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SELF SIMILARITY, MARKOV'S INEQUALITY AND d-SETS

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Abstract. Self-similar sets of fractal type like the Cantor set have been used for a long time in mathematics. Mandelbrot has stressed their importance in models of reality and this inspired Hutchinson to give a mathematical treatment of self-similarity. Self-similarity is here generalized to the concept of polynomial self-similarity. This new concept is inspired by the theory of function spaces on general sets in Euclidean space. In this theory Markov's inequality for polynomials and sets which are in a certain uniform way of the same dimension (d-sets) are important.
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0. Introduction

Sets in Euclidean space which are in some sense geometrically very irregular (fractals) and which are self-similar in the sense that locally they look like they do globally, but on a different scale, have been constructed and used for at least 100 years in mathematics. Examples are Cantor's set and von Koch's curve [6]. Mandelbrot in his book [6] stresses the importance in the description of nature of fractals which are self-similar. With this as a motivation Hutchinson [2] has given a systematic study of self-similarity; in this paper a slightly stronger concept, geometric self-similarity, is used (see §2).

In the general theory of spaces of functions defined on subsets F of Euclidean space as studied by the author jointly with A. Jonsson (see [4] and [7]) two properties of F are important. The first is that F preserves Markov's inequality (see §3) and the second is that F is a d-set in the sense that it has, in a certain uniform way over F, a d-dimensional density $(0<d\leq n)$ which is positive and finite

(see §1). These two properties are satisfied by sets with regular geometry but also by many fractals.

In this paper it is proved that geometrically self-similar sets essentially preserve Markov's inequality and it is pointed out that they are d-sets (§4, Theorem 1 and 2).

In models of reality the concept of geometric self-similarity is usually too strong. Sets preserving Markov's inequality and d-sets may be considered as sets satisfying generalized concepts of self-similarity. The condition involving Markov's inequality is formulated by means of polynomials. In the beginning of Section 5 another concept of self-similarity is formulated using polynomials (polynomial self-similarity), and here the geometric feature is more obvious. Also this concept of self-similarity generalizes the concept of geometric self-similarity (§5, Theorem 5). In Theorem 3 and 4 in Section 5 it is proved that polynomial self-similarity is very closely related to Markov's inequality.

Notation. F is a closed and K a compact subset of $\mathbb{R}^n$ and $B(x_0,r)$ is the closed n-dimensional ball with center $x_0 \in \mathbb{R}^n$ and radius $r>0$. The diameter of a set E is denoted by $\operatorname{diam} E$. $\mathcal{P}_k$ is the set of all real-valued polynomials in n real variables of total degree of most k.

1. Hausdorff measures, d-measures and d-sets

The d-dimensional ($d>0$) Hausdorff measure of $E \subset \mathbb{R}^n$ is

$$m_d(E) := \lim_{\varepsilon \downarrow 0} m_d^{(\varepsilon)}(E)$$

where, for a certain positive constant $\alpha(d)$,

$$m_d^{(\varepsilon)}(E) := \alpha(d) \inf\{\sum_i (\operatorname{diam} E_i)^d : \cup_1^\infty E_i \supset E,\ \operatorname{diam} E_i \le \varepsilon\}.$$

The Hausdorff dimension of E, $\dim E$, is the infimum of the set of $d>0$ such that $m_d(E)=0$. We refer to [1] for details. Sets with non-integral Hausdorff dimension are called fractals as well as some other sets which are in a certain sense very irregular [6].

By definition the set F is a d-set ($0<d\le n$) if there exists a Borel measure μ with support F such that, for some positive constants $c_1=c_1(F)$ and $c_2=c_2(F)$,

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d, \quad \text{for} \quad x \in F \quad \text{and} \quad 0<r\le 1.$$

Such a μ is called a d-measure on F. We refer to [4], Ch.II for these concepts as well as for the following properties. If F is a d-set then $m_d|F$ defined by $m_d|F(E)=m_d(F\cap E)$, for Borel sets E, is a d-measure on F. Also, any d-measure μ on F is equivalent to $m_d|F$ in the sense that for some constants c' and c", $c'\mu \le m_d|F \le c''\mu$. Finally, if F is a d-set, then $\dim F=d$ and $\dim(F\cap B(x,r))=d$ for all $x\in F$ and $r>0$. $\mathbb{R}^n$ itself and a closed domain in $\mathbb{R}^n$ with boundary locally in Lip 1 are examples of d-sets with $d=n$ where the Lebesgue measure gives the d-measure. We shall see in §4, Theorem 2 that a large class of fractals are d-sets.

The concept of d-set and the terminology was originally introduced in [5] in order to prove a Whitney extension theorem in L^p. The terminology may be somewhat confusing since in the theory of Hausdorff measure the term s-set is used for sets E which are measurable m_s and satisfy $0<m_s(E)<\infty$. Clearly, a compact d-set is an s-set with s=d but a compact s-set is not necessarily a d-set as is shown for s=2 by the example $F=F_1\cup F_2$ where F_1 is a (non-degenerated) disk in $\mathbb{R}^2$ and F_2 a disjoint interval. In studying spaces of functions defined on a subset of $\mathbb{R}^n$ in the L^p-metric the stronger concept of d-set is important (see [4]).

2. Geometric self-similarity

We cite some results which we need from Hutchinson's paper [2]. Let N be an integer larger than 1 and let

$$S_i : \mathbb{R}^n \to \mathbb{R}^n \quad (1\le i\le N)$$

be non-constant similitudes on $\mathbb{R}^n$ (similarity transformations) which are contractions, i.e. for constants r_i satisfying $0<r_i<1$, we have

$$|S_i(x)-S_i(y)| = r_i|x-y| \quad \text{for all } x,y\in\mathbb{R}^n.$$

The similitude S_i is a composition of a dilation with scale factor r_i, a translation, a rotation and perhaps a reflection; S_i transforms every subset of $\mathbb{R}^n$ to a geometrically similar set.

Under these assumptions there exists ([2], §3.1) a unique compact set $K \subset \mathbb{R}^n$ such that K is <u>invariant</u> with respect to $\{S_1,\ldots,S_N\}$ in the sense that

$$K = \bigcup_{i=1}^{N} S_i(K) \qquad (2.1)$$

where $S_i(K)$ is the image of K under the mapping S_i. In fact, the above holds even without the assumption that every S_i is a similitude.

The <u>similarity dimension</u> of K is defined as the unique positive number D such that

$$\sum_{i=1}^{N} r_i^D = 1.$$

Then ([2], §4.4) there exists a unique positive Borel regular measure μ with compact support and total mass 1 such that μ is <u>invariant</u> with respect to $\{S_1,\ldots,S_N\}$ in the sense that

$$\mu(E) = \sum_{i=1}^{N} r_i^D \mu(S_i^{-1}(E)) \quad \text{for all Borel sets } E. \qquad (2.2)$$

The support of μ is K.

Now we assume, furthermore, that $\{S_1,\ldots,S_N\}$ satisfies the following condition, the <u>open set condition</u>: there exists a non-empty bounded open set G such that

(i) $\bigcup_{i=1}^{N} S_i(G) \subset G$ and

(ii) $S_i(G) \cap S_j(G) = \emptyset$ if $i \neq j$.

Then, the invariant set K has the following properties ([2], §5):

$\dim K = D$,

i.e. the Hausdorff dimension of K (which is usually difficult to calculate) equals the similarity dimension (which is easy to calculate) Furthermore,

$$0 < m_D(K) < \infty,$$

$$m_D(S_i(K) \cap S_j(K)) = 0 \quad \text{if} \quad i \neq j,$$

and μ is the restriction to K of the normalized D-dimensional

Hausdorff measure in the sense that

$$\mu(E) = m_D(E\cap K)/m_D(K) \quad \text{for all Borel sets } E.$$

In this paper we shall call a set geometrically self-similar if it is the compact invariant set K with respect to $\{S_1,\ldots,S_N\}$ where each S_i is a non-constant similitude which is a contraction and $\{S_1,\ldots,S_N\}$ satisfies the open set condition. Hutchinson uses the term self-similar in a slightly broader sense. Examples of geometrically self-similar sets are the Cantor set and von Koch's curve ([2], §3.3). We give the details for the Cantor set.

Example 1. Let $F=\cap F_i \subset \mathbb{R}^1$ (n=1) be the ordinary Cantor set where $F_0=[0,1]$ and F_i, i=1,2,..., is the union of 2^i closed intervals of length 3^{-i} obtained by removing the middle thirds of the intervals of F_{i-1}. Then F is invariant with respect to $\{S_1,S_2\}$ where $S_1(x)=x/3$ and $S_2(x)=2/3+x/3$. Hence, N=2, $r_1=r_2=1/3$ and the similarity dimension is D=log 2/log 3. As G in the open set condition we may choose]0,1[. The invariant measure μ is the weak limit of μ_i where μ_i is the unit mass uniformly distributed on F_i, i.e. μ_i is the absolutely continuous probability measure with suppport F_i and constant density on F_i.

3. Sets preserving Markov's inequality

We start with the definition and then we collect the results which we need. For more details and for applications we refer to [4], Ch. II and III. By definition $F\subset\mathbb{R}^n$ preserves Markov's inequality if for every positive integer k there exists a constant c=c(n,k,F) such that for all polynomials $P\in P_k$ and balls $B=B(x_0,r)$, $x_0\in F$, $0<r\leq 1$, we have

$$\max_{F\cap B}|\nabla P| \leq \frac{c}{r}\max_{F\cap B}|P|. \tag{3.1}$$

We refer to (3.1) as Markov's inequality on F and note that for $F=\mathbb{R}^n$ it is the ordinary Markov inequality in $\mathbb{R}^n$. We claim that F preserves Markov's inequality iff

$$\max_{B}|P| \leq c\max_{F\cap B}|P|, \quad c=c(n,k,F), \tag{3.2}$$

for all k, all $P\in P_k$, and all $B=B(x_0,r)$, $x_0\in F$, $0<r\leq 1$.

For the only if-part of this claim we refer to [4], p.35 and the if-part follows from the following estimate where we first use a trivial estimate, then Markov's inequality in $\mathbb{R}^n$ ([4], Ch. II, §2.3), and finally (3.2):

$$\max_{F\cap B}|\nabla P| \le \max_{B}|\nabla P| \le \frac{c_1}{r}\max_{B}|P| \le \frac{c_2}{r}\max_{F\cap B}|P|.$$

From the last estimate we also see that if (3.2) holds for $k=1$, then also (3.1) holds for $k=1$, and, by [4], p.37, F preserves Markov's inequality. We sum up: _If F is such that (3.2) holds for all polynomials of degree 1 and all $B=B(x_0,r)$, $x_0\in F$, $0<r\le 1$, then F preserves Markov's inequality._

Furthermore, we have ([8] or [7], §1.2) the following characterization.

Geometrical characterization. F preserves Markov's inequality iff there exists a constant $c^*>0$ so that for every $B=B(x_0,r)$, where $x_0\in F$ and $0<r\le 1$, there are $n+1$ affinely independent points $a_i\in F\cap B$, $i=1,\ldots,n+1$, such that the n-dimensional ball inscribed in the convex hull of $a_1,a_2,\ldots,a_{n+1}$ has radius not less than c^*r.

$\mathbb{R}^n$ itself preserves Markov's inequality; in fact, in this case (3.1) holds for $r>0$ ([4], Ch. II, §2.3) and this is the classical Markov inequality in $\mathbb{R}^n$. The closure of a domain in $\mathbb{R}^n$ with boundary locally in Lip 1 preserves Markov's inequality and from §4, Theorem 1 we get a lot of examples which are fractals and preserve Markov's inequality. If $F\subset\mathbb{R}^n$ is a d-set, then F preserves Markov's inequality if $d>n-1$ but not necessarily if $d\le n-1$ ([4], §2.2). It is easy to construct sets F which preserve Markov's inequality but which are not d-sets. For example, take $F=F_1\cup F_2$ where F_1 and F_2 are closed subsets of disjoint closed balls such that F_1 and F_2 preserve Markov's inequality and $0<\dim F_1<\dim F_2$; note that by the geometric characterization the union of two sets preserving Markov's inequality preserves Markov's inequality.

Remark. If $F\subset\mathbb{R}^n$ preserves Markov's inequality for a certain n, we can use the geometric characterization to conclude that $F\not\subset\mathbb{R}^m$ for any $m<n$. Furthermore, if we consider F as a subset $\mathbb{R}^m$ with $m>n$, then F does not preserve Markov's inequality. This means that if $F\subset\mathbb{R}^n$ preserves Markov's inequality, n is uniquely determined by F and the constant c in (3.1) does not really depend on n.

We shall need two more results related to Markov's inequality which we formulate as lemmas.

Lemma 1. For any constant a, $0<a<1$, and any positive integer k there exists a constant $c=c(n,k,a)$ so that

$$\max\{|P(x)|: x\in B(x_0,r)\}\leq c \max\{|P(x)|: x\in B(x_0,ar)\}$$

for all $P\in P_k$, $x_0\in \mathbb{R}^n$ and $r>0$.

Proof. Repeated application of Markov's inequality in $\mathbb{R}^n$ gives

$$|D^jP(x_0)| \leq c(ar)^{-|j|}\max\{|P(x)|: x\in B(x_0,ar)\},$$

where $j=(j_1,\ldots,j_n)$ is a multiindex, $|j|=j_1+\ldots+j_n$ and D^j the corresponding partial derivative. Using this on the Taylor expansion of P around x_0 we get the desired estimate.

Remark. A similar argument shows that in (3.1) and (3.2) we may change the condition $0<r\leq 1$ to $0<r\leq r_0$ for any $r_0>0$ and get the same class of sets F. Of course, the constant c then depends also on r_0.

For the second lemma we refer to [3], Proposition 4 and its proof.

Lemma 2. Let $F\subset\mathbb{R}^n$ be a compact set and B a closed ball containing F. If k is a positive integer there exists a constant c such that

$$\sup_B|P| \leq c \sup_F|P|, \text{ for all } P\in P_k,$$

iff F is not a subset of any algebraic variety $\{x: P_0(x)=0,\ P_0\in P_k,\ P_0$ not identically zero$\}$.

4. Geometrically self-similar sets

We claim that a geometrically self-similar set is a d-set and that it preserves Markov's inequality if it is not a subset of an (n-1)-dimensional affine subspace of $\mathbb{R}^n$. We start with the latter claim where we do not need the open set condition.

Theorem 1. Let $\{S_1,\ldots,S_N\}$, $N\geq 2$, be a set of non-constant similitudes on $\mathbb{R}^n$ which are contractions and let K be the compact set which is invariant with respect to $\{S_1,\ldots,S_N\}$ in the sense of (2.1). Assume that K is not a subset of an (n-1)-dimensional affine subspace of $\mathbb{R}^n$. Then K preserves Markov's inequality.

Proof. We shall prove that the geometric characterization in Section 3 is satisfied. We start by choosing $n+1$ affinely independent points $y_1,\dots,y_{n+1}\in K$. Then we take any $x_0\in K$ and any r, $0<r\leq \operatorname{diam} K$; note that $\operatorname{diam} K>0$. By [2], §3.1 there exists a sequence $\{i_p\}$, $p=1,2,\dots$, of integers between 1 and N so that

$$x_0 = \bigcap_{p=1}^{\infty} S_{i_1}\circ\dots\circ S_{i_p}(K). \quad (\circ \text{ denotes composition})$$

In fact,

$$K\supset S_{i_1}(K)\supset\dots\supset S_{i_1}\circ\dots\circ S_{i_p}(K)\supset\dots$$

and, since S_i are similitudes,

$$\operatorname{diam}(S_{i_1}\circ\dots\circ S_{i_p}(K)) = r_{i_1}\dots r_{i_p}\cdot(\operatorname{diam} K). \tag{4.1}$$

We choose p so that

$$\operatorname{diam}(S_{i_1}\circ\dots\circ S_{i_p}(K))<r \text{ and } \operatorname{diam}(S_{i_1}\circ\dots\circ S_{i_{p-1}}(K))\geq r \tag{4.2}$$

(if $p=1$, $S_{i_1}\circ\dots\circ S_{i_{p-1}}(K)$ is interpreted as K).

Now we introduce $x_j := S_{i_1}\circ\dots\circ S_{i_p}(y_j)$ for $1\leq j\leq n+1$. Since K is invariant we conclude that $x_j\in K$ and, due to the choice of p, that $x_j\in B(x_0,r)\cap K$ for $1\leq x_j\leq n+1$. The function $S_{i_1}\circ\dots\circ S_{i_p}$ is a similitude with scale $r_{i_1}\dots r_{i_p}$ and, consequently, the simplex spanned by $x_1,\dots,x_{n+1}$ is similar to the one spanned by $y_1,\dots,y_{n+1}$ with scale $r_{i_1}\dots r_{i_p}$. If the n-dimensional ball inscribed in the simplex $\{y_1,\dots,y_{n+1}\}$ has radius $c=c(K)>0$, then the n-dimensional ball inscribed in the simplex $\{x_1,\dots,x_{n+1}\}$ has radius $r^* := r_{i_1}\dots r_{i_p}c>0$. By using (4.1) with p changed to $p-1$ and (4.2) we get

$$r^* = r_{i_1}\dots r_{i_p}c = \operatorname{diam}(S_{i_1}\circ\dots\circ S_{i_{p-1}}(K))\cdot r_{i_p}c/\operatorname{diam} K \geq$$

$$\geq r\cdot \min_{1\leq j\leq N} r_j\cdot c/\operatorname{diam} K.$$

If we introduce $c^*=c^*(K)$ by

$$c^* := c \min_{1\leq j\leq N} r_j/\operatorname{diam} K$$

we have $r^* \geq c^* r$ and from the geometric characterization in Section 3 we conclude that K preserves Markov's inequality.

Remark. The constant c in the above proof is easy to calculate from the geometry of K. This means that for the type of sets considered in Theorem 1 we get an explicit estimate for the constant c^* in the geometric characterization of sets preserving Markov's inequality.

Remark. The method of proof above gives a modified result if K is a subset of an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. Suppose that, in fact, the invariant set K is a subset of an affine subspace A of dimension m but not one of dimension $m-1$; note that m may be 1 but that K consists of more than 1 point. Then we can repeat the proof with n changed to m. This gives that a geometric characterization of the type in Section 3 holds for K in the m-dimensional affine subspace A. This means that K preserves a kind of Markov type inequality in A; see the proof of Theorem 5, §5.

We now turn to the result on d-sets. Let $\{S_1,\ldots,S_N\}$, $N \geq 2$, be a set of non-constant contracting similitudes satisfying the open set condition. Let K be the compact set invariant with respect to $\{S_1,\ldots,S_N\}$ and μ the measure with compact support and total mass 1 invariant with respect to $\{S_1,\ldots S_N\}$ in the sense of (2.2). Then, checking the calculations in [2], pp. 737-738, we see that μ is, in fact, a d-measure on K with $d = \dim K$. We state this as our next theorem.

Theorem 2. <u>If K is geometrically self-similar, then K is a d-set with</u> $d = \dim K$.

5. Polynomial self-similarity

We first give the definition and some examples of polynomial self-similarity and then we explain the relation to sets preserving Markov's inequality (Theorem 3 and 4). Finally, we prove a result (Theorem 5) which implies that geometrically self-similar sets are polynomially self-similar. Note that for fixed $x_0 \in \mathbb{R}^n$ and r, $0 < r \leq 1$, the mapping

$$x \to x_0 + r(x - x_0), \quad x \in \mathbb{R}^n,$$

is a dilation in $\mathbb{R}^n$ with fixed point x_0 and scale r.

Definition. The set $F\subset\mathbb{R}^n$ is <u>polynomially self-similar</u> if for some $\varepsilon_0>0$ the following condition holds: For every positive integer k there exists a constant $c=c(n,k,F,\varepsilon_0)$ such that for all $P\in P_k$ and $0<\varepsilon\le\varepsilon_0$,

$$\max_{x\in F\cap B(x_0,\varepsilon)} |P(x_0+r(x-x_0))| \le c \max_{x\in F\cap B(x_0,r\varepsilon)} |P(x)|, \qquad (5.1)$$

for $x_0\in F$, $0<r\le 1$.

We note that if the condition in the definition is satisfied for a certain value of ε_0 it is satisfied for every smaller value of ε_0.

Example 2. Take $n=2$, $x=(x_1,x_2)$ and $F=F_1\cup F_2$ where F_1 is the closed unit disk with center at the origin and F_2 the interval $[1,2]$ on the x_1-axis. Then F is not polynomially self-similar which is realized by considering the polynomial $P(x_1,x_2)=x_2$ and points $x_0\in F_2$ at distance less than ε_0 from F_1. In fact, when ε is close to ε_0 the left-hand side of (5.1) is different from zero but if r is small the right-hand side is zero.

Example 3. Take $n=2$, $\varepsilon_1>0$ and $F=F_1\cup F_2$ where F_1 is as in Example 2 and F_2 is the set F_2 in Example 2 translated the distance ε_1 in the positive direction of the x_1-axis. Then F_1 preserves Markov's inequality and hence is polynomially self-similar by Theorem 3 below. A direct check shows that F_2 is polynomially self-similar. Since the distance between F_1 and F_2 is ε_1 the set F is polynomially self-similar; in fact, the condition (5.1) in the definition is fulfilled if $\varepsilon_0<\varepsilon_1$, but not if $\varepsilon_0>\varepsilon_1$. Since F does not preserve Markov's inequality this example also shows that polynomial self-similarity does not imply Markov's inequality on F, a fact which should be compared to Theorem 4 below. Obviously, F is also an example of a polynomially self-similar set which is not geometrically self-similar (this follows for instance from the fact that F is not a d-set); this should be compared to Theorem 5 below which shows that geometric self-similarity implies polynomial self-similarity.

Theorem 3. <u>If</u> K <u>preserves Markov's inequality, then</u> F <u>is polynomially self-similar</u>.

<u>Proof</u>. By first making a trivial estimate and then using (3.2) we get for $x_0\in F$, $0<r\le 1$, $0<\varepsilon\le 1$ and $P\in P_k$, with a constant $c=c(n,k,F)$,

$$\max\{|P(x_0+r(x-x_0))|: x\in F\cap B(x_0,\varepsilon)\}$$

$$\leq \max\{|P(x)|: x\in B(x_0,r\varepsilon)\}$$

$$\leq c \max\{|P(x)|: x\in F\cap B(x_0,r\varepsilon)\}.$$

This is (5.1) with $\varepsilon_0=1$.

Theorem 4. Let $F\subset\mathbb{R}^n$ be a compact, polynomially self-similar set. Assume that none of the sets $F\cap B(x_0,r)$, $x_0\in F$, $r>0$, is a subset of an $(n-1)$ dimensional affine subspace of $\mathbb{R}^n$. Then F preserves Markov's inequality.

Proof. In the proof c denotes different constants depending only on n, F and a where $a:=\varepsilon_0/3$, $0<\varepsilon_0\leq 1$, and $\varepsilon_0>0$ is the number in the definition of polynomial self-similarity. Take any point $x_0\in F$, any r so that $0<r\leq 1$, and any polynomial P of degree 1. For any $\rho>0$ let $B(\rho)=B(x_0,\rho)$ be the closed ball with radius ρ and center x_0. For $x\in\mathbb{R}^n$ we introduce the polynomial P_1 by $P_1(x):=P(x_0+r(x-x_0))$. Then

$$\max_{x\in B(ar)} |P(x)| = \max_{x\in B(a)} |P_1(x)|. \qquad (5.2)$$

Since F is compact there exist a positive integer N and points $y_i\in F$, $1\leq i\leq N$, so that

$$F\subset \bigcup_{i=1}^{N} B(y_i,a).$$

We choose an index i such that $x_0\in B(y_i,a)$. Then

$$B(a)\subset B(y_i,2a)\subset B(3a)=B(\varepsilon_0). \qquad (5.3)$$

Our assumption on F means that $F\cap B(y_i,2a)$ is not a subset of any algebraic variety $\{x: P_0(x)=0\}$ where P_0 is a non-zero polynomial of degree 1. Hence, we may apply Lemma 2 in Section 3 to conclude that

$$\max\{|P_1(x)|: x\in B(y_i,2a)\} \leq c \max\{|P_1(x)|: x\in F\cap B(y_i,2a)\}. \qquad (5.4)$$

After these preliminaries we get the following chain of inequalities and equalities where in turn we use Lemma 1 in Section 3, (5.2), (5.3), (5.4), (5.3), transition from P_1 to P, and the assumption on the polynomial self-similarity,

$$\max\{|P(x)|: x\in B(r)\} \le c \max\{|P(x)|: x\in B(ar)\}$$

$$= c \max\{|P_1(x)|: x\in B(a)\} \le c \max\{|P_1(x)|: x\in B(y_i,2a)\}$$

$$\le c \max\{|P_1(x)|: x\in F\cap B(y_i,2a)\}$$

$$\le c \max\{|P_1(x)|: x\in F\cap B(\varepsilon_0)\}$$

$$= c \max\{|P(x_0+r(x-x_0))|: x\in F\cap B(\varepsilon_0)\}$$

$$\le c \max\{|P(x)|: x\in F\cap B(r\varepsilon_0)\}.$$

Since $\varepsilon_0\le 1$ we conclude that (3.2) in Section 3 holds for $k=1$ and by Section 3 we conclude that F preserves Markov's inequality and Theorem 4 is proved.

Remark. From the proof we see that in Theorem 4 we may replace the assumption that F is polynomially self-similar with the weaker assumption that (5.1) holds for $k=1$. By Theorem 3 we may then, in fact, conclude that F is polynomially self-similar.

Theorem 5. Let K be the compact set which is invariant with respect to $\{S_1,\ldots,S_N\}$, $N\ge 2$, where S_i, $1\le i\le N$, are non-constant similitudes on $\mathbb{R}^n$ which are contractions. Then K is polynomially self-similar.

Proof. If K is not a subset of an $(n-1)$-dimensional affine subspace, the desired result follows from Theorem 1 and 3. If K is a subset of an affine subspace of lower dimension, we introduce A and m as in the second remark after the proof of Theorem 1 in §4. From the remark we know that the geometric characterization of Section 3 holds for K in A. We introduce a Cartesian coordinate system in A and consider polynomials in m real variables interpreted as the coordinates introduced. From the geometric characterization of Section 3 we see that Markov's inequality on K holds for these polynomials. Since the restriction to A of $P\in\mathcal{P}_k$ gives a polynomial in the m introduced coordinates we can use Theorem 3 (or its proof) to conclude that K is polynomially self-similar.

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