

**On the Theorem of Whitney-Sendov**

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**Abstract.** For any bounded intergrable in the interval  $[a,b]$  function  $f$  we prove that

$$\sup_{a \leq x \leq b} |f(x) - Q_{n-1}(f;x)| \leq 38 \omega_n(f; \frac{b-a}{n})$$

for any  $n = 1,2,\dots$ , where  $Q_{n-1}(f;x)$  is the interpolation algebraic polynomial of degree  $n-1$  for the function  $f$  in the knots  $a + m \frac{b-a}{n-1}$ , for  $m = 0,1,\dots,n-1$ , and

$$\omega_n(f;h) := \sup \{ |\Delta_h^n f(t)| : t, t+nh \in [0,1] \},$$

where  $\Delta_h^n f(t)$  is the  $n$ -th differences with step size  $h$  defined by

$$\Delta_h^n f(t) := \sum_{i=0}^n (-1)^{n+i} \binom{n}{i} f(t+ih).$$

**1. Introduction.** Let  $f(x)$  be a real valued bounded and measurable in Lebesgue sense function on  $[0,1]$ . We define two interpolation polynomials of  $f$ : first,  $Q_{n-1}(f;x)$  is the Lagrange interpolation polynomial of the function  $f$  which interpolates  $f$  at the points  $q_m = \frac{m}{n-1}$ , for  $m = 0,1,2,\dots,n-1$  and second,  $P_{n-1}(f;x)$  is the Lagrange interpolation polynomial of the function  $f$  which interpolates  $f$  at the points  $p_m = \frac{m}{n+1}$ , for  $m = 1,2,\dots,n$ . We shall also use the notation

$$l_{n,m}(x) = \prod_{\substack{j=0 \\ j \neq m}}^n \frac{x-j}{m-j}, \quad m = 0,1,\dots,n,$$

for the basic Lagrange polynomials for interpolation at the knots  $0,1,2,\dots,n$ .

In [1] it was proved the following

**Theorem 1.** (Whitney- Sendov). For any bounded and intergrable on  $[0,1]$  function  $f$  and for each integer  $n \geq 1$ , we have

$$(1) \quad |f(x) - P_{n-1}(f;x)| \leq 6 \omega_n(f; \frac{1}{n+1}) \text{ for } x \in [0,1].$$

It was Whitney [3] who first proved the existence of the constants  $W_n$

$$\sup_{0 \leq x \leq 1} |f(x) - Q_{n-1}(f;x)| \leq W_n \omega_n(f; \frac{1}{n}).$$

Sendov and Popov [2, p.55] proved that  $W_n = O(\ln n)$ . In this paper we shall prove the following

**Theorem 2.** For any bounded and intergrable on  $[0,1]$  function  $f$  we have

$$(2) \quad |f(x) - Q_{n-1}(f;x)| \leq 38 \omega_n(f; \frac{1}{n+1}) \text{ for } x \in [0,1].$$

**2. Notatoinis and preliminaries.** In the article  $n$  is a fixed integer. We set  $h := \frac{1}{n+1}$ ,  $\sigma_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  and  $\sigma_0 := 0$ . The following integral representation for the function  $f$  was proved in [1]

$$(3) \quad f(x) - P_{n-1}(f;x) = \Delta_h^n f(0) l_{n,0} \left( \frac{x}{h} \right) + \varphi_n(f;x)$$

$$- \sum_{i=0}^n \varphi_n(f; ih) l_{n,i} \left( \frac{x}{h} \right) + \frac{1}{h} \int_0^t \sum_{i=0}^n \varphi_n(f; ih+v) l'_{n,i} \left( \frac{x-v}{h} \right) dv$$

where  $x = ih + t$ , for  $i = 0,1,\dots,n$ ,  $0 \leq t \leq h$ , and

$$\varphi_n(f;x) = \varphi_n(f; ih+t) = \frac{(-1)^{n-i}}{h \binom{n}{i}} \int_0^h \Delta_\tau^n f(t+(h-\tau)i) d\tau.$$

Obviously,

$$(4) \quad |\varphi_n(f;x)| \leq \binom{n}{i}^{-1} \omega_n(f;h).$$

In [1] it was given the estimate

$$(5) \quad |f(x) - P_{n-1}(f;x)| \leq (6 + 7 \min(\sigma_m, \sigma_{n-m})) \binom{n}{m}^{-1} \omega_n(f; \frac{1}{n+1})$$

where  $mh \leq x \leq (m+1)h$  and  $m = 0,1,\dots, [(n-1)/2]$ .

For the interpolation knots we have

$$(6) \quad \gamma_{n,m} := (q_m - p_m)/h = 2m/(n-1)$$

for  $m = 1, \dots, n-1$ . The polynomial  $P_{n-1}(f; x)$  may be written as follows

$$P_{n-1}(f; x) = \sum_{i=1}^n f(ih) l_{n-1,i-1} \left[ \frac{x-h}{h} \right]$$

and it is evident that

$$(7) \quad P_{n-1}(f(1-x); x) = P_{n-1}(f(x); 1-x).$$

$$(8) \quad |f(0) - P_{n-1}(f; 0)| = |\Delta_h^n f(0)| \leq \omega_n(f; h).$$

$$(9) \quad |f(1) - P_{n-1}(f; 1)| = |\Delta_h^n f(h)| \leq \omega_n(f; h).$$

We need the following lemmas:

**Lemma 1.** (see [1])

$$\max_{0 \leq x \leq 1} \left\{ \sum_{i=0}^n \binom{n}{i}^{-1} |l_{n,i}(x)| \right\} = 1.$$

**Lemma 2.** (see [1]) Let

$$\mu_{n,m} := \sum_{i=0}^n \binom{n}{i}^{-1} \max_{m \leq x \leq m+1} |l_{n,i}(x)|.$$

Then  $\mu_{n,m} \leq \binom{n}{m}^{-1} (1 + \sigma_m + \sigma_{m+1})$ ,  $m = 0, 1, 2, \dots, [(n-1)/2]$ .

**Lemma 3.** For  $n = 4, 5, \dots$  we have

$$S_n := \sum_{i=0}^{n-1} \frac{i}{n-1} \binom{n}{i}^{-1} \max_{0 \leq x \leq 1} |l_{n,i}(x)| \leq \frac{1}{\sigma_n - 1}.$$

**Proof.** Let  $\beta$  be the unique root of the equation

$$\frac{1}{x} - \frac{1}{2-x} - \frac{1}{3-x} - \dots - \frac{1}{n-x} = 0$$

in the interval  $[0, 1]$ . We have

$$0 < \beta \leq \left[ \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right]^{-1} = \frac{1}{\sigma_n - 1} < 1 \text{ for } n = 4, 5, \dots$$

Now, we have

$$\begin{aligned}
S_n &= \frac{1}{n!} \sum_{i=1}^{n-1} \frac{i}{n-1} \max_{0 \leq x \leq 1} \frac{x(1-x)\dots(n-x)}{(i-x)} \\
&\leq \frac{1}{n!} \sum_{i=1}^{n-1} \frac{i}{n-1} \max_{0 \leq x \leq 1} \frac{|1-x/i|}{|1-x|} \frac{x(1-x)\dots(n-x)}{(i-x)} \\
&= \frac{1}{n!} \sum_{i=1}^{n-1} \frac{1}{n-1} \max_{0 \leq x \leq 1} x(2-x)\dots(n-x) \\
&= \frac{1}{n!} \beta(2-\beta)\dots(n-\beta) \leq \beta. \square
\end{aligned}$$

**Lemma 4.** For  $0 \leq m \leq [(n-1)/2]$  we have

$$\sum_{i=0}^n \binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})| \leq \binom{n}{m}^{-1} \left[ 1 + \frac{2m}{n-1} (\sigma_m + \sigma_{n-m} + 1) \right].$$

**Proof.** A direct computation gives

$$\begin{aligned}
(10) \quad & \binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})| \\
&= \frac{1}{n!} \frac{2m}{n-1} |m-i + \frac{2m}{n-1}|^{-1} \prod_{j=1}^m (j + \frac{2m}{n-1}) \prod_{j=1}^{n-m} (j - \frac{2m}{n-1}) \\
&= \binom{n}{m}^{-1} \frac{2m}{n-1} |m-i + \frac{2m}{n-1}|^{-1} \prod_{j=1}^m (1 + \frac{2m}{j(n-1)}) \prod_{j=1}^{n-m} (j - \frac{2m}{j(n-1)}) \\
&= \binom{n}{m}^{-1} \frac{2m}{n-1} |m-i + \frac{2m}{n-1}|^{-1} \prod_{j=1}^m \left[ 1 - \left( \frac{2m}{(n-1)m} \right)^2 \right] \prod_{j=m+1}^{n-m} (j - \frac{2m}{j(n-1)}) \\
&\leq \binom{n}{m}^{-1} \frac{2m}{n-1} |m-i + \frac{2m}{n-1}|^{-1}.
\end{aligned}$$

From (10) it is easy to see that

$$(11) \quad |l_{n,m}(m + \frac{2m}{n-1})| \leq 1.$$

Now, using (10) and (11) we get

$$\sum_{i=0}^n \binom{n}{i}^{-1} |l_{n,i}(m + \frac{2m}{n-1})|$$

$$\leq \binom{n}{m}^{-1} \left\{ 1 + \sum_{i=0}^{m-1} \left(m-i + \frac{2m}{n-1}\right)^{-1} + \sum_{i=m+1}^n \left(i-m - \frac{2m}{n-1}\right)^{-1} \right\} \quad (11)$$

$$\leq \binom{n}{m}^{-1} \left\{ 1 + \frac{2}{n-1} \left[ \sigma_m + \sigma_{n-m} + 1 \right] \right\}. \square$$

**3. Proof of the Theorem 2.** In [3,4] Whitney proved  $W_1=W_2=1$ ,  $W_3=14/9$ ,  $W_4 \leq 5.6$ ,  $W_5 \leq 10.4$  and we may suppose that  $n \geq 6$ . We set

$$t_m := q_m - p_m = \frac{2mh}{n-1}$$

for  $m = 1, \dots, n-1$ . Let  $[x]$  denote the integer part of  $x$ , and  $k := [(n-1)/2]$ . For  $m = 1, \dots, k$  following Sendov [1] we get

$$f(q_m) - P_{n-1}(f; q_m) = \Delta_h^n f(0) l_{n,0} \left( \frac{q_m}{h} \right) + \varphi_n(f; q_m) - \sum_{i=0}^n \varphi_n(f; ih) l_{n,i} \left( \frac{q_m}{h} \right) + \frac{1}{h} \int_0^{t_m} \sum_{i=0}^n \varphi_n(f; ih+m) l'_{n,i} \left( \frac{q_m-v}{h} \right) dv.$$

Using (3) for  $x = q_m$  we obtain

$$|f(q_m) - P_{n-1}(f; q_m)| \leq \omega_n(f; h) \binom{n}{m}^{-1} \left\{ 2 + \binom{n}{m} \sum_{i=0}^n \binom{n}{i}^{-1} |l_{n,i}(q_m)| + \binom{n}{m} \int_0^{\gamma_{n,m}} \sum_{i=0}^n \binom{n}{i}^{-1} |l'_{n,i}(m+\gamma_{n,m}-v)| dv \right\}.$$

Applying Lemma 4 and Lemma 2 consecutively, we receive

$$(12) \quad |f(q_m) - P_{n-1}(f; q_m)| \leq \omega_n(f; h) \binom{n}{m}^{-1} \left\{ 3 + \gamma_{n,m} (\sigma_m + \sigma_{n-m} + 1) + 2\gamma_{n,m} \binom{n}{m} \sum_{i=0}^n \binom{n}{i}^{-1} \max_{m \leq x \leq m+1} |l_{n,i}(x)| \right\}$$

$$\leq \omega_n(f; h) \binom{n}{m}^{-1} \left[ 3 + 3\gamma_{n,m} + 6\gamma_{n,m} \sigma_n \right].$$

Using the symmetry (7) we see that (12) holds also for  $m = k+1, \dots, n-2$ . We have that

$$\begin{aligned}
 (13) \quad & |f(x) - Q_{n-1}(f;x)| \\
 &= |f(x) - P_{n-1}(f;x) - Q_{n-1}(f;x) + P_{n-1}(f;x)| \\
 &\leq |f(x) - P_{n-1}(f;x)| + |Q_{n-1}(f - P_{n-1};x)|.
 \end{aligned}$$

Let  $0 \leq x \leq h$ . From (8), (9) and (12) we obtain

$$\begin{aligned}
 |Q_{n-1}(f - P_{n-1};x)| &= \left| \sum_{i=0}^{n-1} [f(q_m) - P_{n-1}(f;q_m)] l_{n-1,i}((n-1)x) \right| \\
 &\leq \omega_n(f;h) \left[ |l_{n-1,0}((n-1)x)| + |l_{n-1,n-1}((n-1)x)| \right. \\
 &\quad \left. + \sum_{i=1}^{n-2} (3+3\gamma_{n,i} + 6\gamma_{n,i} \sigma_n) \binom{n}{i}^{-1} |l_{n-1,i}((n-1)x)| \right].
 \end{aligned}$$

Now, from Lemma 1 and Lemma 4 we get

$$(15) \quad |Q_{n-1}(f - P_{n-1};x)| \leq \omega_n(f;h) \left[ 3 + \frac{6}{\sigma_{n-1}-1} + \frac{12}{\sigma_{n-1}-1} \sigma_n \right] \leq 32 \omega_n(f;h).$$

Analogously, we have (15) for  $nh \leq x \leq 1$ .

For  $h \leq x \leq nh$  using (7), (8), (9), (12) and Lemma 2 we get

$$\begin{aligned}
 |Q_{n-1}(f - P_{n-1};x)| &\leq \omega_n(f;h) \sum_{i=0}^{n-1} (3+3\gamma_{n,i} + 6\gamma_{n,i} \sigma_n) \binom{n}{i}^{-1} |l_{n-1,i}((n-1)x)| \\
 &\leq \omega_n(f;h) \max_{1 \leq t \leq n-1} \sum_{i=0}^{n-1} (6+6\sigma_n) \binom{n}{i}^{-1} |l_{n-1,i}(t)| \\
 &\leq \frac{6+6\sigma_n}{n} \omega_n(f;h).
 \end{aligned}$$

The latter and (15) give

$$(16) \quad |Q_{n-1}(f - P_{n-1};x)| \leq 32 \omega_n(f;h) \text{ for } x \in [0,1].$$

Finally, combining (13), (1) and (16) we receive

$$|f(x) - Q_{n-1}(f;x)| \leq 38 \omega_n(f;h).$$

It is easy to see that for large  $n$  the constant in Theorem 2 becomes 22.  $\square$

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Dedicated to the memory of Vasil Popov

## 1. Introduction

Let  $f: \mathbb{R} \rightarrow \mathbb{R}^2$  be a smooth planar curve and  $\{x_i\}$  be a given increasing sequence of real numbers. The standard geometric Hermite interpolant matches the position and the direction of the tangent at the positions. It is easy to find a quadratic spline which extends along the standard geometric Hermite interpolation for each curve section  $[x_i, x_{i+1}]$ . The resulting spline is  $C^1$  and is  $O(h^2)$  accurate, where  $h = \max\{x_{i+1} - x_i\}$  and  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^2$ . Denote by  $a \times b$  the cross product of two vectors,  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$  in  $\mathbb{R}^2$  and by  $\rho(a) = \frac{1}{|a|} \frac{a_1 \times a_2}{|a|}$  the curvature of  $f(x)$ . An Hermite interpolation by piecewise cubic B-splines which involves curvature in addition to standard Hermite geometric data is described in [2]. The spline provides a  $C^2$  spline which under appropriate conditions preserves the convexity and is  $O(h^3)$ . An important advantage is that the construction is local. It is also demonstrated a technique for estimation of the error which may be applied to various geometric interpolation problems.

Interpolants using nonconsecutive derivative order values at the knots are called Birkhoff or lacunary in the theory of approximations. Making analogy with the geometric Hermite and Birkhoff interpolation natural questions arising are about existence, uniqueness, error estimation and algorithms for geometric lacunary interpolation. For details on Birkhoff interpolation the reader is referred to [3]. In section 2 we study parametric quadratic splines fitting the position and curvature at each knot. We have that tangent direction coincides but the interpolant has exponential

This work is supported by the Ministry of Education and Science under grant no. 88M-15.