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A Multivariate Analog of Fundamental Theorem of Algebra and Hermite Interpolation

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In this paper we continue the study of the normal systems of algebraic equations over $\mathbb{C}^k$ started in [4,7,8]. The case when the systems have maximal number of distinct solutions, was characterized there in terms of certain associated k matrices to be commuting and semisimple. This was treated as a multivariate fundamental theorem of algebra in the case of distinct solutions (MFTA1). The maximal number of solutions: ν, as in the classic fundamental theorem of algebra, equals the number of coefficients of the polynomial (involved in the normal system) that follow the leading monomial. We prove MFTA2, anticipated from MFTA1, concerning the case of multiple solutions. Namely, the above mentioned associated matrices commute if and only if the number of solutions of the algebraic system, counting also the multiplicities, equals the maximal possible ν. This gives also a necessary and sufficient condition for poised systems of points for the multivariate Hermite interpolation (see the Lagrange case in [6,8]). The proof of MFTA2 here is based on two systems of PDEs introduced in [5,7], for which the algebraic system serves as the system of characteristic equations. Meanwhile, the connections of multiple solutions of algebraic systems with corresponding solutions of systems of PDEs as well as the general form of latter's solutions are established. For the relation of MFTA2 and a result of Mourrain [12] see Remark 1.

1. Introduction: the Multivariate Algebraic System and Systems of PDEs

To present the algebraic system, which is a multivariate analog of the polynomial equation

$$x^n - a_{n-1}x^{n-1} - \cdots - a_0 = 0, \tag{1}$$

we need some standard multivariate notation. We use bold fonts to distinguish the vector and matrix quantities. Let $\mathbf{x} = (x_1, \dots, x_k),\ \mathbf{y} = (y_1, \dots, y_k) \in \mathbb{C}^k$

be vectors and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k$ be a multiindex. Then we set

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i, \quad \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_k, \quad \alpha! = \alpha_1! \cdots \alpha_k!.$$

For multiindices $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ the inequality $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$, for $i = 1, \dots, k$. The multiindex whose all entries are zero except the i-th which is one we denote by e_i, $i = 1, \dots, k$. The zero multiindex is $\bar{0} := (0, \dots, 0)$.

The space of univariate polynomials (with complex coefficients) of degree at most m we denote by π_m. Let also Π^k be the space of all polynomials in k variables and let Π_m^k be the subspace of polynomials of total degree at most m.

Next we are going to describe the class of multivariate polynomials we deal.

The finite index set $I \subset \mathbb{Z}_+^k$ is called *down connected (to zero),* if for any $\alpha \in I$ there exists $i_1, \dots, i_n$ with $\alpha = e_{i_1} + \cdots + e_{i_n}$ and $e_{i_1} + \cdots + e_{i_m} \in I$ for $m = 1, \dots, n$. This means that by adding successively suitable e_i we can reach any element of I (see [12]).

From now on, we assume that the set I is down connected.

The space of polynomials, connected with the lower set I, is

$$\Pi_I^k = \left\{ p : p(\mathbf{x}) = \sum_{\alpha \in I} c_\alpha \mathbf{x}^\alpha \right\}.$$

Of course we have

$$\dim \Pi_I^k = \#I.$$

The following will serve as the set of leading monomials for the above polynomial space

$$\partial(I) = \bigcup_{i=1}^{k} (I + e_i) \setminus I.$$

In particular, we have in the total degree case:

$$I = \{\alpha : |\alpha| \leq n - 1\} \quad \text{and} \quad \partial(I) = \{\alpha : |\alpha| = n\}.$$

Of course, in this case Π_I^k coincides with Π_{n-1}^k.

We put briefly $\{y_*\} = \{y_\alpha\}_* := \{y_\alpha\}_{\alpha \in I}$. For the components of this vector we use the lexicographical order. (Any other linear order could be used.)

The algebraic system we will study is

$$\mathbf{x}^\alpha - P_\alpha(\mathbf{x}) := \mathbf{x}^\alpha - \sum_{\beta \in I} a_{\alpha,\beta} \mathbf{x}^\beta = 0, \qquad \alpha \in \partial(I). \tag{2}$$

For example in the case $k = 2$ and $I = \{(i, j) : i + j \leq 1\}$ the system is:

$$\begin{aligned} x^2 - a_0 x - b_0 y - c_0 = 0, \\ xy - a_1 x - b_1 y - c_1 = 0, \\ y^2 - a_2 x - b_2 y - c_2 = 0. \end{aligned} \tag{3}$$

It turns out that the system (2), with certain natural consistency condition of commutation of some associated matrices (see [4,7,8]), is a multivariate generalization of a single univariate polynomial equation (1). The number of roots of the latter equation, according to the fundamental theorem of algebra, is n which is the number of coefficients in (1) following the leading monomial, or in other words it equals $\dim \pi_{n-1}$. Let us denote the corresponding number of coefficients in the system (2) by

$$\nu := \#I = \dim \Pi_I^k.$$

A main result of [4,5,7,8] – MFTA1 (see the forthcoming Theorem 10) states that the number of distinct solutions of the system (2) equals the maximal possible ν, if and only if the associated matrices are semisimple and commuting:

$$\mathcal{A}_i \mathcal{A}_j = \mathcal{A}_j \mathcal{A}_i, \qquad 1 \leq i \neq j \leq k, \tag{4}$$

where $\mathcal{A}_i = \{a_{\alpha,\beta}^{(i)}\}$, in the lexicographical order, with

$$a_{\alpha,\beta}^{(i)} = \begin{cases} 0, & \text{if } \alpha + e_i \in I, \quad \beta \neq \alpha + e_i \\ 1, & \text{if } \alpha + e_i \in I, \quad \beta = \alpha + e_i \\ a_{\alpha+e_i,\beta}, & \text{if } \alpha + e_i \in \partial(I). \end{cases} \tag{5}$$

In other words, for the monomial base $\{\mathbf{x}^\alpha : \alpha \in I\}$, one has

$$[\mathcal{A}_i]^T (\mathbf{x}^\alpha) = \begin{cases} x_i \mathbf{x}^\alpha, & \text{if } \alpha + e_i \in I \\ P_{\alpha+e_i}(\mathbf{x}), & \text{if } \alpha + e_i \in \partial(I). \end{cases}$$

Note that the number of the associated matrices equals the dimension k and otherwise it does not depend on the algebraic system (2). Notice also that some rows of these matrices consist of 0s and one 1 and the others coincide with the rows of coefficients of the algebraic system.

For example in the case of system (3), the two associated matrices are:

$$\begin{pmatrix} 0 & 1 & 0 \\ c_0 & a_0 & b_0 \\ c_1 & a_1 & b_1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{pmatrix}.$$

As one could expect, the condition of semisimplicity in MFTA1 just guarantees the distinctness (or simplicity) of solutions. This leads to MFTA2 (see the forthcoming Theorem 8): The number of solutions of the system (2), counting also multiplicities, equals the maximal possible ν, if and only if the associated matrices are commuting.

For $\mathbb{C}^1$ there is no commuting condition because of only one associated matrix. Thus in this case MFTA2 turns into the classic fundamental theorem of algebra.

Remark 1. Let us mention that the basic statements of MFTA2, except one (coinciding with forthcoming Theorem 7), follow from a result of Mourrain on normal form algorithms [12, Theorem 3.1] and a known fact from ideal theory on the codimension of a $0-$dimensional polynomial ideal (see [1]). Thus, in view of this fact, MFTA2 follows from Theorem 7 and Mourrain's result (see [8] for more details and a short proof of the latter result).

The proof of MFTA2 we bring is based on two linear homogeneous systems of PDEs (see [5,7,8]) with constant coefficients. In what follows till the end of the section we bring results on these systems, which are multivariate analogs of the homogeneous first-order normal system of ordinary differential equations

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \tag{6}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^T$ is the vector of unknown functions, and the homogeneous n-th order normal differential equation:

$$y^{(n)} = a_{n-1}y^{(n-1)} + \cdots + a_0 y. \tag{7}$$

The homogeneous first order normal system of PDEs, the multivariate analog of (6), is the following Pfaff system:

$$\frac{\partial\{z_*\}}{\partial x_i} = \mathcal{A}_i\{z_*\}, \qquad i = 1, \ldots, k. \tag{8}$$

Here $\{z_*\}$ is the unknown vector-function. The initial conditions are

$$z_\alpha(\mathbf{x}^0) = z_\alpha^0, \qquad \alpha \in I, \tag{9}$$

where $\mathbf{x}^0 \in \mathbb{R}^k$ and $\{z_\alpha^0\}_* \in \mathbb{C}^\nu$.

The matrix $\mathcal{A}_i = \{a_{\alpha,\beta}^{(i)}\}$ is the same as in (5).

Let us denote

$$D^\alpha z := z^{(\alpha)} := \frac{\partial^{|\alpha|} z}{\partial x_1^{\alpha_1} \cdots \partial x_k^{\alpha_k}}.$$

The higher order (HO) normal system of PDEs (see [8]) with constant coefficients, the multivariate analog of the equation (7), is

$$D^\alpha z = \sum_{\beta \in I} a_{\alpha,\beta} D^\beta z, \qquad \alpha \in \partial(I). \tag{10}$$

Note that, as in (7), we have a single unknown scalar function z (of k variables). The initial conditions for this system are the following

$$D^\alpha z(\mathbf{x}^0) = z_\alpha^0, \qquad \alpha \in I, \tag{11}$$

where $\mathbf{x}^0 \in \mathbb{R}^k$ and $\{z_\alpha^0\}_* \in \mathbb{C}^\nu$.

Now, by using that the set I is down connected, we obtain (see [8]) that problems (8)–(9) and (10)–(11) are equivalent by the following one-to-one correspondence of solutions.

Theorem 1 (On Equivalence). *To the each solution z of the problem* (10)–(11) *there corresponds a solution $\{z_*\} = \{D^\alpha z\}_*$ of the problem* (8)–(9) *and vice versa, i.e., each solution $\{z_*\}$ of* (8)–(9) *has form $\{D^\alpha z\}_*$ where z is a solution of* (10)–(11).

In view of Theorem 9 of [8] (and the extension to the complex case following it) we have:

Theorem 2. *The problems* (8)–(9) *and* (10)–(11) *have a solution for arbitrary initial data if and only if the consistency condition* (4) *is satisfied. In addition, the solution is unique and is defined on the whole $\mathbb{R}^k$.*

We will use the following multivariate generalization of Wronskian (see [5,7]). Consider a set of ν solutions of the system (10): $\mathcal{G} = \{z_\alpha : \alpha \in I\}$. Then the Wronskian is defined as

$$W(\mathbf{x}) := W(\mathbf{x}|\mathcal{G}) := \det ||D^\beta z_\alpha||_{\alpha,\beta\in I}.$$

Theorem 2 implies (see [7]) the following.

Theorem 3. *Assume that the consistency condition* (4) *is satisfied. Then the dimension of the linear space of solutions of the system of PDEs* (10) *equals ν. Moreover, for any set of solutions $\{z_\alpha : \alpha \in I\}$ either $W = 0$ identically, when they are linearly dependent, or $W(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbb{R}^k$, when they are independent. In the latter case they form a fundamental set of solutions.*

Next we bring the basic relations between the systems of PDEs (8),(10), the algebraic system (2), and the associated matrices, established in [8, Theorems 2 and 10]. Note that the algebraic system is the system of characteristic equations of the system of PDEs (10). Let us mention that the equivalence of (i), (iv), and (v) below, in the special case of common zeros of multivariate orthogonal polynomials, were established earlier by Xu [14].

Theorem 4. *The following assertions are equivalent:*

(i) $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is a solution of the algebraic system (2)*;*

(ii) the function $f = \exp(\boldsymbol{\lambda} \cdot \mathbf{x})$ is a solution of the system (10)*;*

(iii) $\mathbf{z} = \mathbf{h} \exp(\boldsymbol{\lambda} \cdot \mathbf{x})$ is a solution of the system (8)*, with some vector $\mathbf{h}$;*

(iv) $\boldsymbol{\lambda}$ is a collection of eigenvalues corresponding to some common eigenvector $\mathbf{h}$ for the matrices $\mathcal{A}_i,\ 1 \le i \le k$;

(v) $\mathbf{h} = c\{\boldsymbol{\lambda}^\beta\}_*$ is a common eigenvector for the matrices $\mathcal{A}_i,\ 1 \le i \le k$.

Moreover, the common eigendirection is determined uniquely by the corresponding collection of eigenvalues $\boldsymbol{\lambda}$, by (v).

2. Multiple Zeros of Multivariate Polynomials

The concept of multiple zero has been studied in algebraic geometry by many different approaches. For example by means of formal power series, or quadratic transformations and consecutive neighborhoods (see e.g. [13, Chapter 3, Section 7.6 and Chapter 4, Section 5.1]. These approaches are inappropriate for our purpose since they are not transparent.

We follow a natural way, from the point of view of approximation theory, to define multiple or Hermite zero. Namely, we consider it as a result of collapsing of respective number of simple zeros. As it turns out in Proposition 5 below, the result of this approach becomes identical with one based on ideal interpolation schemes, pointed out by Marinari, Möller, and Mora [10] as well as de Boor and Ron [2].

Before we start let us briefly consider a simple approach. The multiplicity here is characterized by means of consecutive chain of directional derivatives: more precisely, for given directions $\mathbf{a}_1, \ldots , \mathbf{a}_m$ in $\mathbb{C}^k$,

$$P = 0,\ D_{\mathbf{a}_1} P = 0,\ \ldots ,\ D_{\mathbf{a}_m} \cdots D_{\mathbf{a}_1} P = 0. \tag{12}$$

The idea here is that the collapsing of one point to another fixed one through a path gives rise to the directional derivative along the tangent line of the path at the fixed point. Nevertheless, as the forthcoming Remark 2 shows this approach cannot be used to describe the Hermite multiplicity, even in the case of two points approaching another fixed one.

Let $R(D)$ be the differential operator given by the polynomial $R \in \Pi^k$.

We say that a *multiple zero at* $\mathbf{x^0}$ *given by* $R(D)$, where $R \in \Pi^k$, is *Hermitian* or is a result of coalescence of m simple zeros, i.e.,

$$\lim_{n\to\infty} \mathbf{x}^{i,n} = \mathbf{x}^0, \qquad i = 1, \ldots , m,$$

where $\mathbf{x}^{i,n}$ are distinct for each fixed n, if for any polynomial $P \in \Pi^k$ with

$$P = \lim_{n\to\infty} P_n, \quad \text{and} \quad P_n(\mathbf{x}^{i,n}) = 0, \quad i = 1, \ldots , m, \tag{13}$$

we have that

$$R(D)P(\mathbf{x}^0) = 0.$$

Let us mention that the convergence in (13) is coefficientwise.

Proposition 1. *Let the multiple zero at* $\mathbf{x^0}$ *given by* $R(D),\ R \in \Pi^k$:

$$R(D)P(\mathbf{x}^0) = 0 \tag{14}$$

be Hermitian. Then we have

$$R^{(\alpha)}(D)P(\mathbf{x}^0) = 0, \qquad \textit{for all } \alpha \in Z_+^k. \tag{15}$$

Proof. The Hermite multiplicity of a zero, according to the adopted approach, depends only on the set of collapsing simple zeros and not on the particular polynomial. Let us show, in view of this, that the condition (14) implies that

$$R(D)(SP)(\mathbf{x}^{(0)}) = 0,$$

where $S \in \Pi^k$ is any polynomial. Indeed, by multiplying the equalities in (13) by $S(\mathbf{x})$ we get that they hold also with P replaced by $\hat{P} = PS$.

Now one could use the above mentioned ideal structure approach presented in [10] and [2], which will end the proof.

Instead, for the sake of completeness, we make use of the following known relation

$$R(D)\left[g(\mathbf{x})f(\mathbf{x})\right] = \sum_{\gamma} \frac{1}{\gamma!} g^{(\gamma)}(\mathbf{x}) R^{(\gamma)}(D) f(\mathbf{x}). \tag{16}$$

Notice that to verify this it suffices to check it for R being a monomial which reduces (16) to Leibniz's rule. By setting here $f = P$ and $g(\mathbf{x}) = S(\mathbf{x}) = x_i$, $i = 1, \dots, k$, we get that

$$\frac{\partial R}{\partial x_i}(D)P(\mathbf{x}^0) = 0.$$

Continuing this way and taking consecutively $R(\mathbf{x}) = \mathbf{x}^\alpha$ with α in lexicographical order, we come to the desired conclusion.

It may seem that the Hermite conditions in the multivariate case must be given at least by homogeneous differential operators $R(D)$, as they are for example in (12). The following example of just three collapsing points, one of which is fixed, shows that this is not true.

Remark 2. Consider the following three distinct points: $(x_0, y_0) = (0, 0)$; $(x_1, y_1) = (0, \varepsilon^2)$; $(x_2, y_2) = (\varepsilon^2, \varepsilon)$ (cf. the example at the end of [9]). Let us show that these simple zeros are transforming to the following three Hermite conditions:

$$P(x_0, y_0) = 0; \quad \frac{\partial}{\partial y}P(x_0, y_0) = 0; \quad \left[\frac{\partial}{\partial x} + \frac{1}{2} \cdot \frac{\partial^2}{\partial y^2}\right]P(x_0, y_0) = 0, \tag{17}$$

where

$$P = \lim_{\varepsilon \to 0} P_\varepsilon,$$

with coefficientwise convergence and

$$P_\varepsilon(0, 0) = 0; \qquad P_\varepsilon(0, \varepsilon^2) = 0; \qquad P_\varepsilon(\varepsilon^2, \varepsilon) = 0.$$

Indeed, the first condition of (17) is obvious. The second condition can be obtained easily by using the points (x_0, y_0) and (x_1, y_1). To obtain the third one we proceed as follows. We take an auxiliary point $(\bar{x}_2, \bar{y}_2) = (0, \varepsilon)$ and consider the following linear combination of the values of P_ε at $(x_i, y_i),\ i = 0, 1, 2$ (below the coefficient of $P_\varepsilon(\bar{x}_2, \bar{y}_2)$ equals zero)

$$\frac{1}{\varepsilon^2}[P_\varepsilon(x_2, y_2) - P_\varepsilon(\bar{x}_2, \bar{y}_2)] \\ + \frac{1-\varepsilon}{\varepsilon}\left[\frac{P_\varepsilon(\bar{x}_2, \bar{y}_2) - P_\varepsilon(x_1, y_1)}{\varepsilon - \varepsilon^2} - \frac{P_\varepsilon(x_1, y_1) - P_\varepsilon(x_0, y_0)}{\varepsilon^2}\right] = 0.$$

By expressing this in terms of divided differences we get that

$$[0, \varepsilon^2]P_\varepsilon(\cdot, \varepsilon) + (1-\varepsilon)[0, \varepsilon, \varepsilon^2]P_\varepsilon(0, \cdot) = 0.$$

This, of course, in the limit gives the third condition.

In view of Proposition 1 we bring

Definition 1. *We shall say that the multiplicity set of* $P(\mathbf{x})$ *or of equation* $P(\mathbf{x}) = 0,$ *at* $\mathbf{x} = \mathbf{x}^0$, *contains* R *if the condition* (15) *holds* $(R, P \in \Pi^k)$.

Let us mention that this definition is in accordance with the Bezout Theorem. Thus, it is equivalent to the other definitions of multiple zeros from algebraic geometry mentioned earlier. Besides, it can be seen that the multiplicity with consecutive directional derivatives (12) is not appropriate from the point of view of the Bezout theorem. Indeed, consider the following system

$$x^2 - x = 0, \qquad y^3 = 0.$$

The number of solutions here, according to the Bezout Theorem, equals $6 = 2 \times 3$ and the solutions are $(0, 0)$ and $(1, 0)$ each with multiplicity $R = y^2, y, 1$. When adopting consecutive directional derivatives (12) as multiplicity, here we would have infinitely many solutions, counting also the multiplicities. In view of (12) these multiplicities, for the same solutions, are $\mathbf{a}_1 = (0, 1),\ \mathbf{a}_2 = (1, 0),\ \mathbf{a}_i = (0, 1),\ i = 3, 4, \ldots$

Remark 3. Let us mention that

(i) In the univariate case the linear span of $R^{(i)}(x) : i = 0, 1, \ldots,$ for any polynomial $R \in \pi_m \setminus \pi_{m-1}$, coincides with π_m. Thus, in this case, starting with any polynomial we arrive at the usual univariate Hermite conditions.

(ii) If $R(D) = (D_{\mathbf{y}})^n$, then the conditions (15) coincide with $(D_{\mathbf{y}})^i P(\mathbf{x^0}) = 0$, $i = 0, \ldots, n$.

(iii) If $R(D) = D^\alpha$, then the conditions (15) coincide with $D^\beta P(\mathbf{x^0}) = 0$, $\beta \leq \alpha$.

(iv) Conditions (15) always include $P(\mathbf{x^0}) = 0$.

(v) If R belongs to a multiplicity set, then $\partial/\partial x_i(R),\ i = 1, \ldots, k$, belongs to it too.

According to the last point the linear space of polynomials R describing the multiplicity is so called D-invariant (see [10],[2], and Section 3).

The following theorem establishes a relation between the multiple solutions of algebraic and partial differential equations. It generalizes the well-known univariate relation between the solutions of homogeneous ordinary differential equation with constant coefficients and the multiple zeros of the characteristic equation.

Theorem 5. *The polynomial R is in the multiplicity set of a solution $\boldsymbol{\lambda}$ of the algebraic equation*

$$Q(\mathbf{x}) = 0$$

if and only if the functions of the collection

$$\mathcal{F} := \{R^{(\alpha)}(\mathbf{x}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x}) : \alpha \in Z_+^k\}, \tag{18}$$

are solutions of the PDE

$$Q(D)z = 0.$$

Proof. First, let us establish the following formula

$$Q(D)\,[R(\mathbf{x}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x})] = \sum_\beta \frac{1}{\beta!}\, R^{(\beta)}(D) Q(\boldsymbol{\lambda}) \mathbf{x}^\beta \exp(\boldsymbol{\lambda} \cdot \mathbf{x}). \tag{19}$$

We start with the following simple relation

$$Q(D) \exp(\boldsymbol{\lambda} \cdot \mathbf{x}) = Q(\boldsymbol{\lambda}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x}).$$

In view of this we get from (16), with $R = Q,\ g = R$, and $f(\mathbf{x}) = \exp(\boldsymbol{\lambda} \cdot \mathbf{x})$:

$$Q(D)\,[R(\mathbf{x}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x})] = \sum_\gamma \frac{1}{\gamma!}\, R^{(\gamma)}(\mathbf{x}) Q^{(\gamma)}(\boldsymbol{\lambda}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x}).$$

Then, assuming that $R(\mathbf{x}) = \sum_\alpha a_\alpha \mathbf{x}^\alpha$, we proceed as follows

$$\begin{aligned}
&\sum_\gamma \frac{1}{\gamma!}\, R^{(\gamma)}(\mathbf{x}) Q^{(\gamma)}(\boldsymbol{\lambda}) = \sum_\gamma \frac{1}{\gamma!} \sum_\alpha a_\alpha \frac{\alpha!}{(\alpha-\gamma)!}\, \mathbf{x}^{\alpha-\gamma} Q^{(\gamma)}(\boldsymbol{\lambda}) \\
&= \sum_\gamma \frac{1}{\gamma!} \sum_\beta a_{\gamma+\beta} \frac{(\gamma+\beta)!}{\beta!}\, \mathbf{x}^\beta Q^{(\gamma)}(\boldsymbol{\lambda}) = \sum_\beta \frac{\mathbf{x}^\beta}{\beta!} \sum_\gamma a_{\gamma+\beta} \frac{(\gamma+\beta)!}{\gamma!}\, Q^{(\gamma)}(\boldsymbol{\lambda}) \\
&= \sum_\beta \frac{\mathbf{x}^\beta}{\beta!} \sum_\delta a_\delta \frac{\delta!}{(\delta-\beta)!}\, Q^{(\delta-\beta)}(\boldsymbol{\lambda}) = \sum_\beta \frac{1}{\beta!}\, R^{(\beta)}(D) Q(\boldsymbol{\lambda}) \mathbf{x}^\beta.
\end{aligned}$$

Thus we established the formula (19) which readily proves the theorem. Indeed, the part $\Rightarrow$ is straightforward. To verify the inverse implication assume that the left hand side of (19) equals zero. Then, since $\exp(\boldsymbol{\lambda} \cdot \mathbf{x}) \neq 0$, the other factor – the polynomial in the right hand side is zero, i.e.,

$$\sum_{\beta} \frac{1}{\beta!} R^{(\beta)}(D)Q(\boldsymbol{\lambda})\mathbf{x}^{\beta} = 0.$$

Thus, all the coefficients here are zero and this ends the proof.

The following theorem which complements the previous one, in view of the above argument, follows immediately from relation (19). It is in an interesting relation with the corresponding property of multiple (or Hermite) zeros of polynomials (Proposition 1).

Theorem 6. *If the function $R(\mathbf{x}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x})$ is a solution of the PDE*

$$Q(D)z = 0,$$

then all the functions of the collection (18) *are so.*

The following application of Theorems 5 and 6 to algebraic system (2) generalizes Theorem 4 to the case of multiple solutions. Of course, the same statement is true also for a single algebraic equation and the corresponding PDE.

Corollary 1. *The polynomial R is in the multiplicity set of a solution $\boldsymbol{\lambda}$ of the algebraic system* (2) *if and only if the function $R(\mathbf{x}) \exp(\boldsymbol{\lambda} \cdot \mathbf{x})$ is a solution of the system of PDEs* (10)*.*

The set I is called *lower* if $\alpha \in I$ and $\beta \leq \alpha$ imply $\beta \in I$.

We shall say that P has *multiindex multiplicity* J, with this latter being a lower set, at $\mathbf{x^0}$ if

$$P^{(\alpha)}(\mathbf{x^0}) = 0, \qquad \text{for all } \ \alpha \in J.$$

We get readily from the above corollary the following result from an earlier version of [8].

Corollary 2. *The solution $\boldsymbol{\lambda}$ of algebraic system* (2) *has multiindex multiplicity J if and only if the functions of the collection*

$$\left\{\mathbf{x}^{\beta} \exp(\boldsymbol{\lambda} \cdot \mathbf{x}) : \ \beta \in J\right\}$$

are solutions of the system (10)*.*

Indeed, we have, for any lower set J,

$$J = \cup_{\alpha \in J} I_{\alpha},$$

where $I_{\alpha} = \{\beta : \beta \leq \alpha\}$. It remains to apply Corollary 1 for $R(\mathbf{x}) = \mathbf{x}^{\alpha}$ where $\alpha \in J$.

Note that the total degree multiplicities, which are considered in algebraic geometry, are characterized in above theorem. Indeed, total degree multiindex sets are obviously lower.

3. The Main Results

As was mentioned in the previous section, after Remark 3, the multiple zero of a polynomial P (or the multiple common zero of several polynomials) at a point $\mathbf{x}$ can be characterized by differential operators from certain D-invariant linear polynomial space which we denote by $\mathcal{M}_{\mathbf{x}}$:

$$R \in \mathcal{M}_{\mathbf{x}} \Rightarrow R^{(\alpha)} \in \mathcal{M}_{\mathbf{x}} \text{ for all } \alpha \in Z_+^k.$$

Following [3] and [11] we call $\mu_{\mathbf{x}} := \dim \mathcal{M}_{\mathbf{x}}$ the (arithmetical) *multiplicity* of $\mathbf{x}$.

Our first aim is proving that the number of solutions of the algebraic system (2), counting also the multiplicities, is always less than or equal to $\nu := \#I$ (see the forthcoming Theorem 7). For this we need a definition and a lemma.

Definition 2. *We call a set of linearly independent vectors*

$$\mathcal{H} = \{\mathbf{h}_1^0, \mathbf{h}_1^1, \dots \mathbf{h}_{i_1}^1, \dots \mathbf{h}_1^m, \dots, \mathbf{h}_{i_m}^m\},$$

with the superscript showing the level, a generalized associated series of the eigenvector $\mathbf{h}_1^0$*, corresponding to the eigenvalue* λ*, of the matrix* $\mathbf{A}$ *if*

$$\mathbf{A}\mathbf{h}_1^0 = \lambda \mathbf{h}_1^0$$

and

$$\mathbf{A}\mathbf{h}_j^l = \lambda \mathbf{h}_j^l + \mathbf{g}_j^l,$$

where $\mathbf{g}_j^l$ *belongs to the linear span of vectors of* $\mathcal{H}$ *of levels* $\leq l-1$.

The following lemma can be considered as a generalization of the well-known result concerning the linear independence of the set of associated series of eigenvectors corresponding to distinct eigenvalues of a single matrix.

Lemma 1. *The set of generalized associated series of common eigenvectors of a set of matrices corresponding to distinct collections of eigenvalues are linearly independent.*

Proof. Let

$$\mathbf{h}_1^{r,0}, \qquad r = 1, \dots, s,$$

be the common eigenvectors of matrices $\mathbf{A}_1, \dots, \mathbf{A}_\sigma$ corresponding to distinct collections of eigenvalues and

$$\mathcal{H}_r, \qquad r = 1, \dots, s,$$

be the set of corresponding associated series.
We use induction on s. The case $s = 1$ is obvious from Definition 2. Assume that the lemma is true for $s-1$ common eigenvectors. Conversely, suppose that the lemma is not true for the case of s common eigenvectors, i.e., a nontrivial

linear combination of vectors of $\cup_{r=1}^{s}\mathcal{H}_r$ is zero, where at least one coefficient of a vector with the highest level from $\mathcal{H}_i$ is nonzero, $i = 1, 2$.
Since the collections of eigenvalues of $\mathbf{h}_1^{10}$ and $\mathbf{h}_1^{20}$ are distinct, there is a matrix $\mathbf{A}_{i_0}$, $1 \leq i_0 \leq \sigma$, with respect to which the above two vectors have distinct eigenvalues, that is,

$$\mathbf{A}_{i_0}\mathbf{h}_1^{j0} = \gamma_j \mathbf{h}_1^{j0}, \qquad j = 1, 2, \ \gamma_1 \neq \gamma_2.$$

Now we apply the operator $[\mathbf{A}_{i_0} - \gamma_1\mathbf{E}]^{m_1}$ to the above nontrivial linear combination, and thus eliminate the series $\mathcal{H}_1$. The resulted zero linear combination of series of $\mathcal{H}_r$, $r = 2, \ldots, s$, is still nontrivial since the coefficients of highest level of $\mathcal{H}_2$, in particular, are proportional to those of original linear combination, with a constant of proportionality $(\gamma_2 - \gamma_1)^{m_1} \neq 0$. This contradicts the induction hypothesis and proves the lemma.

Let us mention that according to Theorem 1 of [8] the algebraic system (2) is equivalent to the following quadratic system

$$\mathcal{A}_i\{x_*\} = x_i\{x_*\}, \qquad i = 1, \ldots, k,$$

where $\{x_*\} = \{\mathbf{x}^\beta\}_*$.

Suppose the set of multiplicity of a solution $\boldsymbol{\lambda}$ of the system (2) contains the polynomial P. Then, by applying here $P(D)$ to the both sides of the above equality and using the relation (16) we get

$$\mathcal{A}_i P(D)\{\lambda_*\} = \lambda_i P(D)\{\lambda_*\} + \frac{\partial}{\partial x_i}P(D)\{\lambda_*\}, \qquad i = 1, \ldots, k, \tag{20}$$

where $\{\lambda_*\} = \{\boldsymbol{\lambda}^\beta\}_*$.

Denote by $\mathcal{S}$ the set of all *distinct* solutions of algebraic system (2).
The following theorem gives the above mentioned bound for the number of solutions of algebraic system (2), counting also the multiplicities. Let us denote this number by μ,

$$\mu := \sum_{\mathbf{x}\in\mathcal{S}} \mu_{\mathbf{x}} = \sum_{\mathbf{x}\in\mathcal{S}} \dim \mathcal{M}_{\mathbf{x}}.$$

Also, we denote by $\mathcal{B}[L]$ a basis of the linear space L.

Theorem 7. *The following set of ν-dimensional vectors*

$$\left\{P(D)\{\boldsymbol{\lambda}^\beta\}_* : \boldsymbol{\lambda} \in \mathcal{S}, P \in \mathcal{B}[\mathcal{M}_\lambda]\right\}$$

is linearly independent and therefore the number of solutions of the algebraic system, counting also the multiplicities, is $\leq \#I$, i.e., $\mu \leq \nu$.

Indeed, according to Theorem 4, the above vectors, in the case $P = 1$, are common eigenvectors of the associated matrices. On the other hand, again by Theorem 4, any different common eigenvectors of associated matrices correspond to different collections of eigenvalues. Thus, to prove the theorem it is enough to arrange the given vectors such that they satisfy the conditions of Definition 2 of the associated series. To do this we choose a so called level basis: $\mathcal{B}^l$ in $\mathcal{M}_\lambda,\ \boldsymbol{\lambda} \in \mathcal{S}$, with the superscripts of elements showing the level, to fit the previous lemma, in the way described below.

Let
$$B_i = \{P_1^i, \ldots, P_{m_i}^i\}, \qquad i = 1, 2, \ldots,$$
be a subset of maximal cardinality of polynomials from $\Pi_i^k \cap \mathcal{M}_\lambda$ such that their homogeneous parts of degree i are linearly independent. Then it is easily seen that the set
$$\mathcal{B}^l = \cup_i B_i$$
is a basis of $\mathcal{M}_\lambda$. Moreover, in view of Theorem 4 and the relation (20),
$$\left\{P(D)\{\boldsymbol{\lambda}^\beta\}_* : \boldsymbol{\lambda} \in \mathcal{S}, P \in \mathcal{B}^l[\mathcal{M}_\lambda]\right\}$$
is a set of generalized associated series of the common eigenvectors $\{\boldsymbol{\lambda}^\beta\}_*$, $\boldsymbol{\lambda} \in \mathcal{S}$, corresponding to the distinct collections of eigenvalues $\boldsymbol{\lambda}$ of the matrices $\mathcal{A}_i$, $i = 1, \ldots, k$. This completes the proof.

The above independence in the case of $\mu = \nu$, of course, is equivalent to
$$V(\mathbf{x}) := V(\mathbf{x}|\mathcal{S}, \mathcal{B}) := \det ||P(D)\{\boldsymbol{\lambda}^\beta\}_*||_{\beta \in I, \lambda \in \mathcal{S}, P \in \mathcal{B}[\mathcal{M}_\lambda]} \neq 0, \qquad (21)$$
where we use the lexicographical order for rows and some fixed order for the columns. Since this is the generalized Vandermonde determinant, the above independence is also equivalent to the poisedness of the Hermite interpolation with the space of polynomials Π_I^k corresponding to the set of points $\mathcal{S}$ with respective multiplicities.

Notice also that every such poised system of points is the set of solutions of a type (2) system. The coefficients of the latter can be expressed via above solutions in a standard way, from the corresponding linear systems as in the Lagrange case described in [7,8].

Now we are in a position to present

Theorem 8 (MFTA2). *The number of solutions of the algebraic system* (2)*, counting also the multiplicities, is the maximal possible, i.e., equals* $\nu := \#I$ *if and only if the associated matrices* $\mathcal{A}_i,\ i = 1, \ldots, k$*, are commuting. Moreover, in the latter case these solutions form a poised set of points for Hermite interpolation with* Π_I^k*. And any poised set is the set of solutions of exactly one algebraic system of type* (2)*, for which the associated matrices are commuting.*

Proof. Assume first that the number of solutions of the algebraic system is the maximal possible: ν. Then we have that the above Vandermonde determinant is different from zero. Our aim is to show that then the Wronskian of the

set of functions
$$\{P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \boldsymbol{\lambda}\in\mathcal{S}, P\in\mathcal{B}[\mathcal{M}_\lambda]\}$$
which, according to Corollary 1, are solutions of the system of PDEs (10), is not zero too:
$$W(\mathbf{x}) := W(\mathbf{x}|\mathcal{S},\mathcal{B}) := \det\|D^\alpha P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x})\|_{\alpha\in I,\lambda\in\mathcal{S},P\in\mathcal{B}[\mathcal{M}_\lambda]} \neq 0. \quad (22)$$
This, according to Theorem 2, will lead to the desired conclusion, i.e., that the associated matrices are commuting. Indeed, as it can be easily seen, (22) means that the system of PDEs (10) has a solution for arbitrary initial values. To simplify the Wronskian we assume that the basis in (22) is the level basis described in the proof of Theorem 7.

Consider a column in the Wronskian:
$$D^\alpha P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \alpha\in I. \quad (23)$$
We make use of the following particular case of the relation (19)
$$D^\alpha\left[P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x})\right] = \sum_\beta \frac{1}{\beta!}P^{(\beta)}(D)\,\boldsymbol{\lambda}^\alpha\mathbf{x}^\beta\exp(\boldsymbol{\lambda}\cdot\mathbf{x}), \qquad \alpha\in I, \quad (24)$$
which we treat as an equality of column vectors with coordinates $\alpha\in I$. Let us show that, in view of (24), by using elementary operations on a determinant with the column (23) and other columns of lower levels, we can reduce this column to the form
$$P(D)\boldsymbol{\lambda}^\alpha\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \alpha\in I. \quad (25)$$
Let us verify this statement by induction on the level $l := degP$. The case $l = 0$ is obvious. Assume that the statement is true for the levels $\leq l-1$ and prove it for l. Then, according to the induction hypothesis, the columns – summands in the right hand side of (24):
$$P^{(\beta)}(D)\boldsymbol{\lambda}^\alpha\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \alpha\in I,$$
with $\beta\neq\bar{0}\in Z_+^k$, are linear combinations of the columns of the Wronskian of levels $\leq l-1$. Hence, by elementary operations we can eliminate all this summands and what remains after this in (24) is the column (25). This completes the proof of the statement.

Now note that columns (25) differ from the corresponding columns of the Vandermonde determinant only by the factors $\exp(\boldsymbol{\lambda}\cdot\mathbf{x})$ and thus we get the following formula
$$W(\mathbf{x}|\mathcal{S},\mathcal{B}^l) = \prod_{\lambda\in\mathcal{S}}\exp(\mu_\lambda\boldsymbol{\lambda}\cdot\mathbf{x})V(\mathbf{x}|\mathcal{S},\mathcal{B}^l), \quad (26)$$
which completes the proof of this part.

Now let us prove the opposite implication. Assume that the associated matrices are commuting. Then the dimension of the linear space of solutions of the system of PDEs (10) equals ν. To complete the proof of the theorem, in view of the above relation between the Wronskian and Vandermonde determinant (26), it is enough to prove the following.

Theorem 9. *All the solutions of the system of PDEs* (10) *have the form*

$$P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x}), \tag{27}$$

where $P \in \Pi^k, \boldsymbol{\lambda} \in \mathbb{C}^k$.

Notice that actually this is more general than we need. Namely, here we do not assume that the consistency condition (4) holds.

Let us mention that this theorem and the arguments of the above proof of Theorem 8 yield the following nice relation between algebraic and PDE systems, where also (4) is not assumed to hold.

Corollary 3. *The dimension of the linear space of solutions of the system of PDEs* (10) *equals the number of solutions of the algebraic system* (2), *counting also the multiplicities. Moreover,*

$$\{P(\mathbf{x})\exp(\boldsymbol{\lambda}\cdot\mathbf{x}) : \boldsymbol{\lambda}\in\mathcal{S}, P\in\mathcal{B}[\mathcal{M}_\lambda]\}$$

is a fundamental set of solutions of the system of PDEs (10).

To establish Theorem 9 we need to prove first the following.

Lemma 2. *Assume that the set of functions*

$$\mathcal{G} := \mathcal{G}_m := \{g_1, \dots, g_m\}$$

is linearly independent. Then there is a set of points $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ *such that the respective Vandermonde determinant does not vanish, i.e.,*

$$V(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathcal{G}) := \det ||g_i(\mathbf{x}_j)||_{i,j=1,\dots,m} \neq 0.$$

Let us prove this by induction on m. The case $m = 1$ is obvious. Assume the lemma is true for $m-1$, i.e., there is a set $\{\mathbf{x}_1^0, \dots, \mathbf{x}_{m-1}^0\}$ such that

$$V(\mathbf{x}_1^0, \dots, \mathbf{x}_{m-1}^0; \mathcal{G}_{m-1}) \neq 0. \tag{28}$$

Now let us verify that the determinant

$$V(\mathbf{x}_1^0, \dots, \mathbf{x}_{m-1}^0, \mathbf{x}_m; \mathcal{G})$$

is not identically zero, which will complete the proof. Indeed, it is easily seen that this determinant is a nontrivial linear combination of set $\mathcal{G}$, since the coefficient of g_m there is $(-1)^m$ times the determinant in (28) and thus is not zero.

Proof of Theorem 9. In view of Theorem 1 (on equivalence) it is enough to prove the proposition with the system (10) replaced by the system (8). Correspondingly, we are to show that the first coordinate $z_{\bar{0}}$ of any solution of (8) has the form (27). We shall prove this by induction on k, i.e., on the dimension. If $k = 1$, then we have the case of system of ordinary differential equations which is well known. Assume the proposition is true for the case $k-1$; we shall prove it for k. Suppose that $z(\mathbf{x})$ is any solution of (8). Then for any fixed last coordinate x_k this is also a solution of the system

$$\frac{\partial\{z_*\}}{\partial x_i} = \mathcal{A}_i\{z_*\}, \qquad i = 1, \dots, k-1. \tag{29}$$

Therefore, according to the induction hypothesis we have

$$z_{\bar{0}}(\mathbf{x}) = \sum_{i=1}^{m} c_i(x_k) P_i(\tilde{\mathbf{x}}) \exp(\tilde{\boldsymbol{\lambda}}_i \cdot \tilde{\mathbf{x}}), \tag{30}$$

where $P_i \in \Pi^{k-1}$, $\tilde{\boldsymbol{\lambda}}_i, \tilde{\mathbf{x}} = (x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1}$. Without loss of generality we can assume that the set of functions

$$\tilde{\mathcal{G}} = \{\tilde{g}_i\}_{i=1}^m,$$

where $\tilde{g}_i = P_i(\tilde{\mathbf{x}}) \exp(\tilde{\boldsymbol{\lambda}}_i \cdot \tilde{\mathbf{x}})$, is linearly independent. Then we apply the previous lemma and get a set of points $\{\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m\}$ such that

$$V(\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m; \tilde{\mathcal{G}}) \neq 0. \tag{31}$$

Now we notice that $z(\mathbf{x})$ with any fixed first $k-1$ coordinates is a solution of the system

$$\frac{\partial\{z_*\}}{\partial x_k} = \mathcal{A}_k\{z_*\}.$$

Therefore we have

$$\sum_{i=1}^{m} c_i(x_k) P_i(\tilde{\mathbf{x}}_j) \exp(\tilde{\boldsymbol{\lambda}}_i \cdot \tilde{\mathbf{x}}_j) = \sum_{i=1}^{m'} c'_{ij} P'_{ij}(x_k) \exp(\lambda_{ij} x_k), \qquad j = 1, \dots, m.$$

We consider this as a linear system with respect to unknowns $c_i(x_k)$, whose main determinant coincides with the one in (31) and thus is not zero. Therefore, the solving of this linear system and substituting the result into (30) will complete the proof.

Thus Theorem 8 is proved, since the part concerning the Hermite interpolation was established earlier.

Let us mention that Theorem 8 (MFTA2) is a complement to the following result of [4,5,7,8].

Theorem 10 (MFTA1). *The number of distinct solutions of the system* (2) *is the maximal possible, which is* $\#I$, *if and only if the matrices* $\mathcal{A}_i$, $1 \leq i \leq k$, *are commuting and each of them is semisimple. Moreover, in the latter case these solutions form a poised set of points for Lagrange interpolation with* Π_I^k. *And any poised set is the set of solutions of exactly one algebraic system of type* (2) *for which the associated matrices are commuting and semisimple.*

This is an important statement for the system (2) since it characterizes the case of distinct solutions of multivariate polynomial system by means of a condition of univariate nature, such is the semisimplicity of a matrix.

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