CONSTRUCTIVE THEORY OF FUNCTIONS, Varna '91
Sofia, 1992, pp. 299-306

ON APPROXIMATION BY CHEBYSHEVIAN SPLINES

Z. Wronicz

Institute of Mathematics, Stanisław Staszic
Academy of Mining and Metallurgy,
Cracow, Poland

1. Introduction. In this paper the investigation of Chebyshevian splines started in the works [6-11] are continued. We are concerned with saturation of Chebyshevian splines and invers theorems for splines systems given in [11]. The obtained results are generalizations of some results of J.H.Ahlberg, E.N.Nilson and J.L.Walsh, and Z.Ciesielski given in [1] and [3] respectively.

Further we need the following notations, definitions and properties of divided differences.

Let $U = U_n = \{u_i\}_{i=0}^{n}$ be a canonical complete Chebyshev system (CCT - system) of functions defined in the interval $I = [a,b]$ as follows:

$$u_0(t) = 1,$$

(1)
$$u_i(t) = \int_a^t w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2) \ldots \int_a^{\tau_{i-1}} w_i(\tau_i)\, d\tau_i \ldots d\tau_1 ,$$

$i = 1,\ldots,n$, where the weight functions w_j are right continuous and satisfy

$$0 < a_j \leqslant w_j(t) \leqslant b_j < \infty \text{ for } t \in I,$$

where a_j and b_j are given constants, $j = 1,\ldots,n$.

Define the following differential operators:

$$D_j f(x) = \lim_{h\to 0} [f(x+h) - f(x)] \left(\int_x^{x+h} w_j(t)dt \right)^{-1},$$

$$D_j^* f(x) = \frac{d}{dx}\left(\frac{f(x)}{w_j(x)}\right), \; j = 0,\ldots,n, \; w_0 = 1.$$

For w_j continuous and f differentiable at x we obtain $D_j f(x) = \frac{f'(x)}{w_j(x)}$.

Put

$$L_j f = D_j \dots D_1 f \ , \ L_j^* f = D_1^* \dots D_j^* f \ , \ j = 1,\dots,n,$$
$$Lf = D_o L_n f \ , \ L^* f = L_n^* D_o^* f \ .$$

The adjoint system $V = V_n = \{v_i\}_{i=0}^n$ is defined as follows:

$$v_o(t) = 1$$
$$(2) \qquad v_i(t) = \int_a^t w_n(\tau_1) \int_a^{\tau_1} w_{n-1}(\tau_2) \dots \int_a^{\tau_{i-1}} w_{n-i+1}(\tau_i) d\tau_i \dots d\tau_1 \ ,$$

$i = 1,\dots,n$ and we put $\tau_o = t$.

The systems U and V span the null spaces of the operators L and L^* respectively.

Define

$$D_U \begin{pmatrix} u_o,\dots,u_k \\ t_o,\dots,t_k \end{pmatrix} = \det \left[L_{d_j} u_i(t_j) \right]_{i,j=o}^k ,$$

$k = 0,\dots,n$, where $d_j = \max \{ l : t_j = \dots = t_{j-l} \}$, $j = 1,\dots,k$ and we assume that for $t_i = t_{i+m}$, $t_{i+j} = t_i$ for $j = 1,\dots,m-1$ and $L_o u = u$.

$$\left[\begin{matrix} u_o,\dots,u_j \\ t_o,\dots,t_j \end{matrix} \middle| f \right] = \frac{D_U \begin{pmatrix} u_o,\dots,u_{j-1},f \\ t_o,\dots,t_{j-1},t_j \end{pmatrix}}{D_U \begin{pmatrix} u_o,\dots,u_j \\ t_o,\dots,t_j \end{pmatrix}} \ , \ j = 1,\dots,n+1,$$

where u_{n+1} is defined by (1) with $w_{n+1}(t) = 1$.

Let $\Delta_N = \{a = t_o \leqslant t_1 \leqslant \dots \leqslant t_N = b\} = \{a = x_o < x_1 < \dots < x_M = b\}$,

where $t_o,\dots,t_N = \overbrace{x_o,\dots,x_o}^{\alpha_o},\dots,\overbrace{x_M,\dots,x_M}^{\alpha_M}$, $\alpha_j \leqslant n+1$ and α_j is the multiplicity of the point x_j, $j = 0,\dots,M$.

A function s is called a <u>Chebyshevian</u> <u>spline</u> w.r.t. the partition Δ_N and the system U if:

(a) s is a polynomial from P_U (a linear combination of the functions from U) in each subinterval (x_{j-1},x_j), $j = 1,\dots,M$.

(b) $\exists_{\varepsilon>0}$: s, $L_k s$ are continuous in the intervals $(x_j-\varepsilon,x_j+\varepsilon)$, $k = 1,\dots,n-\alpha_j$, $j = 1,\dots,M-1$.

We denote the set of these functions by $S_N^U(I)$. We can prove [11] that

$$(3) \qquad \begin{bmatrix} u_o,\dots,u_{n+1} \\ t_j,\dots,t_{j+n+1} \end{bmatrix} f \Bigg] = \int_{t_j}^{t_{j+n+1}} Lf(t)M_{j,n}(t)dt ,$$

for any function f with $L_n f$ absolute continuous in I, where $M_{j,n}$ is the jth B-spline from $S_N^V(I)$ i.e. a spline satisfying the following conditions: (i) supp $M_{j,n} = [t_j, t_{j+n+1}]$, (ii) $M_{j,n}(t) \geq 0$ in I and (iii) $\int_I M_{j,n}(t)dt = 1$.

Put

$$(4) \qquad \Delta_h^{U_k} f(t) = k!\, h^{k+1} \begin{bmatrix} u_o, u_1, \dots, u_{k+1} \\ t, t+h, \dots, t+(k+1)h \end{bmatrix} f \Bigg] .$$

We define the modulus of smoothness of the function f w.r.t. the system U_k by the formula (see [6,11])

$$\omega_{U_k}(f,\delta) = \sup \left\{ |\Delta_h^{U_k} f(t)|,\ 0 < h \leq \delta,\ t, t+(n+1)h \in I \right\} .$$

If $f \in L_p(I)$ for $1 \leq p < \infty$ we put

$$\omega_{U_k}^{(p)}(f,\delta) = \sup_{0<h\leq\delta} \| \Delta_h^{U_k} f \|_p ((k+1)h) ,$$

where

$$\| f \|_p (h) = \left(\int_a^{b-h} |f(t)|^p dt \right)^{1/p}$$

and in the periodic case we integrate over the whole interval I. For $p = \infty$ we put $\omega_{U_k}^{(\infty)}(f,\delta) = \omega_{U_k}(f,\delta)$.

<u>Remark.</u> In the algebraic case we have: $u_j(t) = (t-a)^j$, $j = 0,\dots,n$, $w_i = i$, $i = 1,\dots,n$, $L_k f = L_k^* f = \frac{1}{k!} D_o^k f$, $k = 1,\dots,n$, $Lf = L^* f = D_o^{n+1} f/n!$, and $\omega_{U_k}(f,\delta)$ is the modulus of smoothness of order k+1 in the usual sense.

<u>2. Inverse theorems and saturation.</u> Let $\{\Delta_N\}_{N=2}^{\infty}$ be a given sequence of partitions of the interval I, $\Delta_N = \{t_o = a,\ t_1 = b,\ t_2,\dots,t_N\} = \{a = x_{N,o} < x_{N,1} < \dots < x_{N,N} = b\}$ satisfying the conditions $\Delta_N \subset \Delta_{N+1}$ with $t_N \in \Delta_N \setminus \Delta_{N-1}$ and

$$\sup_N \max_{i,j} (x_{N,i+1} - x_{N,i})/(x_{N,j+1} - x_{N,j}) = R < \infty .$$

Let

$$E_N^{(p)}(f) = \inf\left\{\|f - s\|_p : s \in S_N^U(I)\right\}, \quad 1 \leq p \leq \infty.$$

We have the following estimates (cf. [5,6,11])

$$E_N^{(\infty)}(f) \leq C\,\omega_U(f,1/N), \quad N = 1,2,\ldots,$$

where the constant C depends only on the system U and the mesh ratio R. For $1 \leq p < \infty$ we can only prove the following inequalities [11]:

$$E_N^{(p)}(f) \leq C\, N^{-n-1}\omega_1^{(p)}(Lf, 1/N),$$

where C is a constant depending only on U and R, and $\omega_1^{(p)}(f,\delta)$ is the usual modulus of continuity of f in $L_p(I)$.

To estimate the modulus of smoothness $\omega_U^{(p)}(f,1/N)$ by means of best approximation of f by splines from $S_N^U(I)$ we need the following

Lemma.

$$\left\|\begin{bmatrix} u_0,\ldots,u_{k+1} \\ t\ ,\ldots,t+(k+1)h \end{bmatrix}\Big| f\right\|_p\big((k+1)h\big) \leq C\,\|D_0 L_k f\|_p\big((k+1)h\big), \quad 1 \leq p \leq \infty,$$

where C is a constant depending only on U.

Proof. Applying (3), (4) and Jensen's inequality we obtain

$$\left|\begin{bmatrix} u_0,\ldots,u_{k+1} \\ t_j,\ldots,t_{j+k+1} \end{bmatrix}\Big| f\right|^p \leq \left(\int_{t_j}^{t_{j+k+1}} |D_0L_kf(t)|\, M_{j,k}(t)dt\right)^p$$

$$\leq \int_{t_j}^{t_{j+k+1}} |D_0L_kf(t)|^p\, M_{j,k}(t)dt,$$ where $M_{j,k}$ is the jth B-spline w.r.t. V_k - the adjoint system to U_k defined by means of (2) with $n = k$ and the points $t_j,\ldots,t_{j+k+1}$.

Put $j_h = [(b-a)/h] - (k+1)$, $\tau_j = a+jh$, $j = 0,\ldots,j_h$, $j_h+1 = b-(k+1)h$ and let $M_{t,k}$ denotes the B-spline w.r.t. the system V_k and the points $t, t+h, \ldots, t+(k+1)h$. We have the following estimate (see [11])

$$|M_{t,k}(\tau)| \leq C/(k+1)h,$$

where C is a constant depending only on the system V_k. Hence applying (3) and Jensen's inequality we obtain

$$\int_a^{b-(k+1)h} \left|\begin{bmatrix} u_0,\ldots,u_{k+1} \\ t\ ,\ldots,t+(k+1)h \end{bmatrix}\Big| f\right|^p dt \leq \sum_{j=0}^{j_h}\int_{\tau_j}^{\tau_{j+1}} \left|\begin{bmatrix} u_0,\ldots,u_{k+1} \\ t\ ,\ldots,t+(k+1)h \end{bmatrix}\Big| f\right|^p dt$$

$$\leq \sum_{j=0}^{j_h} \int_{\tau_j}^{\tau_{j+1}} \left(\int_t^{t+(k+1)h} |D_0L_kf(\tau)|^p M_{t,k}(\tau)\, d\tau \right) dt$$

$$\leq \frac{C}{(k+1)h} \sum_{j=0}^{j_h} \int_{\tau_j}^{\tau_{j+1}} \int_t^{t+(k+1)h} |D_0L_kf(\tau)|^p\, d\tau dt$$

$$\leq \frac{C}{(k+1)h} \sum_{j=0}^{j_h} \int_{\tau_j}^{\tau_{j+1}} \int_t^{t+(k+2)h} |D_0L_kf(\tau)|^p\, d\tau dt$$

$$\leq C \int_a^{b-(n+1)h} |D_0L_kf(t)|^p\, dt$$

and we have proved the lemma.

Theorem 1. (cf. [2,3,5]). There exists a constant $C_{U,R}$ depending only on U and R such that for $(b-a)/2^m \leq \delta < (b-a)/2^{m-1}$

$$(5) \qquad \omega_{U_k}^{(p)}(f,\delta) \leq C_{U,R}\left[E_{2^m}^{(p)}(f) + \delta^{k+1} \sum_{i=0}^{m-1} 2^{i(k+1)} E_{2^i}^{(p)}(f)\right],$$

$k = 0,\ldots,n-1$ and

$$(6) \qquad \omega_{U_n}^{(p)}(f,\delta) \leq C_{U,R}\left[E_{2^m}^{(p)}(f) + \delta^{n+\frac{1}{p}} \sum_{i=1}^{m-1} 2^{i(n+\frac{1}{p})} E_{2^i}^{(p)}(f)\right].$$

Proof. In the sequel, we denote by the same letter $C_{U,R}$ different constants depending only on the system U and the constant R.

Let $0 \leq k < n$. Then

$$\Delta_h^{U_k} f(t) = \Delta_h^{U_k}\left[f(t) - s_{2^m}(t)\right] + \sum_{j=1}^{m} \Delta_h^{U_k}\left[s_{2^j}(t) - s_{2^{j-1}}(t)\right]$$

$+ \Delta_h^{U_k} s_1(t)$, where s_{2^j} is a spline from $S_{2^j}^{U_n}(I)$ of best approximation of f in $L_p(I)$. Hence applying the Minkowski inequality, the Markov inequality, properties of $\Delta_h^{U_k} f$ and Lemma (see [11]) we obtain

$$\|\Delta_h^{U_k} f\|_p\big((k+1)h\big) \leq C_{U,R}\left[E_{2^m}^{(p)}(f) + h^{k+1} \sum_{j=1}^{m-1} \|D_0L_k(s_{2^j} - s_{2^{j-1}})\|_p\right]$$

$$\leq C_{U,R}\left[E_{2^m}^{(p)}(f) + h^{k+1} \sum_{j=1}^{m-1} 2^{j(k+1)} \|s_{2^j} - s_{2^{j-1}}\|_p\right]$$

$$\leq C_{U,R}\left[E_{2^m}^{(p)}(f) + h^{k+1} \sum_{j=0}^{m-1} 2^{j(k+1)} E_{2^j}^{(p)}(f)\right]$$

and we have proved (5).

Let now k = n. Applying the Mühlbach theorem [4,11] we obtain

$$\Delta_h^{U_n} f(t) = \frac{nh\left(\Delta_h^{U_{n-1}} f(t+h) - \Delta_h^{U_{n-1}} f(t)\right)}{\begin{bmatrix} u_0 \ ,\ldots, u_n \\ t+h,\ldots,t+(n+1)h \end{bmatrix} u_{n+1} - \begin{bmatrix} u_0,\ldots,u_n \\ t \ ,\ldots,t+nh \end{bmatrix} u_{n+1}}$$

and by Lemma 4.1 of [11] (see also [8]) we have

$$\|\Delta_h^{U_n} f\|_p\left((n+1)h\right) \leqslant c_{U,R} \|\Delta_h^{U_{n-1}} f(t+h) - \Delta_h^{U_{n-1}} f(t)\|_p\left((n+1)h\right) .$$

Let $0 < (n+1)h \leqslant x_{N,j+1} - x_{N,j}$ for some j. Then by (3) we conclude that $\Delta_h^{U_{n-1}} s_N(t)$ is a constant in the interval $[x_{N,j}, x_{N,j+1} - nh]$.

Hence for $nh < \min_j (x_{N,j+1} - x_{N,j})$

$$\|\Delta_h^{U_n} s_N\|_p^p\left((n+1)h\right) \leqslant c_{U,R} \frac{nh}{\min_j (x_{N,j+1} - x_{N,j})} \|\Delta_h^{U_{n-1}} s_N\|_p^p(nh).$$

Hence

$$\|\Delta_h^{U_n} s_N\|_p\left((n+1)h\right) \leqslant c_{U,R}(Nh)^{1/p} \|\Delta_h^{U_{n-1}} s_N\|_p(nh) .$$

Now proceeding as in the proof of (5) and applying properties of the modulus of smoothness (see [11]) we obtain (6).

Corollary 1. (cf.[2,3]). Let $f \in L_p(I)$ if p is finite, $1 \leqslant p < \infty$, and let $f \in C(I)$ if $p = \infty$. Moreover, let $0 < \alpha < n + \frac{1}{p}$ and $n \geqslant 1$. Then

$$\omega_U^{(p)}(f,\delta) = O(\delta^\alpha) \text{ if } E_N^{(p)}(f) = O(1/N^\alpha) , \ 1 \leqslant p < \infty,$$

$$\omega_U(f,\delta) = O(\delta^\alpha) \text{ iff } E_N^{(\infty)}(f) = O(1/N^\alpha) , \ p = \infty.$$

Remark. Corollary 1 remains true after replacing capital O by small o.

Corollary 2. Let $n > 1$. Then $\omega_{U_{n-1}}(f,\delta) = O(\delta^{n-1})$ if and only if $E_N^{(\infty)}(f) = N^{-n+1}$. The same holds with O replaced by o.

Define the following system of functions: $\{\varphi_N\}_{N=1}^\infty$, $\varphi_N \in S_N^{U_1}(I)$, $\varphi_N(t_N) = 1$ and $\varphi_N(t) = 0$ for $t \in \Delta_{N-1}$ and put

$$\varphi(t) = \sum_{m=1}^\infty \frac{1}{2^{m+1}} \sum_{k=1}^{2^m} \varphi_{2^m+k}(t) , \ t \in I.$$

If $\{\Delta_N\}$ is a dyadic system of partitions, in the algebraic case we have the function defined by Z.Ciesielski in [2] p. 316. For this function we have

$$\Delta^{U_1}_{1/2^{m+1}}\varphi(t_m) = \Delta^2_{1/2^{m+1}}\varphi(t_m) = m/2^m, \text{ where } t_m = \frac{a+b}{2} - \left(\frac{1}{2}\right)^{m+1}.$$

Hence $\omega_{U_1}(\varphi, 1/2^{m+1}) \geqslant m/2^m$ but $E_N^{(\infty)}(1/N) = O(1/N)$ and for $p = \infty$ we cannot put $\alpha = n$ in Corollary 1.

For any $\alpha > 1$ define

$$\varphi_\alpha(t) = \sum_{m=1}^{\infty}\left(\frac{1}{2^{m+1}}\right)^{\alpha}\sum_{k=1}^{2^m}\varphi_{2^m+k}(t), \quad t \in I.$$

We have

$$\varphi_\alpha \in C(I),\ E_N^{(\infty)}(\varphi_\alpha) = O(1/N^\alpha) \text{ and } \omega_2(f,\delta) = O(\delta).$$

On the other hand we have

<u>Theorem 2.</u> Let f, $L_k f$, $k = 1,\dots,n$ and Lf be continuous in I and $E_N^{(\infty)}(f) = o(1/N^{n+1})$. Then $f \in P_U(I)$.

<u>Proof.</u> We shall show that $Lf(t) = 0$ for $t \in I$. Put

$$[t_0,\dots,t_{n+1}:f]_U = \begin{bmatrix} u_0,\dots,u_{n+1} \\ t_0,\dots,t_{n+1}\end{bmatrix} f .$$

Given $t \in I$ and $\varepsilon > 0$. There exists h_0 such that $|[t,\dots,t+(n+1)h:f]_U - Lf(t)| < \varepsilon/2$ for $|h| \leqslant h_0$ (see [11]) and a spline $s_{N,f} \in S_N^U(I)$ with $\|f - s_{N,f}\| < C\,N^{-n-1}\varphi(1/N) < \varepsilon/2$ $(\lim_{N\to\infty}\varphi(1/N) = 0)$. Take N such that $\max_i (x_{N,i+1} - x_{N,i}) < h_0$ and h such that $t, t+(n+1)h \in [x_{N,i}, x_{N,i+1}]$ with $|h| \geqslant (x_{N,i+1} - x_{N,i})/2(N+1)$. Hence by properties of divided differences we have

$|Lf(t)| \leqslant |Lf(t) - [t,\dots,t+(n+1)h:f]_U| + |[t,\dots,t+(n+1)h:f-s_{N,f}]_U|$
$+ |[t,\dots,t+(n+1)h:s_{N,f}]_U| < \varepsilon$ since the last expression is equal to zero $\left(s_{N,f}\big|_{[x_{N,i},x_{N,i+1}]} \in P_U([x_{N,i},x_{N,i+1}])\right)$. And we have proved the theorem.

On the other hand, without the assumption $Lf \in C(I)$, we conclude from (6) that $E_N^{(p)}(f) = O(1/N^{n+\alpha})$, $\alpha > 1/p$ implies $\omega_U^{(p)}(f,\delta) = O(\delta^{n+\frac{1}{p}})$.

References

[1] J.H.Ahlberg, E.N.Nilson, and J.L.Walsh, The theory of splines and their applications, Academic Press 1967.

[2] Z.Ciesielski, Properties of the orthonormal Franklin system II, Studia Math. 27(1966), 289 - 323.

[3] —— , Constructive function theory and spline systems, ibidem 53(1975), 278 - 302.

[4] G.Mühlbach, A recurrence formula for generalized divided differences and some applications, J. Approx. Th. 9(1973), 165 - 172.

[5] L.L.Schumaker, Spline functions: basic theory, Wiley and Sons, New York 1981.

[6] Z.Wronicz, Moduli of smoothness associated with Chebyshev systems and approximation by L-splines, Constructive Theory of Functions'84, Sofia 1984, 906 - 916.

[7] —— , On some properties of LB-splines, Ann. Polon. Math. 46 (1985), 379 - 388.

[8] —— , The Marhaud inequality for generalized moduli of smoothness, Rational Approximation and its Applications in Mathematics and Physics, Lecture Notes in Math. 1237, Springer Verlag 1987, 134 - 144.

[9] —— , On equivalence of spline bases in L_p spaces, Bull. Pol. Acad. Sci., Math. 36(1988), 273 - 278.

[10] —— , Systems conjugate to biorthogonal spline systems, ibidem 36(1988), 279 - 288.

[11] —— , Chebyshevian splines, Dissertationes Mathematicae nr 305, Warszawa 1990.