in N_c}} |r^*_{mn}(z)| = \infty \qquad \text{for all } z \notin [-1,1]. \tag{17}$$

Since the analytic continuation of $f(z) = |z|^\alpha$ is finite in H_+ and H_-, the corollary shows that we do not have overconvergence beyond $[-1,1]$ for any $c > 1$.

Theorem 2 is proved by using the fact that $r^*_{mn}(\sqrt{z})$, m,n even, is a multipoint Padé approximant to the function $\int_0^\infty t^{\alpha/2}\,dt/(t+z)$, $z \notin \mathbb{R}_-$, which is a Stieltjes function for $0<\alpha<2$. The main work is then done by potential-theoretic arguments. For more details see §3 and §4 in [SSW]. As a byproduct of the analysis it turns out that the overconvergence stated in (13) has an analogue for all $\alpha > 0$.

Theorem 4 *(see [SSW], Theorem 1.4): Let $\alpha > 0$, $[\alpha/2]$ be the greatest integer not larger than $\alpha/2$, and let $r^*_{mn} = r^*_{mn}(|x|^\alpha, [-1,1]; .)$, $m,n \in \mathbb{N}$.*

(a) *We have*

$$\lim_{n\to\infty} r^*_{n+2[\alpha/2],n}(z) = \begin{cases} z^\alpha & \text{for } z \in H_+ \\ (-z)^\alpha & \text{for } z \in H_- \end{cases} \tag{18}$$

uniformly on compact subsets of $H_- \cup H_+$.

(b) *Let n be even and $\alpha \notin 2\mathbb{N}$. Then the n poles and $n-2$ of the $n+2[\alpha/2]$ zeros of $r^*_{n+2[\alpha/2],n}$ lie on the imaginary axis. The poles are simple, and zeros and poles interlace on each half of the imaginary axis. The $2[\alpha/2]+2$ remaining zeros of $r^*_{n+2[\alpha/2],n}$ cluster at $z=0$.*

Remark. Since z^α has a zero of order at least $2[\alpha/2]$ and at most $2[\alpha/2]+2$ at $z=0$, it is natural to use approximants with a numerator degree that exceeds the denominator degree by $2[\alpha/2]$. In case that $\alpha = 0,2,4,\ldots$ we have $r^*_{n+2[\alpha/2],n}(z) \equiv z^\alpha$ for all $n \geq 2$, and Theorem 4 holds trivially.

We next come to the investigation of extreme points of the error function $f - r^*_{mn}$ on $[-1,1]$. We have seen in (5) that all approximants $r^*_{mn}(z) = r^*_{mn}(|x|^\alpha, [-1,1]; z)$ are even functions.

Lemma 5. *For $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and even $m, n \in \mathbb{N}$ there exist $m+n+3$ points*

$$-1 = x_1 < x_2 < \ldots < x_{m+n+3} = 1 \tag{19}$$

such that

$$(-1)^{(m+n)/2}(-1)^k(|x_k|^\alpha - r^*_{mn}(x_k)) = E_{m,n}(|x|^\alpha, [-1,1]) \tag{20}$$

for $k = 1, \ldots, m+n+3$, and the equality (21) holds on $[-1,1]$ only for these $m+n+3$ points.

The lemma follows from Chebyshev's theorem on alternation points (see [Me], Theorem 23, or [Ri], Theorem 5.2) and the fact that

$$W_{mn} := \text{span}\left\{1, x, \ldots, x^{m/2}, x^{\alpha/2}, x^{1+\alpha/2}, \ldots, x^{(n+\alpha)/2}\right\}$$

forms a Haar space of dimension $(m+n)/2+1$ on $[0,1]$ if $\alpha \notin 2\mathbb{N}$. For more details see [SSW], §2.

Let the set of extreme points $\{x_1, \ldots, x_{m+n+3}\}$ be denoted by A_{mn}, and the counting measure of this set by

$$\nu_{A_{mn}} := \sum_{x \in A_{mn}} \delta_x, \tag{21}$$

where δ_z is Dirac's measure at $z \in \mathbb{C}$. By $\omega_{[-1,1]}$ we denote the equilibrium distribution of the set $[-1,1]$, i.e.

$$d\omega_{[-1,1]}(x) = (1/\pi)dx/\sqrt{1-x^2}, \quad x \in [-1,1]. \tag{22}$$

Theorem 6. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$. For any ray sequence $N_c \in \mathbb{N}^2$ with numerator-denominator ratio $c \in [1,\infty]$ and $m \geq n + 2[\alpha/2]$ for all $(m,n) \in \mathbb{N}_c$ we have*

$$\frac{1}{m+n+3}\nu_{A_{mn}} \xrightarrow{*} \left(1 - \frac{|c-1|}{c+1}\right)\delta_0 + \frac{|c-1|}{c+1}\omega_{[-1,1]} \quad \text{as } m+n \to \infty, \quad (m,n) \in N_c. \tag{23}$$

where $\xrightarrow{*}$ denotes the convergence of measures in the weak topology.

Remarks (1) We see that the asymptotic distribution of the extreme points changes continuously with the ratio c. Actually, it is a convex combination of the two measures δ_0 and $\omega_{[-1,1]}$, which hold in the extreme cases of the paradiagonal rational approximants $r^*_{n+2[\alpha/c],n}$ and the polynomial approximants r^*_{m0}, respectively.

(2) In the paradiagonal case $m = n + 2[\alpha/2]$ we have $c = 1$, and therefore the asymptotic distribution is δ_0. Hence, almost all extreme points of $f - r^*_{n+2[\alpha/2],n}$ converge to $z = 0$.

(3) In the cases of polynomial approximation ($c = \infty$) assertion (23) has been proved in [BS]. However, Theorem 6 is somewhat more general since it also covers sequences $\{r^*_{m,n_m}\}_{m=1}^{\infty}$ with $m/n_m \to \infty$.

The proof of Theorem 6 is closely connected with that of Theorem 2 and follows from the same analysis. For details see [SSW], §4.

We now come to the last result of our investigation: the asymptotic distribution of zeros and poles of the approximants $r^*_{mn} = r^*_{mn}(|x|^\alpha, [-1,1])$.

Theorem 7. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and let N_c be a ray sequence with $c \in [1,\infty)$, $m \geq n + 2[\alpha/2]$, and m, n even for all $(m,n) \in N_c$.*

(a) *The approximant r^*_{mn} is of exact degree m and n, and all n poles of r^*_{mn} are simple and lie on the imaginary axis.*

(b) *Let $P_{mn} := \{y_1, \ldots, y_n\}$ be the set of all poles of r^*_{mn}. They can be ordered so that*

$$\mathrm{Im}(y_1) < \mathrm{Im}(y_2) < \ldots < \mathrm{Im}(y_{n/2}) < 0 < \mathrm{Im}(y_{n/2+1}) < \ldots < \mathrm{Im}(y_n). \tag{24}$$

In each of the n segments $(y_1, y_2), \ldots, (y_{n/2-1}, y_{n/2}), (y_{n/2+1}, y_{n/2+2}), \ldots, (y_{n-1}, y_n)$ there lies at least one zero of r^*_{mn}, $2[\alpha/2] + 2$ zeros cluster near $z = 0$, and the remaining $m - n - 2[\alpha/2]$ zeros surround the interval $[-1,1]$.

(c) *Denote the set of all zeros of r^*_{mn} by Z_{mn}. Then we have*

$$\frac{1}{n}\nu_{P_{mn}} \xrightarrow{*} \delta_0 \quad \text{as } m+n \to \infty,\ (m,n) \in N_c \tag{25}$$

$$\frac{1}{m}\nu_{Z_{mn}} \xrightarrow{*} \left(1 - \frac{|c-1|}{c+1}\right)\delta_0 + \frac{|c-1|}{c+1}\omega_{[-1,1]}$$

as $m + n \to \infty$, $(m,n) \in N_c$.

(d) *If $c > 1$, then all poles of r^*_{mn} converge to $z = 0$, and all zeros to $[-1,1]$, i.e.*

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{(m,n)\in N_c \\ m+n\geq k}} P_{mn}} = \{0\}, \quad \bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{(m,n)\in N_c \\ m+n\geq k}} Z_{mn}} = [-1,1]. \tag{26}$$

Remarks. (1) The additional assumption that $m \geq n + 2[\alpha/2]$ is always satisfied, up to a finite number of indices if $c > 1$.

(2) If $c = 1$, then assertion (d) is in general not true. For instance, if $m = n + 2[\alpha/2]$, then the pole with largest modulus tends to infinity as n tends to infinity.

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