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THE CONVOLUTIONAL METHOD FOR THE CONSTRUCTING OF INTEGRAL ANALOGS OF LEIBNIZ RULE

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Leibniz rule for the operator of differentiation has form

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (f(x))^{(k)} (g(x))^{(n-k)}$$

and it is widely known and used in mathematical analysis. There are the important generalizations of this rule on the case of operator of fractional differentiation $D^{\alpha}_{a+}f$ [12]. They have form

$$D^{\alpha}_{a+}(f(x)g(x)) = \sum_{k=0}^{+\infty} \binom{\alpha}{k} \left[D^{\alpha-k}_{a+}f(x)\right](g(x))^{(k)},$$

$$D^{\alpha}_{a+}(f(x)g(x)) = \sum_{k=-\infty}^{+\infty} \binom{\alpha}{k+\beta} \left[D^{\alpha-\beta-k}_{a+}f(x)\right]\left[D^{k+\beta}_{a+}g(x)\right],$$

$$D^{\alpha}_{a+}(f(x)g(x)) = \int_{-\infty}^{+\infty} \binom{\alpha}{\tau+\beta} \left[D^{\alpha-\tau-\beta}_{a+}f(x)\right]\left[D^{\tau+\beta}_{a+}g(x)\right]d\tau.$$

Leibniz rules for the operator of fractional differentiation and their integral analog were considered by several authors (see, for example, [9]-[11]). This operator is known to belong to the class of Mellin's convolution transforms [12]. One of the most general representations of this class is the G-transform [7]. It seems very natural to try to obtain some new Leibniz rules and their integral analogs for the others G-transforms. We have found a general method for constructing such Leibniz rules and their integral analogs based on the notion of G-convolution [1] and various representations of the kernels of G-convolutions. Several Leibniz rules obtained by this method were represented in [2] and integral analogs of Leibniz rule will be constructed in this report.

THE NOTION OF G-CONVOLUTION

We introduce the following general notions:

DEFINITION 1. [7]. Let $c, \gamma \in R$, and $2\mathrm{sgn}(c) + \mathrm{sgn}(\gamma) \geq 0$. Denote by $\mathfrak{M}^{-1}_{c,\gamma}(L)$ the space of functions $f(x)$, $x>0$, representable in the form

$$f(x)=\frac{1}{2\pi i}\int_{\sigma} f^{*}(s)x^{-s}ds, \quad x>0,$$

where $f^{*}(s)|s|^{\gamma}\, e^{\pi c|\mathrm{Im}\, s|} \in L(\sigma)$, $\sigma = \{s, \mathrm{Re}\ s = \frac{1}{2}\}$.

The space $\mathfrak{M}^{-1}_{c,\gamma}(L)$ is a Banach space with the norm

$$\|f\|_{\mathfrak{M}^{-1}_{c,\gamma}(L)}=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\pi c|\tau|}\,|\tau|^{\gamma}\,|f^{*}(\tfrac{1}{2}+i\tau)d\tau|.$$

Obviously, the set $\mathfrak{M}^{-1}_{c,\gamma}(L)$ of spaces is well-ordered:

$$\mathfrak{M}^{-1}_{c_1,\gamma_1}(L) \subset \mathfrak{M}^{-1}_{c,\gamma}(L)$$

for $2\ \mathrm{sgn}\ (c_1-c) + \mathrm{sgn}\ (\gamma_1-\gamma) \geq 0$.

DEFINITION 2. [7]. G-transform of function f(x) is called the value of the next integral

$$(Gf)(x)\equiv G^{m,n}_{p,q}\left(\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)\circ\Big[f(u)\Big](x)=\frac{1}{2\pi i}\int_{\sigma}\Phi(s)f^{*}(s)x^{-s}ds, \tag{1}$$

where $x>0$, $f^{*}(s)=\mathfrak{M}\{f(x);s\}$ - Mellin transform of function f(x), $\sigma = \{s, \mathrm{Re}s = \frac{1}{2}\}$, components of p-and q-dimensional vectors (α_p) and (β_q) are a complex parameters, for which the next conditions take place

$$\mathrm{Re}\ \beta_j > -\frac{1}{2}, \ j=1,\dots,m; \quad \mathrm{Re}\ \beta_j < \frac{1}{2}, \ j=m+1,\dots,q;$$

$$\mathrm{Re}\ \alpha_j < \frac{1}{2}, \ j=1,\dots,n; \quad \mathrm{Re}\ \alpha_j > -\frac{1}{2}, \ j=n+1,\dots,p$$

and $\Phi(s)$ is determined as follows:

$$\Phi(s) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j+s)\prod_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod_{j=1+n}^{p}\Gamma(\alpha_j+s)\prod_{j=1+m}^{q}\Gamma(1-\beta_j-s)}, \quad 0\leq m\leq q,\ 0\leq n\leq p. \tag{2}$$

G-transform is known to exist in the space of functions $\mathfrak{M}^{-1}_{c,\gamma}(L)$ under condition:

$$2\mathrm{sign}(c+c^{*}) + \mathrm{sign}(\gamma+\gamma^{*}) \geq 0,$$

where $c^{*}=m+n-(p+q)/2$, $\gamma^{*}=\mathrm{Re}(\sum_{j=1}^{p}\alpha_j - \sum_{j=1}^{q}\beta_j)$ (3)

and one can represent it in the form

$$(Gf)(x)\equiv G^{m,n}_{p,q}\left(\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)\circ\Big[f(u)\Big](x) = \int_{0}^{\infty}G^{m,n}_{p,q}\left[x/t\,\middle|\,\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right] f(t)\,\frac{dt}{t}, \tag{4}$$

where $G^{m,n}_{p,q}\left[x\left|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right.\right]$ - Meijer's G-function [14] under additional condition: $4\mathrm{sign}(c^*)+2\mathrm{sign}(\gamma^*)+\mathrm{sign}|p-q| > 0$.

NOTE 1. G-transform includes many of particular cases of integral transforms [12], which we will use in the further discussion:

1) The modified operators of fractional calculus:

$$(x^{\beta}I^{\alpha}_{0+}x^{-\alpha-\beta}\ f)(x)=G^{0,1}_{1,1}\left[\begin{matrix}\alpha+\beta\\ \beta\end{matrix}\right]\circ\Big[f(u)\Big](x)=\frac{x^{\beta}}{\Gamma(\alpha)}\int_0^x\frac{f(t)t^{-\alpha-\beta}}{(x-t)^{1-\alpha}}\,dt,\ \mathrm{Re}\alpha > 0.$$

$$(x^{\beta}I^{\alpha}_{0+}x^{-\alpha-\beta}\ f)(x)=G^{0,1}_{1,1}\left[\begin{matrix}\alpha+\beta\\ \beta\end{matrix}\right]\circ\Big[f(u)\Big](x)=\frac{x^{\beta}}{\Gamma(\alpha+n)}\left(\frac{d}{dx}\right)^n\int_0^x\frac{f(t)t^{-\alpha-\beta}}{(x-t)^{1-\alpha-n}}\,dt,$$

$-n < \mathrm{Re}\alpha \le 0$, $n=[-\mathrm{Re}\alpha]+1$.

$$(x^{\beta}I^{\alpha}_{-}\ x^{-\alpha-\beta}\ f)(x)=G^{1,0}_{1,1}\left[\begin{matrix}\alpha+\beta\\ \beta\end{matrix}\right]\circ\Big[f(u)\Big](x)=\frac{x^{\beta}}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)t^{-\alpha-\beta}}{(t-x)^{1-\alpha}}\,dt,\ \mathrm{Re}\alpha > 0.$$

2) The operator of modified Laplace transformation and its inverse:

$$(x^{\alpha}\Lambda_{+}x^{-\alpha}\ f)(x)=G^{1,0}_{0,1}\left[\begin{matrix}-\\ \alpha\end{matrix}\right]\circ\Big[f(u)\Big](x)=x^{\alpha}\int_0^{\infty}e^{-(x/t)}f(t)t^{-\alpha-1}\,dt,$$

$$(x^{\alpha}\Lambda^{-1}_{+}x^{-\alpha}\ f)(x)=G^{0,0}_{1,0}\left[\begin{matrix}\alpha\\ -\end{matrix}\right]\circ\Big[f(u)\Big](x),$$

3) The inverse operator to the generalized Stieltjes transformation:

$$(x^{\alpha}\left\{(1+x)^{-\rho}\right\}^{-1}x^{-\alpha}\ f)(x)=\Gamma(\rho)G^{0,0}_{1,1}\left[\begin{matrix}\alpha\\ \alpha+1-\rho\end{matrix}\right]\circ\Big[f(u)\Big](x).$$

4) The operator with Tricomi function $\Psi(a,b,x)$ in the kernel:

$$(x^{\alpha}\Psi^{b}_{a}x^{-\alpha}f)(x)=G^{2,0}_{1,2}\left[\begin{matrix}\alpha+1+a-b\\ \alpha,\alpha+1-b\end{matrix}\right]\circ\Big[f(u)\Big](x)=x^{\alpha}\int_0^{\infty}e^{-x/t}\Psi(a,b,x/t)t^{-\alpha-1}f(t)dt.$$

5) The operator with algebraic function in the kernel:

$$(x^{\alpha}Al_{\upsilon}x^{-\alpha}f)(x)=x^{\alpha}\int_x^{+\infty}\frac{1}{\sqrt{1-x/t}}\left[(1+\sqrt{1-x/t})^{\upsilon}+(1-\sqrt{1-x/t})^{\upsilon}\right]t^{-\alpha-1}f(t)dt=$$

$$=2\sqrt{\pi}\; G^{2,0}_{2,2}\left[\begin{array}{c}\alpha+\nu/2,\alpha+\nu/2+1/2\\ \alpha+\nu,\alpha\end{array}\right]\circ\Big[f(u)\Big](x).$$

DEFINITION 3. [1]. Let $f(x)\in\mathfrak{M}^{-1}_{c_1,\gamma_1}(L)$, $g(x)\in\mathfrak{M}^{-1}_{c_2,\gamma_2}(L)$. Then the G-convolution of these functions is called the value of the next integral

$$(f*g)(x)=\frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_t}\frac{\Phi_1(s)\Phi_2(t)}{\Phi_3(s+t)}f^*(s)g^*(t)x^{-s-t}dsdt, \tag{5}$$

where functions $\Phi_i(\tau)$ $i=1,2,3$ have the form (2) , $\sigma_s=\{s,\mathrm{Re}\ s=\frac{1}{2}\}$, $\sigma_t=\{t,\mathrm{Re}\ t=\frac{1}{2}\}$ and the complex numbers $\alpha_j^{(k)}$, $\beta_j^{(k)}$ satisfy the following conditions of the poles separation supposition for the line $\sigma=(1/2-i\infty,\ 1/2+i\infty)$

$$\left\{\begin{array}{l}\mathrm{Re}(\beta_j^{(k)})+1/2>0,\ j=1,2,\dots,m_k,\ k=1,2,3;\\ 1/2-\mathrm{Re}(\alpha_j^{(k)})>0,\ j=1,2,\dots,n_k,\ k=1,2;\\ \mathrm{Re}(\alpha_j^{(3)})<0,\ j=1,2,\dots,n_3,\\ \mathrm{Re}(\alpha_j^{(3)})+1>0,\ \ j=n_3+1,\dots,p_3;\\ \mathrm{Re}(\beta_j^{(3)})<0,\ j=m_3+1,\dots q_3.\end{array}\right. \tag{6}$$

NOTE 2. If we consider the following convolution

$$(f\overset{L}{*}g)(x)=\frac{x}{(2\pi i)^2}\int_{\sigma_t}\int_{\sigma_s}\frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(2-s-t)}f^*(s)g^*(t)x^{-s-t}dsdt,$$

then we obtain usual Laplace convolution for functions $f(x)$ and $g(x)$ using integral representation for B-function and changing the order of integration:

$$(f\overset{L}{*}g)(x)=\int_0^x f(t)g(x-t)dt.$$

The existence conditions of convolution (5) and the conditions for the following factorization property

$$(G_3(f*g))(x)=(G_1f)(x)(G_2g)(x), \tag{7}$$

where $(G_if)(x)$ is the G-transform with the kernel $\Phi_i(s)$ are given in [1].

We will use G-convolutions with $\Phi_1(\tau)\equiv\Phi_2(\tau)\equiv1$ and the following known formulas [15] in the further discussion:

$$\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z). \tag{8}$$

$$\int_{-\infty}^{+\infty}\Gamma\left[\begin{matrix}-\\ a+x,b+x,c-x,d-x\end{matrix}\right]dx=\Gamma\left[\begin{matrix}a+b+c+d-3\\ a+c-1,a+d-1,b+c-1,b+d-1\end{matrix}\right], \tag{9}$$

$\mathrm{Re}(a+b+c+d)>3.$

$$\int_{-\infty}^{+\infty}\Gamma\left[\begin{matrix}-\\ c+x,d-x\end{matrix}\right]dx=\frac{2^{c+d-2}}{\Gamma(c+d-1)},\ \mathrm{Re}(c+d)>1. \tag{10}$$

$$\int_{0}^{+\infty}\Gamma\left[\begin{matrix}a+ix,b+ix,a-ix,b-ix\\ -\end{matrix}\right]dx=2^{1-2a-2b}\pi^{3/2}\Gamma\left[\begin{matrix}2a,2b,a+b\\ a+b+1/2\end{matrix}\right], \tag{11}$$

$\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0.$

THEOREM 1. *Suppose that* $f(x)\in\mathfrak{M}^{-1}_{c_1,\gamma_1}(L)$, $g(x)\in\mathfrak{M}^{-1}_{c_2,\gamma_2}(L)$ *and G-convolution has the kernel* $\Phi_3(\tau)=H(\tau)$. *Then the following integral analogs of Leibniz rule hold true:*

1. *If* $H(\tau)=\Gamma(a+b-1+\tau)$ *then*

$$(f*g)(x)=(x^{a+b-1}\Lambda_+^{-1}x^{-a-b+1}fg)(x)=\int_{-\infty}^{+\infty}2^{2-a-b}(x^{a+\tau}\Lambda_+^{-1}x^{-a-\tau}f)(2x)\times \tag{12}$$

$$\times(x^{b-\tau}\Lambda_+^{-1}x^{-b+\tau}g)(2x)d\tau$$ *under these conditions:*

$\mathrm{Re}(a+b)>1,\ 2\mathrm{sgn}(c_i-1)+\mathrm{sgn}(\gamma_i)\ge0,\ i=1,2.$

2. *If* $H(\tau)=\Gamma(a+b+c+d+1-\tau)/\Gamma(b+d+1-\tau)$ *then*

$$(f*g)(x)=(x^{-b-d}I_{0+}^{-a-c}x^{a+b+c+d}fg)(x)=$$

$$=\int_{-\infty}^{+\infty}\frac{\Gamma(a+c+1)}{\Gamma(a+1+\tau)\Gamma(c+1-\tau)}(x^{-b-\tau}I_{0+}^{\tau-c}x^{b+c}f)(x)(x^{-d+\tau}I_{0+}^{-\tau-a}x^{a+d}g)(x)d\tau \tag{13}$$

under these conditions: $\mathrm{Re}(a+b+c+d)>0$, $\mathrm{Re}(b+d)>0$, $\mathrm{Re}(a+c)>-1$,

$\min\{\mathrm{Re}(a+d),\mathrm{Re}(b+c)\}>-1/2$, $2\mathrm{sgn}(c_i-1/2)+\mathrm{sgn}\left(\gamma_i-\mathrm{Re}\begin{Bmatrix}(b+c)\\(a+d)\end{Bmatrix}\right)\ge0,\ i=\begin{Bmatrix}1\\2\end{Bmatrix}.$

3. *If* $H(\tau)=\Gamma(a+b+c+d-1+\tau)$ *then*

$$(f*g)(x)=(x^{a+b+c+d-1}\Lambda_{+}x^{-a-b-c-d+1}\,fg)(x)=$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\Gamma(c-\tau)\Gamma(d-\tau)}(x^{a+c}\Psi^{c-d+1}_{\tau-d+1}x^{-a-c}f)(x)(x^{b+d}\Psi^{d-c+1}_{\tau-c+1}x^{-b-d}g)(x)d\tau \qquad (14)$$

under these conditions: $\min\left\{Re(a+c), Re(a+d), Re(b+c), Re(b+d)\right\}>-1/2$,

$Re(a+b+c+d)>1/2$, $2sgn(c_i)+sgn(\gamma_i-Re\begin{Bmatrix}(2a+c+d)\\(2b+c+d)\end{Bmatrix})\geq 0$, $i=\begin{Bmatrix}1\\2\end{Bmatrix}$.

4. *If* $H(\tau)=1/(\Gamma(a+d+1-\tau)\Gamma(b+c-1+\tau))$ *then*

$$\Gamma(a+b+c+d)\times(f*g)(x)=(x^{b+c-1}\left\{(1+x)^{-a-b-c-d}\right\}^{-1}x^{-b-c+1}\,fg)(x)=\int_{-\infty}^{+\infty}\frac{(a+b+c+d-1)}{(b+d)(a+c)}\times$$

$$\times(x^{b+\tau}\left\{(1+x)^{-b-d-1}\right\}^{-1}x^{-b-\tau}f)(x)(x^{c-\tau}\left\{(1+x)^{-a-c-1}\right\}^{-1}x^{\tau-c}g)(x) \qquad (15)$$

under these conditions: $Re(a+b+c+d)>1$, $Re(b+c)>1/2$,

$\max\left\{Re(a+d), Re(a+c), Re(b+d)\right\}>0$, $2sgn(c_i-1)+sgn(\gamma_i)\geq 0$, $i=1,2$.

5. *If* $H(\tau)=\Gamma(a+b+\tau)/\Gamma(a+b+1/2+\tau)$ *then*

$$(f*g)(x)=(x^{a+b}I_{-}^{1/2}x^{-a-b-1/2}\,fg)(x)=\int_{0}^{+\infty}\frac{1}{2\pi^{3/2}}(x^{a-i\tau}A\,l_{2i\tau}x^{-a+i\tau}f)(x)\times \qquad (16)$$

$$\times(x^{b-i\tau}Al_{2i\tau}x^{-b+i\tau}g)(x)d\tau$$ *under these conditions:*

$Re(a+b)>-1/2$, $Re(a)>-1/2$, $Re(b)>-1/2$, $2sgn(c_i-1)+sgn(\gamma_i-1)\geq 0$, $i=1,2$.

PROOF: We will prove formula (12) only, since the others are proved in the same manner.

Let us take $c=s+\alpha$, $d=t+\beta$ in the formula (10). Then we obtain the following representation:

$$\frac{1}{\Gamma(\alpha+\beta-1+s+t)}=2^{2-\alpha-\beta-s-t}\int_{-\infty}^{+\infty}\frac{d\tau}{\Gamma(s+\alpha+\tau)\Gamma(t+\beta-\tau)},\quad Re(\alpha+\beta)>0. \qquad (17)$$

Now we consider the next G-convolution:

$$(f*g)(x)=\frac{1}{(2\pi i)^2}\int_{\sigma_t}\int_{\sigma_s}\frac{f^{*}(s)g^{*}(t)x^{-s-t}.dsdt}{\Gamma(\alpha+\beta-1+s+t)}. \qquad (18)$$

It follows from results obtained in [1] that G-convolution (18) exists under conditions as in Theorem 1. and we can represent it by means of Note 1. in the form

$$(f*g)(x)=(x^{\alpha+\beta-1}\Lambda_{+}^{-1}x^{-\alpha-\beta+1}\,fg)(x). \qquad (19)$$

Now we use representation (17) of the kernel of G-convolution (18) and obtain:

$$(f*g)(x)=\frac{1}{(2\pi i)^2}\int\limits_{\sigma_s}\int\limits_{\sigma_t} 2^{2-\alpha-\beta-s-t} f^*(s)g^*(t)x^{-s-t}\int\limits_{-\infty}^{+\infty}\frac{d\tau}{\Gamma(s+\alpha+\tau)\Gamma(t+\beta-\tau)}\,dsdt. \tag{20}$$

We will show that one can change the order of operations of integration in the right part of (20) by means of Fubini's theorem. For the sake of brevity we introduce the following notation:

$$F(s,t,\tau) = \frac{2^{2-\alpha-\beta-s-t} f^*(s)\, g^*(t)\, x^{-s-t}}{\Gamma(s+\alpha+\tau)\Gamma(t+\beta-\tau)}. \tag{21}$$

We consider the following multiple integral now:

$$I(x) = \int\limits_{\sigma_t}\int\limits_{\sigma_s}\int\limits_{-\infty}^{+\infty} F(s,t,\tau)\ dsdtd\tau.$$

It is convenient to represent $I(x)$ in the form:

$$I(x) = \int\limits_{\sigma_t}\int\limits_{\sigma_s}\int\limits_{-\infty}^{-N} F(s,t,\tau)\ dsdtd\tau + \int\limits_{\sigma_t}\int\limits_{\sigma_s}\int\limits_{-N}^{N} F(s,t,\tau)\ dsdtd\tau +$$

$$+ \int\limits_{\sigma_t}\int\limits_{\sigma_s}\int\limits_{N}^{+\infty} F(s,t,\tau)\ dsdtd\tau = I_1(x) + I_2(x) + I_3(x), \tag{22}$$

where N is a constant, which we will define in the further discussion. At first to estimate $I_2(x)$ we represent it in the form:

$$I_2(x)= \int\limits_{|u|<r}\int\limits_{|v|<r}\int\limits_{-N}^{N} F(s,t,\tau)dsdtd\tau+\int\limits_{|u|>r}\int\limits_{|v|<r}\int\limits_{-N}^{N} F(s,t,\tau)dsdtd\tau+$$

$$+\int\limits_{|u|<r}\int\limits_{|v|>r}\int\limits_{-N}^{N} F(s,t,\tau)dsdtd\tau + \int\limits_{|u|>r}\int\limits_{|v|>r}\int\limits_{-N}^{N} F(s,t,\tau)dsdtd\tau = I_{21}(x) +$$

$$+ I_{22}(x) + I_{23}(x) + I_{24}(x), \tag{23}$$

where $s=1/2+iu$, $t=1/2+iv$, r is a constant.

Then we have:

$$\left|I_{21}(x)\right| \le \int\limits_{|u|<r} \int\limits_{|v|<r} \int\limits_{-N}^{N} |F(s,t,\tau)| ds dt d\tau, \tag{24}$$

Since $f(x) \in \mathfrak{M}^{-1}_{c_1,\gamma_1}(L)$, $g(x) \in \mathfrak{M}^{-1}_{c_2,\gamma_2}(L)$ then

$$|f^*(s)| = F(\xi), \text{ where } F(\xi) \in L(\sigma_s), \tag{25}$$

$$|g^*(t)| = G(t), \text{ where } G(t) \in L(\sigma_t), \tag{26}$$

$$\left|\frac{2^{2-\alpha-\beta-s-t} x^{-s-t}}{\Gamma(s+\alpha+\tau)\Gamma(t+\beta-\tau)}\right| \le B_1 \text{ if } |u|<r, |v|<r, -N<\tau<N. \tag{27}$$

From (24)-(27) we obtain:

$\left|F(s,t,\tau)\right| \le B_1 F(s)G(t)$, if $|u|<r$, $|v|<r$, $-N<\tau<N$ and, consequently, $F(s,t,\tau)$ is integrable in the domain $|u|<r$, $|v|<r$, $-N<\tau<N$ and we have:

$$\left|I_{21}(x)\right| \le B_2. \tag{28}$$

Then we have using asymptotic behavior of Γ-function:

$$\left|I_{22}(x)\right| \le \int\limits_{|u|>r} \int\limits_{|v|<r} \int\limits_{-N}^{N} |F(s,t,\tau)| ds dt d\tau, \tag{29}$$

$$|f^*(s)/\Gamma(s+\alpha+\tau)| \le F_1(t), \text{ where } F_1(t) \in L(\sigma_s), |u|>r, -N<\tau<N, \tag{30}$$

$$\left|\frac{2^{2-\alpha-\beta-s-t} x^{-s-t}}{\Gamma(t+\beta-\tau)}\right| \le B_3 \text{ if } |u|>r, |v|<r, -N<\tau<N. \tag{31}$$

From (26), (29)-(31) we obtain:

$\left|F(s,t,\tau)\right| \le B_3 F_1(s)G(t)$ if $|u|>r$, $|v|<r$, $-N<\tau<N$ and, consequently, $F(s,t,\tau)$ is integrable in the domain $|u|>r$, $|v|<r$, $-N<\tau<N$ and we have:

$$\left|I_{22}(x)\right| \le B_4. \tag{32}$$

We estimate $I_{23}(x)$ in the same manner as $I_{22}(x)$ and obtain:

$$\left|I_{23}(x)\right| \le B_5. \tag{33}$$

For $I_{24}(x)$ we have:

$$\left|I_{24}(x)\right| \le \int\limits_{|u|>r} \int\limits_{|v|>r} \int\limits_{-N}^{N} |F(s,t,\tau)| ds dt d\tau, \tag{34}$$

$$|g^*(t)/\Gamma(t+\beta-\tau)| \le G_1(t), \text{ where } G_1(t) \in L(\sigma_t), |v|>r, -N<\tau<N, \tag{35}$$

$$\left|2^{2-\alpha-\beta-s-t}x^{-s-t}\right| \le B_\sigma \text{ if } |u|>r,\ |v|>r,\ -N<\tau<N. \tag{36}$$

From (30), (34)-(36) we obtain:

$\left|F(s,t,\tau)\right| \le B_\sigma F_1(s)G_1(t)$ if $|u|>r$, $|v|>r$, $-N<\tau<N$ and, consequently, $F(s,t,\tau)$ is integrable in the domain $|u|>r$, $|v|>r$, $-N<\tau<N$ and we have:

$$\left|I_{24}(x)\right| \le B_7. \tag{37}$$

Finally, in view of (23), (28), (32), (33) and (37) we have the following inequality:

$$|I_2(x)| \le B_8. \tag{38}$$

Then we will estimate $I_1(x)$.

$$\left|I_1(x)\right| \le \int_{\sigma_s}\int_{\sigma_t}\int_{-\infty}^{-N} |F(s,t,\tau)|dsdtd\tau = \int_{\sigma_s}\int_{\sigma_t}\int_{N}^{+\infty} |F(s,t,-\tau)|dsdtd\tau. \tag{39}$$

Then using formula (8) we obtain the following representation:

$$|F(s,t,-\tau)| = \left|\frac{2^{2-\alpha-\beta}\Gamma(1-s-\alpha+\tau)\sin(\pi(1-s-\alpha+\tau))f^*(s)g^*(t)(2x)^{-s-t}}{\Gamma(t+\beta+\tau)}\right|. \tag{40}$$

Then we have:

$$|\sin(\pi(1-s-\alpha+\tau))f^*(s)| \le F_2(s), \text{ where } F_2(s) \in L(\sigma_s),\ N<\tau,\ s \in \sigma_s. \tag{41}$$

We will use in the further discussions the following simple inequalities, which are corollaries from integral representations of Γ- and B-functions:

$$|\Gamma(x)| \le |\Gamma(\text{Re } x)| \text{ if Re } x > 0. \tag{42}$$

$$|B(x,y)| = \left|\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right| \le |B(\text{Re } x, \text{Re } y)| \text{ if Re } x>0, \text{ Re } y>0. \tag{43}$$

If $\tau=\tau_1+\tau_2$, where $\tau_1>0$, $\tau_2>0$, $1/2+\text{Re}(\beta)+\tau_1>0$, τ_1 is a constant, when

$$|1/\Gamma(t+\beta+\tau)| = \left|\frac{\Gamma(t+\beta+\tau_1)\Gamma(\tau_2)}{\Gamma(t+\beta+\tau)\Gamma(t+\beta+\tau_1)\Gamma(\tau_2)}\right| \le$$

$$\le \left|\frac{\Gamma(1/2+\text{Re}\beta+\tau_1)\Gamma(\tau_2)}{\Gamma(1/2+\text{Re}\beta+\tau)\Gamma(t+\beta+\tau_1)\Gamma(\tau_2)}\right| = \left|\frac{\Gamma(1/2+\text{Re}\beta+\tau_1)}{\Gamma(1/2+\text{Re}\beta+\tau)\Gamma(t+\beta+\tau_1)}\right|. \tag{44}$$

$$|g^*(t)/\Gamma(t+\beta+\tau_1)| \le G_2(t), \text{ where } G_2(t) \in L(\sigma_t),\ N<\tau,\ t \in \sigma_t. \tag{45}$$

$$\left|2^{2-\alpha-\beta}(2x)^{-s-t}\right| \le B_9 \text{ if } N<\tau,\ t \in \sigma_t,\ s \in \sigma_s. \tag{46}$$

$$|\Gamma(1-s-\alpha+\tau)| \le \Gamma(1/2-\text{Re}\alpha+\tau) \text{ if } 1/2 - \text{Re}\alpha + \tau > 0. \tag{47}$$

If we choose $N=\max\{1,\mathrm{Re}\alpha-1/2,-1/2-\mathrm{Re}\beta\}$, when estimates (41), (44)-(47) hold true and we have:

$$|F(s,t,-\tau)|\le B_9F_2(s)G_2(t)\left|\frac{\Gamma(1/2-\mathrm{Re}\alpha+\tau)}{\Gamma(1/2+\mathrm{Re}\beta+\tau)}\right|=B_9F_2(s)G_2(t)\tau^{-\mathrm{Re}\alpha-\mathrm{Re}\beta}\Big[1+ + O(1/\tau)\Big]. \tag{48}$$

Consequently, $|F(s,t,-\tau)|$ is integrable function in the domain $\sigma_s\times\sigma_t\times$ $\times\,[N,+\infty]$ and we obtain:

$$|I_1(x)|\le B_{10}. \tag{49}$$

We estimate $I_3(x)$ in the same manner as $I_1(x)$ and obtain:

$$|I_3(x)|\le B_{11}\ \text{ if } N=\max\{1,-\mathrm{Re}\alpha-1/2,\mathrm{Re}\beta-1/2\}. \tag{50}$$

Finally, if we choose $N=\max\{1,-\mathrm{Re}\alpha-1/2,\mathrm{Re}\beta-1/2,\mathrm{Re}\alpha-1/2,-1/2-\mathrm{Re}\beta\}$, then in view of (22), (38), (49), (50) we have the following inequality:

$$|I(x)|\le B_{12}. \tag{51}$$

Then using Fubini's theorem and Note 1. we can rewrite (20) in the form:

$$(f*g)(x)=\int_{-\infty}^{+\infty}2^{2-\alpha-\beta}\frac{1}{2\pi i}\int_{\sigma_s}\frac{f^*(s)(2x)^{-s}}{\Gamma(s+\alpha+\tau)}ds\,\frac{1}{2\pi i}\int_{\sigma_t}\frac{g^*(t)(2x)^{-t}}{\Gamma(t+\beta-\tau)}dt\,d\tau=$$

$$=\int_{-\infty}^{+\infty}2^{2-\alpha-\beta}(x^{\alpha+\tau}\Lambda_+^{-1}x^{-\alpha-\tau}f)(2x)(x^{\beta-\tau}\Lambda_+^{-1}x^{-\beta+\tau}g)(2x). \tag{52}$$

Comparing (19) and (52) we conclude, that integral analog of Leibniz rule (12) hold true under conditions as in Theorem 1.∎

NOTE 3. The obtained integral analogs of Leibniz rule have many important applications, in particular, to the evaluation of integrals with respect to the indices(parameters) of hypergeometric functions [3].

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