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On the Theorem of Whitney-Sendov

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Abstract. For any bounded intergrable in the interval [a,b] function f we prove that

$$\sup_{a \le x \le b} |f(x) - Q_{n-1}(f;x)| \le 38\, \omega_n(f;\frac{b-a}{n})$$

for any n = 1,2,..., where $Q_{n-1}(f;x)$ is the interpolation algebraic polynomial of degree n-1 for the function f in the knots $a + m\frac{b-a}{n-1}$, for m = 0,1,...,n-1, and

$$\omega_n(f;h) := \sup\,\{|\Delta_h^n f(t)|:\ t, t+nh \in [0,1]\},$$

where $\Delta_h^n f(t)$ is the n-th differences with step size h defined by

$$\Delta_h^n f(t) := \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} f(t+ih).$$

1. Introduction. Let f(x) be a real valued bounded and measurable in Lebesque sense function on [0,1]. We define two interpolation polynomials of f: first, $Q_{n-1}(f;x)$ is the Lagrange interpolation polynomial of the function f which interpolates f at the points $q_m = \frac{m}{n-1}$, for m = 0,1,2,...,n-1 and second, $P_{n-1}(f;x)$ is the Lagrange interpolation polynomial of the function f which interpolates f at the points $p_m = \frac{m}{n+1}$, for m = 1,2,...,n. We shall also use the notation

$$l_{n,m}(x) = \prod_{\substack{j=0 \\ j \ne m}} \frac{x-j}{m-j}, \quad m = 0,1,\ldots,n,$$

for the basic Lagrange polynomials for interpolation at the knots 0,1,2,...,n.

In [1] it was proved the following

Theorem 1. (Whitney- Sendov). For any bounded and intergrable on [0,1] function f and for each integer $n \geq 1$, we have

$$|f(x) - P_{n-1}(f;x)| \leq 6\,\omega_n(f;\frac{1}{n+1}) \text{ for } x \in [0,1]. \tag{1}$$

It was Whitney [3] who first proved the existence of the constants W_n

$$\sup_{0\leq x\leq 1} |f(x) - Q_{n-1}(f;x)| \leq W_n\,\omega_n(f;\frac{1}{n}).$$

Sendov and Popov [2, p.55] proved that $W_n = O(\ln n)$. In this paper we shall prove the following

Theorem 2. For any bounded and intergrable on [0,1] function f we have

$$|f(x) - Q_{n-1}(f;x)| \leq 38\,\omega_n(f;\frac{1}{n+1}) \text{ for } x \in [0,1]. \tag{2}$$

2. Notatoins and preliminaries. In the article n is a fixed integer. We set $h := \frac{1}{n+1}$, $\sigma_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ and $\sigma_0 := 0$. The following integral representation for the function f was proved in [1]

$$f(x) - P_{n-1}(f;x) = \Delta_h^n f(0) l_{n,0}\left[\frac{x}{h}\right] + \varphi_n(f;x) \tag{3}$$

$$- \sum_{i=0}^{n} \varphi_n(f;ih) l_{n,i}\left[\frac{x}{h}\right] + \frac{1}{h}\int_0^t \sum_{i=0}^{n} \varphi_n(f;ih+v) l'_{n,i}\left[\frac{x-v}{h}\right] dv$$

where $x = ih + t$, for $i = 0,1,\dots,n$, $0 \leq t \leq h$, and

$$\varphi_n(f;x) = \varphi_n(f;ih+t) := \frac{(-1)^{n-i}}{h\binom{n}{i}} \int_0^h \Delta_\tau^n f(t+(h-\tau)i)\,d\tau.$$

Obviously,

$$|\varphi_n(f;x)| \leq \binom{n}{i}^{-1} \omega_n(f;h). \tag{4}$$

In [1] it was given the estimate

$$|f(x) - P_{n-1}(f;x)| \leq (6 + 7\min(\sigma_m, \sigma_{n-m})) \binom{n}{m}^{-1} \omega_n(f;\frac{1}{n+1}) \tag{5}$$

where $mh \leq x \leq (m+1)h$ and $m = 0,1,\dots,[(n-1)/2]$.

For the interpolation knots we have

(6) $$\gamma_{n,m} := (q_m - p_m)/h = 2m/(n-1)$$

for $m = 1,\dots,n-1$. The polynomial $P_{n-1}(f;x)$ may be written as follows

$$P_{n-1}(f;x) = \sum_{i=1}^{n} f(ih) l_{n-1,i-1}\left(\frac{x-h}{h}\right)$$

and it is evident that

(7) $$P_{n-1}(f(1-x);x) = P_{n-1}(f(x);1-x).$$

(8) $$| f(0) - P_{n-1}(f;0) | = | \Delta_h^n f(0) | \le \omega_n(f;h).$$

(9) $$| f(1) - P_{n-1}(f;1) | = | \Delta_h^n f(h) | \le \omega_n(f;h).$$

We need the following lemmas:

Lemma 1. (see [1])

$$\max_{0 \le x \le 1} \left\{ \sum_{i=0}^{n} \binom{n}{i}^{-1} |l_{n,i}(x)| \right\} = 1.$$

Lemma 2. (see [1]) Let

$$\mu_{n,m} := \sum_{i=0}^{n} \binom{n}{i}^{-1} \max_{m \le x \le m+1} |l_{n,i}(x)|.$$

Then $\mu_{n,m} \le \binom{n}{m}^{-1}(1 + \sigma_m + \sigma_{m+1})$, $m = 0,1,2,\dots,[(n-1)\backslash 2]$.

Lemma 3. For $n = 4,5,\dots$ we have

$$S_n := \sum_{i=0}^{n-1} \frac{i}{n-1} \binom{n}{i}^{-1} \max_{0 \le x \le 1} |l_{n,i}(x)| \le \frac{1}{\sigma_n - 1}.$$

Proof. Let β be the unique root of the equation

$$\frac{1}{x} - \frac{1}{2-x} - \frac{1}{3-x} - \dots - \frac{1}{n-x} = 0$$

in the interval $[0,1]$. We have

$$0 < \beta \le \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)^{-1} = \frac{1}{\sigma_n - 1} < 1 \text{ for } n = 4,5,\dots$$

Now, we have

$$S_n = \frac{1}{n!} \sum_{i=1}^{n-1} \frac{i}{n-1} \max_{0 \le x \le 1} \frac{x(1-x)\ldots(n-x)}{(i-x)}$$

$$\le \frac{1}{n!} \sum_{i=1}^{n-1} \frac{i}{n-1} \max_{0 \le x \le 1} \frac{|1-x/i|}{|1-x|} \frac{x(1-x)\ldots(n-x)}{(i-x)}$$

$$= \frac{1}{n!} \sum_{i=1}^{n-1} \frac{1}{n-1} \max_{0 \le x \le 1} x(2-x)\ldots(n-x)$$

$$= \frac{1}{n!}\beta(2-\beta)\ldots(n-\beta) \le \beta. \square$$

Lemma 4. For $0 \le m \le [(n-1)/2]$ we have

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})| \le \binom{n}{m}^{-1} \left[1 + \frac{2m}{n-1}\left[\sigma_m + \sigma_{n-m} + 1\right]\right].$$

Proof. A direct computation gives

$$\binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})| \tag{10}$$

$$= \frac{1}{n!} \frac{2m}{n-1} \, |m-i+\frac{2m}{n-1}|^{-1} \prod_{j=1}^{m}(j + \frac{2m}{n-1}) \prod_{j=1}^{n-m}(j - \frac{2m}{n-1})$$

$$= \binom{n}{m}^{-1} \frac{2m}{n-1} \, |m-i+\frac{2m}{n-1}|^{-1} \prod_{j=1}^{m}(1 + \frac{2m}{j(n-1)}) \prod_{j=1}^{n-m}(j - \frac{2m}{j(n-1)})$$

$$= \binom{n}{m}^{-1} \frac{2m}{n-1} \, |m-i+\frac{2m}{n-1}|^{-1} \prod_{j=1}^{m} \left[1 - \left[\frac{2m}{(n-1)m}\right]^2\right] \prod_{j=m+1}^{n-m}(j - \frac{2m}{j(n-1)})$$

$$\le \binom{n}{m}^{-1} \frac{2m}{n-1} \, |m-i+\frac{2m}{n-1}|^{-1}.$$

From (10) it is easy to see that

$$|l_{n,m}(m + \frac{2m}{n-1})| \le 1. \tag{11}$$

Now, using (10) and (11) we get

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})|$$

$$\leq \binom{n}{m}^{-1}\left\{1+\sum_{i=0}^{m-1}(m-i+\frac{2m}{n-1})^{-1}+\sum_{i=m+1}^{n}(i-m-\frac{2m}{n-1})^{-1}\right\}$$

$$\leq \binom{n}{m}^{-1}\left\{1+\frac{2}{n-1}\left[\sigma_m+\sigma_{n-m}+1\right]\right\}.\square$$

3. Proof of the Theorem 2. In [3,4] Whitney proved $W_1=W_2=1$, $W_3=14/9$, $W_4 \leq 5.6$, $W_5 \leq 10.4$ and we may suppose that $n\geq 6$. We set

$$t_m := q_m - p_m = \frac{2mh}{n-1}$$

for $m = 1,\dots,n-1$. Let $[x]$ denote the integer part of x, and $k:=[(n-1)/2]$. For $m = 1,\dots,k$ following Sendov [1] we get

$$f(q_m) - P_{n-1}(f;q_m) = \Delta_h^n f(0) l_{n,0}\left(\frac{q_m}{h}\right) + \varphi_n(f;q_m)$$

$$-\sum_{i=0}^{n}\varphi_n(f;ih) l_{n,i}\left(\frac{q_m}{h}\right) + \frac{1}{h}\int_0^{t_m}\sum_{i=0}^{n}\varphi_n(f;ih+m) l'_{n,i}\left(\frac{q_m-v}{h}\right)dv.$$

Using (3) for $x = q_m$ we obtain

$$|f(q_m) - P_{n-1}(f;q_m)| \leq \omega_n(f;h)\binom{n}{m}^{-1}\left\{2+\binom{n}{m}\sum_{i=0}^{n}\binom{n}{i}^{-1}|l_{n,i}(q_m)|\right.$$

$$\left.+\binom{n}{m}\int_0^{\gamma_{n,m}}\sum_{i=0}^{n}\binom{n}{i}^{-1}|l'_{n,i}(m+\gamma_{n,m}-v)|dv\right\}.$$

Applying Lemma 4 and Lemma 2 consecutively, we receive

(12) $\quad |f(q_m) - P_{n-1}(f;q_m)|$

$$\leq \omega_n(f;h)\binom{n}{m}^{-1}\left\{3+\gamma_{n,m}(\sigma_m+\sigma_{n-m}+1)+2\gamma_{n,m}\binom{n}{m}\sum_{i=0}^{n}\binom{n}{i}^{-1}\max_{m\leq x\leq m+1}|l_{n,i}(x)|\right\}$$

$$\leq \omega_n(f;h)\binom{n}{m}^{-1}\left[3+3\gamma_{n,m}+6\gamma_{n,m}\sigma_n\right].$$

Using the symmetry (7) we see that (12) holds also for $m = k+1,\dots, n-2$. We have that

$$(13)\qquad |f(x) - Q_{n-1}(f;x)|$$

$$= |f(x) - P_{n-1}(f;x) - Q_{n-1}(f;x) + P_{n-1}(f;x)|$$

$$\leq |f(x) - P_{n-1}(f;x)| + |Q_{n-1}(f - P_{n-1};x)|.$$

Let $0 \leq x \leq h$. From (8), (9) and (12) we obtain

$$|Q_{n-1}(f - P_{n-1};x)| = |\sum_{i=0}^{n-1} [f(q_m) - P_{n-1}(f;q_m)] l_{n-1,i}((n-1)x)|$$

$$\leq \omega_n(f;h)\Big[|l_{n-1,0}((n-1)x)| + |l_{n-1,n-1}((n-1)x)|$$

$$+ \sum_{i=1}^{n-2} (3+3\gamma_{n,i}+6\gamma_{n,i}\sigma_n)\binom{n}{i}^{-1} |l_{n-1,i}((n-1)x)|\Big].$$

Now, from Lemma 1 and Lemma 4 we get

$$(15)\qquad |Q_{n-1}(f - P_{n-1};x)| \leq \omega_n(f;h)\left[3 + \frac{6}{\sigma_{n-1}-1} + \frac{12}{\sigma_{n-1}-1}\sigma_n\right] \leq 32\ \omega_n(f;h).$$

Analogiously, we have (15) for $nh \leq x \leq 1$.

For $h \leq x \leq nh$ using (7), (8), (9), (12) and Lemma 2 we get

$$|Q_{n-1}(f - P_{n-1};x)| \leq \omega_n(f;h) \sum_{i=0}^{n-1} (3+3\gamma_{n,i}+6\gamma_{n,i}\sigma_n)\binom{n}{i}^{-1} |l_{n-1,i}((n-1)x)|$$

$$\leq \omega_n(f;h) \max_{1\leq t\leq n-1} \sum_{i=0}^{n-1} (6+6\sigma_n)\binom{n}{i}^{-1} |l_{n-1,i}(t)|$$

$$\leq \frac{6+6\sigma_n}{n}\ \omega_n(f;h).$$

The latter and (15) give

$$(16)\qquad |Q_{n-1}(f - P_{n-1};x)| \leq 32\ \omega_n(f;h) \text{ for } x \in [0,1].$$

Finally, combining (13), (1) and (16) we receive

$$|f(x) - Q_{n-1}(f;x)| \leq 38\,\omega_n(f;h).$$

It is easy to see that for large n the constant in Theorem 2 becomes 22.□

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