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Geometric Birkhoff Interpolation with Parametric Quadratic Splines*

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Dedicated to the memory of Vasil Popov

1. Introduction

Let $\mathbf{f} : \mathbf{R} \to \mathbf{R}^2$ be a smooth planar curve and $\{t_i\}$ be a given increasing sequence of real numbers. The standard geometric Hermite interpolant matches the positions $\mathbf{f}(t_i)$ and direction of the tangent at the positions. It is easy to find a quadratic Bézier polynomial solving the standard geometric Hermite interpolation for each curve segment $\mathbf{f}(t), t \in [t_i, t_{i+1}]$. The resulting spline is G^1 and is $O(h^4)$ accurate, where $h := \max_i \|\mathbf{f}(t_{i+1}) - \mathbf{f}(t_i)\|$ and $\| \circ \|$ is the Euclidean norm in $\mathbf{R}^2$. Denote by $\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1$ the cross product of two vectors $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ in $\mathbf{R}^2$ and by $\mathbf{f}^{**}(t) := \frac{\mathbf{f}'(t) \times \mathbf{f}''(t)}{\|\mathbf{f}'(t)\|^3}$ the curvature at $\mathbf{f}(t)$. An Hermite interpolation by piecewise cubic Bézier polynomials involving curvature in addition to standard Hermite geometric data is described in [2]. This scheme provides a G^2 spline which under appropriate conditions preserves the convexity and is $O(h^6)$. An important advantage is that the construction is local. It is also demonstrated a technique for estimation of the error which may be applied to various geometric interpolation problems.

Interpolations using nonconsecutive derivative order values at the knots are called Birkhoff or lacunary in the theory of approximations. Making analogy with the algebraic polynomial Hermite and Birkhoff interpolation natural questions arising are about existence, uniqueness, error estimation and algorithms for geometric lacunary interpolations. For details on Birkhoff interpolation the reader is refered to [3]. In Section 2 we study parametric quadratic splines fitting the position and curvature at each knot. We loss the tangent direction continuity but the interpolant has continuous

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curvature. Applying some results of [4] we prove $O(h^3)$ accuracy. Another high accuracy local scheme based on the usual two point (0,2) Birkhoff interpolation is given in Section 3. The resulting parametric quadratic spline interpolant is explicitely found and is $O(h^4)$. The error analysis in this case is done using the technique of [2].

2. C^0 interpolation with continuous curvature

Let $\mathbf{f} : [t_0, t_n] \to \mathbf{R}^2$ be a smooth planar curve with nonvanishing curvature and the real numbers $t_0 < t_1 < \ldots < t_n$ be given. Setting $\mathbf{f}_i := \mathbf{f}(t_i)$, $k_i := \mathbf{f}^{**}(t_i)$ we consider the interpolation problem

$$\mathbf{g}(i) = \mathbf{f}_i, \quad \mathbf{g}^{**}(i) = k_i, \quad i = 0, \ldots, n, \tag{2.1}$$

for piecewise quadratic Bézier polynomials $\mathbf{g}$. More precisely, on each of the parameter intervals $[i, i+1]$ we seek the interpolant in the form [1]:

$$\mathbf{p}(t) = \sum_{j=0}^{2} \mathbf{b}_j \binom{2}{j} t^j (1-t)^{2-j}, \quad 0 \le t \le 1. \tag{2.2}$$

So, each piece of the spline $\mathbf{g}$ can be computed individualy using the data at two consecutive positions. Consider, for example, the problem (2.1) for the first curve segment:

$$\mathbf{p}(i) = \mathbf{f}_i, \quad \mathbf{p}^{**}(i) = k_i, \quad i = 0, 1. \tag{2.3}$$

Evidently $\mathbf{b}_0 = \mathbf{f}_0$, $\mathbf{b}_2 = \mathbf{f}_1$ and it remains to find $\mathbf{b}_1 = (x, y)$. Without loss of generality we may assume that $\mathbf{f}_0 = (-\delta, 0)$, $\mathbf{f}_1 = (\delta, 0)$, where $h := 2\delta := \|\mathbf{f}_1 - \mathbf{f}_0\|$. It is not difficult to verify that the interpolation conditions (2.3) give the nonlinear system

$$\begin{aligned} (x+\delta)^2 + y^2 &= (-\delta y / k_0)^{2/3}, \\ (x-\delta)^2 + y^2 &= (-\delta y / k_1)^{2/3}. \end{aligned} \tag{2.4}$$

Let us concentrate to the case $|k_1| > |k_0|$ first. Subtracting the equalities in (2.4) we find

$$y = 8\delta^{1/2} k_0 k_1 \left(\frac{x}{k_1^{2/3} - k_0^{2/3}} \right)^{3/2}. \tag{2.5}$$

Then, using (2.5) we obtain from (2.4) an algebraic equation with respect to x:

$$Q(x) := a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0, \tag{2.6}$$

where

$$\begin{aligned} a_0 &= 64\delta k_0^2 k_1^2, \\ a_1 &= (k_1^{2/3} - k_0^{2/3})^3, \\ a_2 &= -2\delta(k_0^{2/3} + k_1^{2/3})(k_1^{2/3} - k_0^{2/3})^2, \\ a_3 &= \delta^2 (k_1^{2/3} - k_0^{2/3})^3, \end{aligned} \tag{2.7}$$

but since by (2.5) sign$x$=sign$(k_1^{2/3} - k_0^{2/3})$ we are interested in positive roots only. Obviously, $a_i > 0$, $i = 0, 1, 3$, $a_2 < 0$ and Descartes rule gives that Q has one negative and either two simple, or one double or no positive zeros. Suppose that $Q'(\xi) = 0$, $\xi = \frac{-a_1+\sqrt{a_1^2-3a_0a_2}}{3a_0}$ being the only positive extremal point of the algebraic cubic polynomial Q. Note that Q has a local minimum at the point ξ. Therefore Q has a positive zero if and only if $Q(\xi) \le 0$. From

$$Q(\xi) = \frac{-2(a_1^2 - 3a_0a_2)^{3/2} + 27a_0^2a_3 - 9a_0a_1a_2 + 2a_1^3}{27a_0^2} \le 0$$

we get the inequality

$$\frac{(27a_0^2a_3 - 9a_0a_1a_2 + 2a_1^3)^2}{4(a_1^2 - 3a_0a_2)^3} \le 1 \tag{2.8}$$

In the case $|k_1| < |k_0|$, x has to be a negative root of the equation (2.6), with coefficients (2.7). Since $a_0 > 0$, $a_i < 0$, $i = 1, 2, 3$, the polynomial Q has one positive zero, and it also has a negative zero if and only if $Q(\eta) \ge 0$, where $Q'(\eta) = 0$, $\eta = \frac{-a_1-\sqrt{a_1^2-3a_0a_2}}{3a_0} < 0$. The inequality $Q(\eta) \ge 0$ yields (2.8).

If $k_0 = k_1$ we have $x = 0$, sign$y$=−signk_0 and we obtain the equation

$$Q(z) := a_0z^3 + a_1z^2 + a_2z + a_3 = 0, \tag{2.9}$$

with respect to $z := y^2$, where the coefficients are defined as follows

$$a_0 = 1, \quad a_1 = 3\delta^2, \quad a_2 = \delta^2(3\delta^2 - k_0^{-2}), \quad a_3 = \delta^6. \tag{2.10}$$

Using the same arguments as in the case when $|k_1| > |k_0|$ it is seen that inequality (2.8) is necessary and sufficient condition for (2.9) to have a positive root.

Given data, condition (2.8) is easily verified and if it holds then there exists at least one solution of the system (2.4). We now prove that for sufficiently dense data there exist interpolants.

Theorem 1. *If* $\mathbf{f}$ *is a smooth planar curve with nonvanishing curvature and* $h := \max_i \|\mathbf{f}(t_{i+1}) - \mathbf{f}(t_i)\|$ *is sufficiently small, then solutions of the interpolation problem* (2.1) *exist and each interpolant is* $O(h^3)$ *acurate.*

Proof. To prove existence part it is sufficient to show that (2.8) holds for small h. Let us parametrize $\mathbf{f}$ by arclength s, so that

$$\mathbf{f}_0 = \mathbf{f}(-\epsilon), \quad \mathbf{f}_1 = \mathbf{f}(\epsilon), \quad \epsilon > 0,$$

$$\mathbf{f}^*(s) = \mathbf{f}'(s) := (\cos\theta(s), \sin\theta(s)), \quad \theta'(s) = k(s),$$

with θ the indefinite integral of the curvature. Expanding $\theta(s)$ to second order at the point $s = 0$:

$$\theta(s) = \theta_0 + \theta_1 s + \theta_2 s^2 + O(s^3)$$

and taking into account that $\theta_1 \neq 0$ since the curvature of $\mathbf{f}$ does not vanish we obtain

$$k_0 = \theta'(-\epsilon) = \theta_1 - 2\theta_2\epsilon + O(\epsilon^2), \quad k_1 = \theta'(\epsilon) = \theta_1 + 2\theta_2\epsilon + O(\epsilon^2),$$

(2.11) $$k_1^{2/3} - k_0^{2/3} = \frac{8\theta_2}{3\theta_1^{1/3}}\epsilon + O(\epsilon^2), \quad h = 2\delta = 2\epsilon + O(\epsilon^3).$$

Using (2.11) we find $a_0 = O(h)$, $a_1 = O(h^3)$, $a_2 = O(h^3)$, $a_3 = O(h^5)$, for the coefficients (2.7), and $a_0 = 1$, $a_1 = O(h^2)$, $a_2 = O(h^2)$, $a_3 = O(h^6)$, for the coefficients (2.10). Hence, $\frac{(27a_0^2a_3 - 9a_0a_1a_2 + 2a_1^3)^2}{4(a_1^2 - 3a_0a_2)^3} = O(h^2)$ in both cases and (2.8) holds for sufficiently small $h > 0$.

By Lemma 6.1 [4] the angle $\alpha_0(\mathbf{f})$ between the tangent to $\mathbf{f}$ at $\mathbf{f}(s_0)$ and a chord $\mathbf{f}(s_1) - \mathbf{f}(s_0)$ is

$$\alpha_0(\mathbf{f}) = \frac{1}{2}k(s_0)h + \frac{1}{3}k'(s_0)h^2 + O(h^3),$$

for $s_1 > s_0$, $s_1 \to s_0$, where $k(s)$ is the curvature of the curve $\mathbf{f}(s)$. In accordance with Lemma 6.2 [4] the angle $\alpha_1(\mathbf{f})$ between the tangent to $\mathbf{f}$ at $\mathbf{f}(s_1)$ and $\mathbf{f}(s_1) - \mathbf{f}(s_0)$ is

$$\alpha_1(\mathbf{f}) = \frac{1}{2}k(s_0)h + \frac{2}{3}k'(s_0)h^2 + O(h^3).$$

Clearly, similar formulae hold for the quadratic Bézier interpolant. Hence, denoting by $\alpha_0(\mathbf{p})$ and $\alpha_1(\mathbf{p})$ the corresponding angles for $\mathbf{p}$ we have

(2.12) $$\alpha_0(\mathbf{f}) - \alpha_0(\mathbf{p}) = O(h^2), \quad \alpha_1(\mathbf{f}) - \alpha_1(\mathbf{p}) = O(h^2),$$

because $\mathbf{f}$ and $\mathbf{p}$ have equal curvatures at the positions. We recall one more statement noted by Schaback. Namely, Theorem 6.2 [4] asserts that if a local Hermite interpolation of a smooth planar curve is done by quadratic (Bézier) polynomials using exact positions and approximate tangent directions with error $O(h^m)$, $m = 2, 3$, the error (of the interpolation) will be $O(h^{m+1})$ in terms of the distance h between the interpolation points. Applying this result the equalities (2.12) immediately give $O(h^3)$ for the error of interpolation (2.3).

3. C^0 interpolation with $O(h^4)$ accuracy

Suppose that $\mathbf{f} := (U, V)$ is a smooth planar curve with nonvanishing curvature and the coordinate functions $U, V : [\tau_0, \tau_1] \to \mathbf{R}$ are such that at least one of the following conditions holds:

(3.1) $$U'(t) \neq 0, \quad t \in [\tau_0, \tau_1],$$

(3.2) $$V'(t) \neq 0, \quad t \in [\tau_0, \tau_1].$$

Let us set

$$\mathbf{f}_i := (x_i, y_i) := \mathbf{f}(\tau_i), \quad \sigma_i := \hat{\mathbf{f}}(\tau_i) := \begin{cases} \frac{U'V''-U''V'}{(U')^3}\big|_{t=\tau_i} & \text{in case of (3.1)}, \\ \frac{V'U''-V''U'}{(V')^3}\big|_{t=\tau_i} & \text{otherwise.} \end{cases}$$

Theorem 2. *There exists a unique quadratic Bézier polynomial* $\mathbf{p}(t) = \sum_{j=0}^{2} \binom{2}{j} t^j (1-t)^{2-j}$, $0 \le t \le 1$, *satisfying conditions*

$$(3.3) \qquad \mathbf{p}(i) = \mathbf{f}_i, \quad \hat{\mathbf{p}}(i) = \sigma_i, \quad i = 0, 1.$$

Moreover, the error of interpolation is $O(h^4)$, $h = \|\mathbf{f}_1 - \mathbf{f}_0\|$.

Proof. Without loss of generallity we may assume that $U'(t) > 0$, $t \in [\tau_0, \tau_1]$. Clearly, $\mathbf{b}_0 = \mathbf{f}_0$, $\mathbf{b}_2 = \mathbf{f}_1$ and denoting $\mathbf{b}_1 = (x, y)$ we obtain the system

$$(3.4) \qquad \begin{aligned} 2\sigma_0(x - x_0)^3 &= (x - x_0)(y_1 - y_0) - (y - y_0)(x_1 - x_0), \\ 2\sigma_1(x_1 - x)^3 &= (x - x_0)(y_1 - y_0) - (y - y_0)(x_1 - x_0), \end{aligned}$$

from the interpolation conditions (3.3), as is not difficult to verify. The solution is explicitely given by the formulae

$$(3.5) \qquad x = \frac{x_0\sigma_0^{1/3} + x_1\sigma_1^{1/3}}{\sigma_0^{1/3} + \sigma_1^{1/3}}, \quad y = \frac{y_0\sigma_0^{1/3} + y_1\sigma_1^{1/3}}{\sigma_0^{1/3} + \sigma_1^{1/3}} - \frac{2\sigma_0\sigma_1(x_1 - x_0)^2}{(\sigma_0^{1/3} + \sigma_1^{1/3})^3}.$$

To estimate the error of interpolation we apply the technique used in [2]. Suppose that h is sufficiently small and the curve $\mathbf{f}$ is parametrized by arclength s in the manner of Section 2:

$$\mathbf{f}_0 = \mathbf{f}(-\epsilon), \quad \mathbf{f}_1 = \mathbf{f}(\epsilon), \quad \epsilon > 0,$$

$$\mathbf{f}^*(s) = \mathbf{f}'(s) := (\cos\theta(s), \sin\theta(s)),$$

$$\theta'(s) = k(s), \quad \theta(s) = \theta_0 + \theta_1 s + \theta_2 s^2 + O(s^3).$$

Having in mind that $\theta_1 \neq 0$ since the curvature k of $\mathbf{f}$ does not vanish we obtain

$$(x_1 - x_0, y_1 - y_0) = \Big(\int_{-\epsilon}^{\epsilon} \cos\theta(s) ds, \int_{-\epsilon}^{\epsilon} \sin\theta(s) ds\Big) = 2(\cos\theta_0, \sin\theta_0)\epsilon + O(\epsilon^3),$$

$$h = 2\epsilon + O(\epsilon^3),$$

$$\sigma_1^{1/3} - \sigma_0^{1/3} = \frac{2(3\theta_1^2 \sin\theta_0 + \theta_2\cos\theta_0)}{3\theta_1^{2/3}\cos^2\theta_0}\epsilon + O(\epsilon^2),$$

$$\mathbf{b}_1 - \mathbf{b}_0 = (x - x_0, y - y_0) = (\epsilon\cos\theta_0, \epsilon\sin\theta_0) + O(\epsilon^2),$$

$$\mathbf{b}_2 - \mathbf{b}_1 = (x_1 - x, y_1 - y) = (\epsilon\cos\theta_0, \epsilon\sin\theta_0) + O(\epsilon^2),$$

$$\mathbf{b}_0 - 2\mathbf{b}_1 + \mathbf{b}_2 = O(\epsilon^2).$$

Note that $2(\mathbf{b}_1 - \mathbf{b}_0)$, $2(\mathbf{b}_2 - \mathbf{b}_1)$ are the Bézier coefficients of $\mathbf{p}'(t)$ and $4(\mathbf{b}_0 - 2\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{p}''(t)$. Hence,

$$\mathbf{p}'(t) = (\epsilon \cos\theta_0, \epsilon \sin\theta_0) + O(\epsilon^2), \tag{3.6}$$

$$|\mathbf{p}''(t)| = O(\epsilon^2). \tag{3.7}$$

Because of (3.6) and $\mathbf{f}'(s) = (\cos\theta_0, \sin\theta_0) + O(\epsilon)$ the functions $\mathbf{p}(t) := (X(t), Y(t))$, $\mathbf{f}(s) := (U(s), V(s))$ could be parametrized with respect to the first coordinate:

$$\mathbf{p}(\xi) := (\xi, Z(\xi)), \quad \mathbf{f}(\xi) := (\xi, W(\xi)), \quad \xi \in [\alpha, \beta],$$

where $\alpha = X(0) = U(-\epsilon)$, $\beta = X(1) = U(\epsilon)$, $\beta - \alpha = h\cos\theta_0$, $Z := Y \circ X^{-1}$, $W := V \circ U^{-1}$. By the interpolation conditions (3.3) we have

$$\begin{aligned} Z(\alpha) &= W(\alpha), & \left.\frac{d^2 Z}{d\xi^2}\right|_{\xi=\alpha} &= \left.\frac{d^2 W}{d\xi^2}\right|_{\xi=\alpha}, \\ Z(\beta) &= W(\beta), & \left.\frac{d^2 Z}{d\xi^2}\right|_{\xi=\beta} &= \left.\frac{d^2 W}{d\xi^2}\right|_{\xi=\beta}, \end{aligned}$$

which is the usual two-point (0,2) Birkhoff interpolation. It is well known that given a real valued function F there exists a unique cubic polynomial Q satisfying

$$\begin{aligned} F(\alpha) &= Q(\alpha), & F''(\alpha) &= Q''(\alpha), \\ F(\beta) &= Q(\beta), & F''(\beta) &= Q''(\beta), \end{aligned} \tag{3.8}$$

and under the assumption $\max_{\alpha\leq\xi\leq\beta} |F^{(4)}(\xi)| \leq Const.$ the estimate $\max_{\alpha\leq\xi\leq\beta} |F(\xi) - Q(\xi)| = O((\beta - \alpha)^4)$ holds for the error of interpolation (3.8). Since $\frac{d^4 Z}{d\xi^4} = \frac{15(X'')^2(X'Y'' - X''Y')}{(X')^7}$ the relations (3.6) and (3.7) imply $\frac{d^4 Z}{d\xi^4}$ is bounded in $[\alpha, \beta]$. By the smoothness of the curve $\mathbf{f}$ it follows that $\frac{d^4 W}{d\xi^4}$ is also bounded in $[\alpha, \beta]$. Therefore, we have

$$\max_{\alpha\leq\xi\leq\beta} |W(\xi) - Z(\xi)| = O((\beta - \alpha)^4) = O(h^4)$$

for the error of the interpolation problem (3.3). This proves the theorem.

Let an increasing sequence of real numbers $\{t_i\}_0^n$ and a smooth planar curve $\mathbf{f} := (U, V)$ with nonvanishing curvature be given, where $U, V : [t_0, t_n] \to \mathbf{R}$ are such that at least one of the conditions $U'(t) \neq 0$, $V'(t) \neq 0$ holds in each interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$. Consider now the interpolation problem

$$\mathbf{p}(i) = \mathbf{f}(t_i), \quad \hat{\mathbf{p}}(i) = \hat{\mathbf{f}}(t_i), \quad i = 0, \ldots, n, \tag{3.9}$$

where $\mathbf{p}$ is a piecewise quadratic Bézier polynomial. Note that $\hat{\mathbf{f}}$ gives the second derivative of the function $V \circ U^{-1}$ if $U' \neq 0$, resp., of $U \circ V^{-1}$ if U' vanishes in the corresponding interval. Let us set $h := \max_i \|\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})\|$.

As a consequence from Theorem 2 we obtain the following assertions.

Corollary. (i) *An interpolant* $\mathbf{p}$ *for the problem* (3.9) *exists and each segment may be explicitely found as in* Theorem 2.
(ii) *The error of interpolation is* $O(h^4)$.
(iii) *If in addition we have* $U'(t) \neq 0$ $(V'(t) \neq 0)$ *for all* $t \in [t_{i-1}, t_i]$, *then parametrizing* $\mathbf{p}$ *with respect to the first (resp., second) coordinate, the resulting function has continuous second derivative.*

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