VASIL ATANASOV POPOV

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Vasil Popov was born in Sofia, Bulgaria, January 14, 1942. His father, Dr. in Social Sciences, was harassed years long by the communist authorities for his activities as an opposition journalist. This caused many problems in Vasil Popov's life and career. His mother, a very intelligent and gifted person, one of the best translators of the French philosophers, is continuing to work till now.

Vasil Popov grew up as a very polite and kind man, smart and humorous, open and straight person with rather broad background: playing piano, taking the full university courses on physics and mathematics simultaneously in only three and half years. After graduating in

1965 from the Mathematical Department of the Sofia University, major Mathematics, he took his only permanent position in the Institute of Mathematics of the Bulgarian Academy of Sciences, Sofia, as scientist. Despite the troubles caused by his origin, he became senior scientist in 1974, full professor in 1981, and corresponding member of the Bulgarian Academy of Sciences in 1984. He visited for longer periods the Steklov Institute in Moscow, University of South Carolina, Columbia, South Carolina, and Temple University, Philadelphia, Pennsylvania.

The fruitful life of V. Popov was divided between his mathematical researches, his very constructive and enthusiastic work on the development of the Bulgarian school in Approximation theory and the teaching activities in the University of Sofia. His Ph.D. Thesis, 1971 was devoted to *Convex Approximations.* He got a Doctor of Sciences degree in 1976 with a dissertation on *Direct and Converse Theorems in Approximation Theory*. V. Popov took part in the organization of all Bulgarian conferences on Approximation theory. He left more than 15 students, which were influenced very much by his ideas and had the advantage to benefit from his spirit.

The big talent for mathematics of Vasil Popov and his creative energy made him a burning star of the Bulgarian mathematics, that set very early. His death is a big loss for the Bulgarian mathematical community, but his results and ideas will live and will inspire many followers.

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Popov's scientific work is in various fields of approximation theory – approximation in Hausdorff metric, rational and spline approximation, constrained approximations and averaged moduli of smoothness, applications in numerical analysis.

V. Popov was first attracted by problems in *Hausdorff approximations*. Most of his papers written in the period 1966–1975 were in this field and were inspired by his teacher Bl. Sendov.

The Hausdorff distance $r(f,g)$ between two bounded in an interval $[a,b]$ functions f and g is defined as the distance between their completed graphs $F(f)$ and $F(g)$ in the Hausdorff metric in the plane. The completed graph $F(f)$ of a function f can be considered as the smallest closed set in the plane that contains the graph of f and is convex with respect to the y-axis. Thus, the completed graph of a continuous function is simply its graph.

One of the most interesting results in Hausdorff approximations is the exact asymptotic behavior of the best polynomial approximations [30]: *If B^M denotes the class of all bounded by M in the uniform norm functions defined in the interval $[a,b]$, then*

$$\lim_{n\to\infty} \sup_{f\in B^M} \frac{n}{\ln n} E_{n,r}(f) = \frac{b-a}{2},$$

where $E_{n,r}(f)$ *is the best approximation of* f *by algebraic polynomials of degree* n *in Hausdorff metric.* It is shown [30] that the above limit is 1 when approximating 2π-periodic functions by trigonometric polynomials.

Another result that has to be mentioned is the generalization of Jackson theorem to Hausdorff approximation [35]: *For every* 2π*-periodic function* f *and every positive* α *the inequality*

$$E^*_{n,r}(f) \le \frac{c}{\alpha n} \max\left\{ \ln\left(\alpha n \omega\left(f, \frac{1}{n}\right)\right), \right\}$$

holds, where $\omega(f,\delta)$ *is the modulus of continuity and* $E^*_{n,r}(f)$ *is the best approximation of* f *in Hausdorff metric with parameter* α *by trigonometric polynomials of degree* n. A similar result is also proved [35] for approximation by algebraic polynomials.

A classical inverse theorem in approximation theory is the Bernstein's result: *For every monotonically decreasing to zero sequence* $\{\varepsilon_n\}$ *there is a continuous function* f *such that* $E_n(f) = \varepsilon_n$, *where* $E_n(f)$ *is the best uniform approximation to* f *by algebraic polynomials of degree* n. The same result holds true in an arbitrary normed space with minimal requirements on the approximation tool. In a very ingenious way V. Popov established that there is no similar result for the approximations in Hausdorff metric – a case of metric non-normable space. Namely, he showed that [22]:

There is a decreasing sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n \le n^{-1} \ln n$, *for which there are not an interval* $[a,b]$ *and a bounded in this interval function* f *such that* $E_{n,r}(f) = \varepsilon_n$ *for every* n.

The history of the solution of this problem is very interesting. Bl. Sendov suggested to V. Popov to investigate it prior to his graduation from the university. It turned out that the problem was rather complicated. Vasil Popov kept thinking and finally managed to solve it after preparing the Ph.D. thesis.

Vasil Popov's mathematical intuition culminates in his results on *rational approximation and approximation by free knot splines.* Starting in the late sixties he continued to work in this field during his whole life and got some remarkable results. Together with G. Freud he obtained [9, 20] that

$$E^k_n(f)_\infty \le c_k n^{-k-1} \bigvee f^{(k)}, \qquad k \ge 0,$$

where $E^k_n(f)_\infty$ is the best uniform approximation to f by splines of degree k with $n+1$ free knots in and $\bigvee f^{(k)}$ is the variation of $f^{(k)}$. It turned out that this result was of significant importance for non-linear approximations, specially for free knot splines and rational ones.

V. Popov improved this result [45] by replacing the variation with the best L_1 spline approximation of the derivative. As a consequence a "small o" effect for the approximation of any individual functions is obtained, i.e. the estimate for an individual function with absolute continuous $k-1$-st derivative is $o(n^{-k})$, while for the whole class the estimate is $O(n^{-k})$ [20] and it is exact [26].

In several papers [19, 36, 49, 55] V. Popov obtained a characterization of the best approximation by free knot splines using suitable defined moduli of smoothness.

The problem for obtaining an estimate similar to the above mentioned result of G. Freud and V. Popov for the rational approximation has been treated by many mathematicians, among them A. Bulanov, G. Freud, A. Gonchar, J. Szabados, P. Szusz, P. Turan. In a cycle of papers [39, 40, 43, 52, 56, 65] V. Popov improved their results and, developing a remarkable method, he finally got [56] the exact estimate:

$$R_n(f)_\infty \le c_k n^{-k-1} \mathrm{V} f^{(k)}, \quad k \ge 1,$$

where $R_n(f)_\infty$ is the best uniform approximation of f by rational functions of degree n. Popov's method is a basic tool for proving upper bound estimates for rational approximations. In [56] V. Popov also proves Newman's conjecture. Namely,

For any Lipschitz function f one has $R_n(f)_\infty = o(n^{-1})$.

Impressed by his remarkable solution D. J. Newman refer to V. Popov as "a brilliant young Bulgarian mathematician".

In [60] the exact estimate $O(n^{-1})$ for the uniform rational approximation of the class of all uniformly bounded convex and continuous functions is obtained and "small o" effect is established as well. Jackson type estimates are proved [71] for rational approximation

$$R_n(f)_\infty \le c_k n^{-1} \omega_k(f', \tfrac{1}{n})_p, \quad p > 1; \qquad \text{or} \qquad R_n(f)_\infty \le c_k n^{-2} \omega_k(f'', \tfrac{1}{n})_1.$$

In several papers [37, 55, 92, 94, 96, 98, 100] Vasil Popov reveals deep relations between rational approximations, approximations by free knot splines and interpolation of Besov spaces.

Many of the mentioned results found their place in P. Petrushev and V. Popov's monograph "Rational Approximation of Real Functions" [97].

V. Popov and R. DeVore [92] introduced a reasonable definition of free knot splines in d-dimensional case, $d > 1$. Namely, they defined the free spline as a linear combination of n simple functions, which are dilations and translations of a single generating function, such as B-spline or box-spline. They proved direct and inverse theorems for approximation by splines, which are piecewise constants, involving Besov spaces. These results substantially influenced the further investigations of spline and wavelets approximations.

Direct and inverse theorems for *the best one-sided approximations* by trigonometric polynomials or splines with fixed knots are obtained in [58, 59, 62, 64, 66, 69]. These results are similar to the corresponding theorems due to Jackson, Zygmund, Timan and Stechkin in the unconstrained case and the role of the usual moduli of smoothness ω is played by *the averaged moduli of smoothness* τ, which for an integer k are given by [58]

$$\tau_k(f,\delta)_p = \|\omega_k(f,\cdot,\delta)\|_p,$$

where

$$\omega_k(f,x,\delta) = \sup\left\{\left|\Delta_h^k f(t)\right| : t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\right\}.$$

Similar type of characteristics had earlier appeared in papers of Bl. Sendov, P. P. Korovkin, E. P. Dolzhenko and E. A. Sevast'yanov. The complete theory of these moduli was build mainly by the Bulgarian group in approximation theory and is the topic of the monograph of V. Popov and Bl. Sendov "Averaged Moduli of Smoothness" [78].

Using τ-moduli V. Popov obtained a quantitative theorem of Korovkin type [76], direct theorems for approximation in integral metrics and the saturation classes for a number of discrete operators [70, 85]. Some of these results are generalized in the multidimensional case [77, 80, 95]. It is proved in [62, 58] that

$$\tilde{E}_n(f)_p \le c_k \tau_k(f,n^{-1})_p \quad \text{and} \quad \tau_k(f,n^{-1})_p \le c_k n^{-k} \sum_{s=0}^{n} (s+1)^{k-1} \tilde{E}_s(f)_p,$$

where $\tilde{E}_n(f)_p$ denotes the best one-sided L_p-approximation of a 2π-periodic function f by trigonometric polynomials of degree n.

The function spaces generated by the averaged moduli are systematically studied in [79, 81, 82]. In order to investigate their interpolation properties and to obtain imbedding theorems for Sobolev and Besov spaces and equivalent norms for them, V. Popov introduced the following *one-sided K-functional*

$$\tilde{K}_k(f,t)_p = \inf\left\{\|\varphi - \psi\|_p + t\|\varphi^{(k)}\|_p + t\|\psi^{(k)}\|_p : \varphi,\psi \in W_p^k, \varphi \le f \le \psi\right\}.$$

He has shown that the averaged moduli of smoothness are equivalent to the one-sided K-functionals [84, 91]

$$c_k^{-1}\tau_k(f,t)_p \le \tilde{K}_k(f,t^k)_p \le c_k \tau_k(f,t)_p.$$

These results together with many other applications of the averaged moduli to approximation by discrete operators in integral metrics, estimates of the errors of quadrature formulae or numerical methods for differential equations are given in [78].

This is only a brief survey on Vasil Popov's basic results that strongly influenced the approximation theory in the past two decades and specially the Bulgarian mathematics.

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