CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 19-32.

Classical Polynomial Inequalities in Several Variables

András Kroó *

Classical polynomial inequalities of Chebyshev, Markov, Bernstein, Remez dealing with extremal properties of univariate algebraic and trigonometric polynomials play a central role in Constructive Function Theory. In the last 15–20 years extensions of these inequalities to the multivariate case have been widely investigated. The transition to several variables requires a combination of analytic and geometric methods, since the multivariate results are intricately related to the geometry of underlying sets. In this paper we shall give a brief survey of these results.

1. Introduction

Classical polynomial inequalities play an important role in Constructive Function Theory. They describe extremal properties of algebraic and trigonometric polynomials under suitable normalization. Basic polynomial inequalities in univariate case were proved by Chebyshev, Markov, Bernstein and Remez in the end of 19-th beginning of 20-th century. In the last 15–20 years extensions of the classical polynomial inequalities to the multivariate case have been widely investigated. In this paper we shall give a survey of these results. The transition from one variable to several variables leads to a new phenomenon: the results are intricately connected to the geometry of underlying sets on which the corresponding extremal problems are considered. This interplay of analytic and geometric properties will be a constant theme in our considerations.

For the sake of completness we shall recall first the classical inequalities which will be the center of our attention. The classical polynomial inequalities are closely related to the extremal properties of the Chebyshev Polynomials given by

$$T_n(x) := \frac{1}{2}\{(x + \sqrt{x^2 - 1}\,)^n + (x - \sqrt{x^2 - 1}\,)^n\} = 2^{n-1}x^n + \dots$$

*Supported by the OTKA grant # T034531.

Let us denote by

$$P_n^1 := \left\{ \sum_{k=0}^{n} a_k\, x^k : a_k \in \mathbb{R} \right\}$$

the set of algebraic polynomials of degree $\le n$ and one real variable $x \in \mathbb{R}$. Furthermore,

$$\|p_n\|_K := \max_{x \in K} |p_n(x)|$$

is the usual supremum norm on the compact set $K \subset \mathbb{R}$.

Now we list the main inequalities that we are going to consider.

1. Chebyshev Inequalities. There are two classical inequlities due to Chebyshev concerning extremal properties of $p_n \in P_n^1$ with $\|p_n\|_{[-1,1]} = 1$. One of them estimates the size of polynomials *outside* $[-1,1]$, the second gives a bound on the *leading coefficient* of p_n. Here are the corresponding statements:

A) For any $p_n \in P_n^1$ and $x \in \mathbb{R} \setminus [-1,1]$,

$$|p_n(x)| \le |T_n(x)| \cdot \|p_n\|_{[-1,1]} \ . \tag{1.1}$$

B) For any $p_n(x) = a_n x^n + \cdots + a_0 \in P_n^1$,

$$|a_n| \le 2^{n-1} \|p_n\|_{[-1,1]} \ . \tag{1.2}$$

The next two inequalities due to Markov and Bernstein are dealing with the size of derivatives of polynomials.

2. Markov Inequality:

$$\|p_n'\|_{[-1,1]} \le n^2 \|p_n\|_{[-1,1]}, \qquad p_n \in P_n^1 \ . \tag{1.3}$$

3. Bernstein Inequality:

$$|p_n'(x)| \le \frac{n}{\sqrt{1-x^2}} \|p_n\|_{[-1,1]}, \qquad x \in (-1,1), \quad p_n \in P_n^1 \ . \tag{1.4}$$

Markov and Bernstein Inequalities complement each other in the sense that while estimate (1.3) is *uniform* on $[-1,1]$, inequality (1.4) yields a sharper bound for $x \in (-1,1)$ suitably separated from the endpoints ± 1.

Next we consider the Remez Inequality which gives a bound for the norm of polynomial on $[-1,1]$ provided that its size on a "large" subset on $[-1,1]$ is known. (Remez-type inequalities are instrumental in verifying further important polynomial inequalities, for instance Nikolski or Schur-type inequalities.) From now on we shall denote by $\mu_d(\dots)$ the Lebesgue measure in $\mathbb{R}^d$.

4. Remez Inequality: For any $p_n \in P_n^1$ and $E \subset [-1,1]$ with $\mu_1(E) \ge 2(1-\varepsilon)$ we have

$$\|p_n\|_{[-1,1]} \le T_n\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \|p_n\|_E \ . \tag{1.5}$$

It should be noted that the explicit formula for T_n given above yields that $T_n\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \sim e^{c\,n\sqrt{\varepsilon}}$. Hence (1.5) yields that

$$\|p_n\|_{[-1,1]} \leq e^{c\,n\sqrt{\varepsilon}}\,\|p_n\|_E \ .$$

whenever $\mu_1(E) \geq 2(1-\varepsilon)$ (c is an absolute constant).

In all of the above inequalities by a standard linear transformation the interval $[-1,1]$ can be replaced by $[a,b]$. In addition, it should be noted that inequalities (1.1)–(1.4) are *sharp* with the Chebyshev Polynomial $T_n(x)$ providing the extremal polynomial. (In case of Bernstein Inequality this is true for certain $x \in (-1,1)$, while for Remez Inequality one has to take $E = [-1, 1-2\varepsilon]$ and the translation of T_n into this interval.)

Now let us consider the set of *multivariate* algebraic polynomials of total degree $\leq n$ and d real variables given by

$$P_n^d := \left\{ \sum_{|\mathbf{k}|_1 \leq n} a_{\mathbf{k}}\, \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R},\ \mathbf{k} \in \mathbb{Z}_+^d \right\}.$$

Here for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ we use the standard notations

$$\mathbf{x}^{\mathbf{k}} := \prod_{j=1}^{d} x_j^{k_j}, \quad |\mathbf{x}|_p := \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p}, \qquad p > 0 \ .$$

Consider now a compact set K in $\mathbb{R}^d$, and as above let

$$\|p_n\|_K := \max_{\mathbf{x} \in K} |p_n(\mathbf{x})|.$$

As it was indicated in the introduction in multivariate case the results in general heavily depend on the *geometry* of underlying set K. First we shall consider the case when K is a *convex body* in $\mathbb{R}^d$, i.e., K is a closed bounded convex set with a nonempty interior in $\mathbb{R}^d$. In order to formulate the results we shall need some auxiliary notations.

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$ and consider a unit vector $\mathbf{v} \in S^{d-1}$. Then the *directional width* of the convex body K in direction $\mathbf{v}$ is defined as

$$w_{\mathbf{v}}(K) := \sup\left\{ |\mathbf{a} - \mathbf{b}|_2 : \mathbf{a}, \mathbf{b} \in K,\ \mathbf{a} - \mathbf{b} = \lambda \mathbf{v} \ \text{ for some } \ \lambda \in \mathbb{R} \right\} \ .$$

Clearly, $w_{\mathbf{v}}(K)$ is just the length of the longest line segment having direction $\mathbf{v}$ and imbedded into K. The global *width* of K can be defined by

$$w(K) := \inf\{ w_{\mathbf{v}}(K) : \mathbf{v} \in S^{d-1} \}.$$

It can be shown that $w(K)$ is also equal to the minimal distance between two parallel supporting hyperplanes of K.

If K possesses parallel supporting hyperplanes at two points $\mathbf{A}, \mathbf{B}$ of its boundary $\operatorname{Bd} K$, and $\mathbf{h} \in S^{d-1}$ is the common normal to these hyperplanes oriented so that $\langle \mathbf{A}, \mathbf{h} \rangle < \langle \mathbf{B}, \mathbf{h} \rangle$, then the set

$$L := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{A}, \mathbf{h} \rangle \le \langle \mathbf{x}, \mathbf{h} \rangle \le \langle \mathbf{B}, \mathbf{h} \rangle\}$$

is called a *supporting layer* of K. Clearly, K is the intersection of its supporting layers. Furthermore, given a supporting layer L its α-*dilation*, $\alpha > 0$, is defined as

$$L_\alpha := \left\{\mathbf{x} \in \mathbb{R}^d : \left\langle \mathbf{A} - (\alpha - 1)\frac{\mathbf{B} - \mathbf{A}}{2}, \mathbf{h} \right\rangle \le \langle \mathbf{x}, \mathbf{h} \rangle \le \left\langle \mathbf{B} + (\alpha - 1)\frac{\mathbf{B} - \mathbf{A}}{2}, \mathbf{h} \right\rangle \right\}.$$

Note that if the distance between the boundary hyperplanes of the supporting layer L is ϱ, then for L_α this distance becomes $\alpha\varrho$. Hence L_α is the enlargment ($\alpha > 1$) or reduction ($\alpha < 1$) of the layer L performed symmetrically around the "middle" hyperplane of L. Using this notation we introduce the α-*dilation of convex body* K by

$$K_\alpha := \bigcap \{L_\alpha : L \text{ is a supporting layer of } K\}, \qquad \alpha > 0 .$$

Clearly, $K_\alpha \subset K$ if $\alpha < 1$, and $K_\alpha \supset K$ if $\alpha > 1$ ($K_1 = K$).

After these preparations we can introduce the $\alpha(K, x)$ functional by

$$\alpha(K, \mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x} \in K_\alpha\}.$$

This functional measures the "distance" from any $\mathbf{x} \in \mathbb{R}^d$ to the boundary $\operatorname{Bd} K$ of K. Obviously, $\alpha(K, \mathbf{x}) > 1$ if $\mathbf{x} \notin K$, $\alpha(K, \mathbf{x}) < 1$ if $\mathbf{x} \in \operatorname{Int} K$ and $\alpha(K, \mathbf{x}) = 1$ if $\mathbf{x} \in \operatorname{Bd} K$. In the special case when K is *central-symmetric* about $\mathbf{0}$, i.e., $\mathbf{x} \in K$ if and only if $-\mathbf{x} \in K$, we have

$$K_\alpha = \alpha K = \{\alpha \mathbf{x} : \mathbf{x} \in K\}$$

and

$$\alpha(K, \mathbf{x}) = \varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}.$$

Here $\varphi_K(x)$ is the usual Minkowski functional centered at $\mathbf{0}$ of the convex body K. Thus $\alpha(K, \mathbf{x})$ can be considered as a generalization of this functional to the nonsymmetric case.

2. Chebyshev-Type Inequalities on Convex Bodies

Using the above definitions we can extend now Chebyshev Inequalities (1.1) and (1.2) for the convex bodies in $\mathbb{R}^d$.

Theorem 1 (Kroó-Schmidt, [10]). *For any* $\mathbf{x} \in \mathbb{R}^d \setminus K$ *we have*

$$\sup_{p \in P_n^d} \frac{|p_n(\mathbf{x})|}{\|p_n\|_K} = T_n(\alpha(K, \mathbf{x})).$$

(A partial result was given earlier by Rivlin-Shapiro [22].)

Now for any $p_n \in P_n^d$ denote by p_n^* its n-th order homogeneous part, that is

$$p_n(\mathbf{x}) = \sum_{|\mathbf{k}|_1 \le n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} = \sum_{|\mathbf{k}|_1 = n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} + \sum_{|\mathbf{k}|_1} p_n^*(\mathbf{x}) = \sum_{|\mathbf{k}|_1 = n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \ .$$

For a fixed $\mathbf{v} \in S^{d-1}$ consider the problem of estimating from above the quantity $|p_n^*(\mathbf{v})|$ under condition that $\|p_n\|_K \le 1$. Since $p_n^*(\mathbf{v})$ is a linear functional of the leading coefficients of p_n this question is related to the second Chebyshev Inequality (1.2).

Theorem 2 (Kroó, [12]). *For any* $\mathbf{v} \in S^{d-1}$

$$\sup_{p \in P_n^d} \frac{|p_n^*(\mathbf{v})|}{\|p_n\|_K} = \frac{2^{2n-1}}{w_{\mathbf{v}}^n(K)} \ .$$

This theorem also yields the next

Corollary 1 ([12]). *For any convex body* $K \subset \mathbb{R}^d$

$$\sup_{p \in P_n^d} \frac{\|p_n^*\|_{S^{d-1}}}{\|p_n\|_K} = \frac{2^{2n-1}}{w^n(K)} \ .$$

Theorems 1–2 provide analogues of Chebyshev Inequalities (1.1)–(1.2) for convex bodies in $\mathbb{R}^d$. (Recently in [25] an extension for $\mathbb{R}^\infty$ was also given.)

Let us briefly describe now the extremal polynomials in Theorems 1 and 2. The extremal polynomials can be described using the next two Lemmas.

Lemma 1 ([10]). *Given a convex body* $K \subset \mathbb{R}^d$ *and any* $\mathbf{x} \in \mathbb{R}^d \setminus K$ *there exists a line* ℓ *through* $\mathbf{x}$ *intersecting* $Bd\,K$ *at* $\mathbf{A}$ *and* $\mathbf{B}$, *so that* K *possesses parallel supporting hyperplanes at* $\mathbf{A}, \mathbf{B}$.

Lemma 2 ([11]). *Let* $\mathbf{v} \in S^{d-1}$ *and* $K \subset \mathbb{R}^d$ *be a convex body. Furthermore, let* $\mathbf{A}, \mathbf{B} \in \mathrm{Bd}\,K$ *be such that* $\mathbf{A} - \mathbf{B} = \lambda \mathbf{v}$ *with some* $\lambda \in \mathbb{R}$, *and* $|\mathbf{A} - \mathbf{B}|_2 = w_{\mathbf{v}}(K)$. *Then* K *possesses parallel supporting hyperplanes at* $\mathbf{A}, \mathbf{B}$.

Consider now the "ridge" polynomial $T_n\Big(\frac{2}{\langle \mathbf{h}, \mathbf{B} - \mathbf{A} \rangle} \Big\langle \mathbf{h}, \mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2} \Big\rangle\Big)$, where $\mathbf{h}$, is a normal to the parallel supporting hyperplanes of Lemma 1 or Lemma 2. Then this polynomial will be extremal for Theorem 1 or Theorem 2, respectively.

Finally, let us mention that while in the univariate case the extremal polynomial for Chebyshev Inequalities is *unique*, the situation is more complex in the multivariate setting, with both uniqueness and nonuniqueness occuring in different cases. (The question of uniqueness for $d = 2$ and $d > 2$ is considered in [12] and [23], respectively.)

3. Markov, Bernstein and Remez-Type Inequalities on Convex Bodies

Now we turn our attention to analogues of inequalities (1.3)–(1.4) in multivariate setting. We shall consider the directional derivatives

$$D_{\mathbf{v}}\, p_n := \langle \boldsymbol{\partial} p_n, \mathbf{v} \rangle, \qquad \mathbf{v} \in S^{d-1},$$

and denote by

$$D\, p_n := \max_{\mathbf{v}\in S^{d-1}} |D_{\mathbf{v}}\, p_n| = |\boldsymbol{\partial} p_n|_2$$

the Eucledian norm of the gradient $\boldsymbol{\partial} p_n$ of p_n. Then the n-th Markov Factor of the set $K \subset \mathbb{R}^d$ is given by

$$M_n(K) := \sup_{p_n \in P_n^d} \frac{\|D\, p_n\|_K}{\|p_n\|_K} \ .$$

(Note that Markov Inequality (1.3) can be rewritten now as $M_n([a,b]) = 2n^2/b-a$, since estimate (1.3) is attained by the Chebyshev polynomial translated to $[a,b]$.)

The first sharp multivariate estimate of $M_n(K)$ was given by Kellogg ([20]), who showed that $M_n(B_R) = n^2/R$, where B_R stands for the Eucledian ball of radius R. Later Wilhelmsen [30] gave the following estimates for any convex body $K \subset \mathbb{R}^d$:

$$\frac{2n^2}{w(K)} \le M_n(K) \le \frac{4n^2}{w(K)} \ .$$

(This was also proved for a triangle $\triangle \subset \mathbb{R}^2$ in [28].)

(Note that $w(B_R) = 2R$, i.e., Kellog's result can be written as $M_n(B_R) = 2n^2/w(B_R)$.) The question of sharp constants in the above inequality remained open for some time. Subsequently Sarantopoulos [26] and Baran [1] showed that if K is a **0**-*symmetric convex body* then

$$M_n(K) = \frac{2n^2}{w(K)} \ , \tag{3.1}$$

and for every $p_n \in P_n^d$ and $\mathbf{x} \in \operatorname{Int} K$

$$|D_{\mathbf{v}}\, p_n(\mathbf{x})| \le \frac{2n\|p_n\|_K}{w_{\mathbf{v}}(K)\sqrt{1-\varphi_K^2(\mathbf{x})}} \ , \tag{3.2}$$

where $\varphi_K(\mathbf{x})$ is the Minkowski functional of K centered at $\mathbf{0}$. The last inequality is a sharp equivalent of (1.4) for central symmetric convex bodies.

Surprisingly, the situation turned out to be quite different in nonsymmetric case. It was shown in [7] that $M_n(K) > \frac{2n^2}{w(K)}$, in general, when $K \subset \mathbb{R}^2$ is a triangle. Recently, Skalyga [27] settled the question of sharp Markov-type inequality for nonsymmetric convex bodies. He verified that for every convex body $K \subset \mathbb{R}^d$

$$M_n(K) \leq \frac{2n \cot \frac{\pi}{4n}}{w(K)} \ ,$$

and this inequality is sharp, in general, in the class of all convex bodies in $\mathbb{R}^d$. (Note that for large n $\cot \frac{\pi}{4n} \sim \frac{4n}{\pi}$, i.e. we obtain a constant $\sim\ 8n^2/\pi w(K)$.)

A sharp Bernstein-type inequality for non-symmetric convex bodies was given by Kroó, Révész [11]: whenever $\mathbf{x} \in \operatorname{Int} K$,

$$|D_{\mathbf{v}}\, p_n(\mathbf{x})| \leq \frac{2n \|p_n\|_K}{w_{\mathbf{v}}(K)\sqrt{1-\alpha(K,\mathbf{x})}} \ , \qquad p_n \in P_n^d \ . \tag{3.3}$$

Note that the same "distance" functional used for Chebyshev-type inequalities comes into play here. In general, (3.2) can not be deduced from (3.3), but nevertheless the constant 2 in (3.3) can not be replaced by $2-\varepsilon,\ \varepsilon > 0$. (This can be seen by taking K and $\mathbf{x}$ such that $\alpha(K,\mathbf{x})$ is small.)

Since by the Baran-Sarantopoulos theorem $M_n(K) = \frac{2n^2}{w(K)}$ whenever K is central-symmetric about $\mathbf{0}$ this leads to the natural question of extremal polynomials in this equality. It can be easily shown that there exist points $\mathbf{A} \in \operatorname{Bd} K$ for which $|\mathbf{A}|_2 = w(K)/2$ and that polynomial $T_n\big(\frac{2\langle \mathbf{x},\mathbf{A}\rangle}{w(K)}\big)$ is extremal for relation (3.1) with largest derivatives attained in radial directions. It is shown in [14] that any polynomial extremal for (3.1) must coincide along some line passing through $\mathbf{0}$ with $T_n\big(\frac{2\langle \mathbf{x},\mathbf{A}\rangle}{w(K)}\big)$. This "weak" uniqueness of extremal polynomials can be extended further only under some additional conditions ([14], [24]).

Now we present a Remez–type inequality first given in Brudnyi and Ganzburg [6]. For any convex body $K \subset \mathbb{R}^d$ and a subset $F \subset K$ satisfying $\mu_d(F) \geq (1-\varepsilon)\mu_d(K)$ we have

$$\|p_n\|_K \leq T_n\left(\frac{1+\varepsilon^{1/d}}{1-\varepsilon^{1/d}}\right)\|p_n\|_F, \qquad p_n \in P_n^d \ . \tag{3.4}$$

Moreover, equality in (3.4) is attained for some p_n and F if and only if K is a convex cone. (Estimate (3.4) easily extends to starlike domains, see [10].) As it was mentioned above $T_n\big(\frac{1+\varepsilon}{1-\varepsilon}\big) \sim e^{c\,n\sqrt{\varepsilon}}$, i.e., we obtain from (3.4) that

$$\|p_n\|_K \leq e^{c\,n\,\varepsilon^{1/2d}}\|p_n\|_F \ . \tag{3.5}$$

Thus the increase in dimension d clearly has an adverse effect on the rate of convergence $\|p_n\|_F \to \|p_n\|_K$ as $\mu_d(F) \to \mu_d(K)$, i.e., $\varepsilon \to 0$.

Inequalities (3.1)–(3.4) of this section give sharp Markov, Bernstein and Remez-type inequalities on convex bodies. Clearly they can be extended to the sets which are unions of convex bodies K with $w(K) \geq \delta > 0$. Sets of this type are called *noncuspidal.* Of course, exact constants can not be given anymore, but the asymptotic value with respect to n (and ε in case of (3.4)–(3.5)) remains the same for noncuspidal sets.

4. Effects of Smooth Boundary on Markov and Remez-Type Inequalities

In this section we shall adress the following natural question: what are the effects of smoothness of the boundary of the underlying set $K \subset \mathbb{R}^d$ on resulting Markov and Remez-type inequalities? First we look at the Markov problem on a *regular* convex body K, that is we shall assume that K possesses a unique supporting hyperplane at every point of its boundary. In addition, let us assume that $\mathbf{0} \in \operatorname{Int} K$, and consider the $\mathbf{0}$-centered Minkowski functional

$$\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}.$$

Then $\varphi_K(\mathbf{x}) \leq 1$ if and only if $\mathbf{x} \in K$, and the gradient $\boldsymbol{\partial}\varphi_K(\mathbf{x})$ of the Minkowski functional (which provides the outer normal to $\operatorname{Bd} K$ at $\mathbf{x}$) is continuous on $\operatorname{Bd} K$. Hence

$$\omega(\boldsymbol{\partial}\varphi_K, t) := \max_{\mathbf{x},\mathbf{y} \in \operatorname{Bd} K,\ |\mathbf{x}-\mathbf{y}|_2 \leq t} |\boldsymbol{\partial}\varphi_K(\mathbf{x}) - \boldsymbol{\partial}\varphi_K(\mathbf{y})|_2 \to 0$$

as $t \to 0^+$.

Clearly, smoothness of the boundary does not effect the rate of the usual Markov Factor $M_n(K)$, since even for the unit ball B_1 we have $M_n(B_1) = n^2$. The situation changes dramatically if only *tangential* derivatives on $\operatorname{Bd} K$ are considered. Namely, we introduce the so-called Tangential Markov Factor:

$$M_n^T(K) := \sup\big\{|D_{\mathbf{v}} p_n(\mathbf{x})| : \mathbf{x} \in \operatorname{Bd} K,\ \mathbf{v} \in S^{d-1} \text{ is tangent to } \operatorname{Bd} K \text{ at } \mathbf{x},\\ p_n \in P_n^d,\ \|p_n\|_K \leq 1\big\}\ .$$

It should be noted that the crucial difference in the definition of $M_n^T(K)$ compared to $M_n(K)$ consists in the fact that intstead of estimating $D_{\mathbf{v}}\, p_n(\mathbf{x})$ in *all* directions $\mathbf{v} \in S^{d-1}$ we only consider the derivatives in *tangential* directions. This notation leads to the following

Theorem 3 (Kroó, [16]). *If $K \subset \mathbb{R}^d$ is a regular convex body then*

$$M_n^T(K) = O\left(\frac{1}{w_K^{-1}(1/n^2)}\right), \tag{4.1}$$

where $\omega_K(t) = t\omega(\boldsymbol{\partial}\varphi_K, t)$, and ω_K^{-1} stands for its inverse. Moreover this estimate is asymptotically sharp, in general.

In particular the above estimate yields that $M_n^T(K) = o(n^2)$ for every regular convex body. Furthermore, if the boundary of K is C^q-smooth, 1 covered by an ℓ_q-ball inscribed into K), then $\omega_K(t) \sim t^q$, and we obtain from (4.1) that $M_n^T(K) = O(n^{2/q})$. In case of C^2-smoothness ($q = 2$) this yields $M_n^T(K) = O(n)$. All of the above exhibit a significant improvement compared to the rate $M_n(K) \sim n^2$ of the usual Markov Factors.

It is also interesting to consider the Markov problem for *homogeneous polynomials* of degree n

$$H_n^d := \left\{ \sum_{|\mathbf{k}|_1=n} a_{\mathbf{k}}\, \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R} \right\}.$$

Clearly, $h(t\mathbf{x}) = t^n h(\mathbf{x})$ for any $h \in H_n^d$, $\mathbf{x} \in \mathbb{R}^d$, $t \in \mathbb{R}$. Therefore *radial* derivatives of $h \in H_n^d$ have magnitude n. Since in case of ordinary polynomials of total degree $\leq n$ the largest derivatives of order n^2 are attained precisely in radial directions it is natural to expect that better bounds should hold for Markov Factors of homogeneous polynomials defined by

$$M_n^*(K) := \sup_{h \in H_n^d} \frac{\|Dh\|_K}{\|h\|_K} \ .$$

It was proved by Harris [9] that for any convex body $K \subset \mathbb{R}^d$

$$M_n^*(K) = O(n \log n) \ . \tag{4.2}$$

In a recent paper [17] we show that for *regular* convex bodies

$$M_n^*(K) \leq c\, n \int_{1/n}^{1} \frac{\omega(\boldsymbol{\partial}\varphi_K, t)}{t}\, dt \ . \tag{4.3}$$

This clearly yields that $M_n^*(K) = o\,(n \log n)$ for every regular convex body. Moreover, under the mild restriction

$$\int_0^1 \frac{\omega(\boldsymbol{\partial}\varphi_K, t)}{t}\, dt < \infty$$

we obtain $M_n^*(K) = O(n)$.

Let us also mention two converse results from [17]. One of them complements (4.2) by stating that $M_n^*(K) \geq c\,n\, \log n$ whenever K is a *nonregular* $\mathbf{0}$-symmetric convex body. Furthermore, for arbitrary $\varepsilon_n \downarrow 0^+$ one can construct a regular convex body $K \subset \mathbb{R}^d$ ($\mathbf{0} \in \operatorname{Int} K$) so that $M_n^*(K) \geq \varepsilon_n\, n \log n$ for every $n \in \mathbb{N}$. This shows that estimate $M_n^*(K) = o\,(n \log n)$ is sharp in the class of all regular convex bodies. The question of reversing the more delicate inequality (4.3) remains open.

Now let us consider the effects of the smooth boundaries on the Remez-type inequalities (3.4)–(3.5). It is proved in [13] that in case when $K \subset \mathbb{R}^d$ has a C^q-boundary (i.e., points of K can be covered by inscribed ℓ_q-balls) $1 \leq q \leq 2$ then (3.5) can be replaced by the estimate

$$\|p_n\|_K \leq e^{c\, n\, \varepsilon^{q/(2d+2q-2)}} \|p_n\|_F \ . \tag{4.4}$$

Since $\frac{q}{2d+2q-2} > \frac{1}{2d}$ whenever $q > 1$ the estimate (4.4) is sharper than (3.5) for $q > 1$ (smooth boundaries). The best rate in (4.4) is attained when $q = 2$ with $\varepsilon^{\frac{1}{d+1}}$ replacing $\varepsilon^{\frac{1}{2d}}$. The estimate (4.4) is also shown to be sharp, in general.

5. Markov-Type Inequalities for Cuspidal Sets

As it was mentioned above the rate of the Markov Factors remains n^2 for those sets which have the property that their points can be covered by inscribed convex bodies (of width separated from 0). The situation changes substantially in presense of *cusps*, that is, points which can not be covered by convex bodies inscribed into underlying set. For cuspidal domains the rate of the Markov Factors can increase substantially. The first example of this nature was provided by Goetgheluck [31] who showed that for

$$D_p := \{(x, y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \ 0 \le y \le x^p\}, \qquad p > 1,$$

we have $M_n(D_p) \sim n^{2p}$. Clearly, $\mathbf{0}$ is a cuspidal point for D_p an this results in a dramatic increase of the rate of Markov Factors.

In order to give a systematic study of Markov Factors for cuspidal domains we need to introduce some structural properties of cuspidal points. We shall assume that any point $\mathbf{x}$ of the underlying set K can be connected to $\operatorname{Int} K$ by a curve $\boldsymbol{\gamma}(t), \ 0 \le t \le 1$, so that $B(\boldsymbol{\gamma}(t), \varphi(t)) \subset K, \ 0 \le t \le 1, \ \boldsymbol{\gamma}(0) = \mathbf{x}$. Here $\varphi(t)$ is a fixed "width" function which measures cross-sections of cusps, $B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{a}|_2 \le r\}$ is the ball of radius r centered at $\mathbf{a}$. Typically, $\varphi(t) = o(t)$ for cuspidal domains. The connecting curves $\boldsymbol{\gamma}$ can be taken to be polynomial, analytic, etc. Correspondingly, we say that K is polynomially or analytically connected. Thus the structure of cuspidal domains will depend on the analytic properties of connecting curves and the size of the width function $\varphi(t)$. The above concept was introduced by Pawlucki and Plesniak [21] with polynomial curves. They also gave first estimates of Markov Factors for polynomially connected cuspidal domains.

Let us list now some asymptotically sharp bounds for *polynomially connected* domains given in Baran [2], and Kroó, Szabados [18]:

1) if the width function $\varphi(t)$ has polynomial growth (i.e. $t^{-\beta}\varphi(t)$ is a decreasing function for some $\beta > 1$), then

$$M_n(K) = O(1/\varphi(n^{-2})) \ ;$$

2) in case of an arbitrary width function

$$M_n(K) = O(e^{n\,\delta_n}) \ ,$$

where δ_n is the solution of $n\sqrt{\delta} = \log \frac{1}{\varphi(\delta)}$.

Note that δ_n defined above always tends to 0 as $n \to \infty$, i.e., we obtain from the second statement that

$$\limsup_{n\to\infty} \ M_n^{1/n}(K) = 1 \tag{5.1}$$

whenever K is *polynomially connected*. The growth condition (5.1) is usually called subexponential.

This leads to the natural question of what conditions guarantee the subexponential growth (5.1) of the Markov Factors? It was shown independently by Kroó [15] and Totik [29] that subexponential increase (5.1) holds whenever K is *analytically connected*. Moreover, Totik [29] also gives an example of a C^∞-connected domain for which Markov Factors grow exponentially.

Thus subexponential growth (5.1) holds for analytically connected domains, and fails, in general, for C^∞-connected sets. On the other hand, in general, analytic connectedness is not necessary for (5.1) to hold: there exist simple transcendental domains without analytic connectedness whose Markov Factors are subexponential. An example of this phenomenon is given in Erdélyi, Kroó [8] where it is shown that for the transcendental domain

$$K_\alpha := \{(x,y) \in \mathbb{R}^2 :\ 0 \le x \le 1,\ x^\alpha \le y \le 2x^\alpha\},$$

$\alpha > 1$ *irrational*, the relation

$$M_n(K_\alpha) \le n^{c \log n}$$

holds.

The exact rate of Markov Factors of "banana" shaped domains K_α is not known even for $\alpha = \frac{p}{q}$ rational. The existing estimates (see [18]) yield

$$c_2 n^{2p/q} \le M_n(K_\alpha) \le c_1 n^{2p}, \qquad \alpha = \frac{p}{q} > 1\ .$$

6. Tangential Markov-Bernstein Inequalities on Curves

In this section we shall consider the following *tangential* Markov and Bernstein Factors on curves $\Gamma \subset \mathbb{R}^2$,

$$M_n^T(\Gamma) := \sup_{p_n \in P_n^2} \frac{\|D_T\, p_n(\mathbf{x})\|_\Gamma}{\|p_n\|_\Gamma},$$

$$B_n^T(\Gamma, \mathbf{x}) := \sup_{p_n \in P_n^2} \frac{|D_T\, p_n(\mathbf{x})|}{\|p_n\|_\Gamma}, \qquad \mathbf{x} \in \Gamma\ ,$$

where $D_T\, p_n(\mathbf{x})$ denotes the *tangential derivative* of p_n at $\mathbf{x} \in \Gamma$, i.e., the derivative in (unit) tangential direction to Γ at $\mathbf{x}$. It turns out that the magnitude of the quantities $M_n^T(\Gamma)$ and $B_n^T(\Gamma, x)$ can vary substantially depending on the analytic properties of the underlying curve Γ. It is known (see [3]) that the optimal rates $M_n^T(\Gamma) \sim n^2$ and $B_n^T(\Gamma, x) \sim n$ can be attained essentially only for *algebraic curves*. In order to illustrate the dependence between structural properties of Γ and rates of $M_n^T(\Gamma)$ and $B_n^T(\Gamma, x)$ we shall consider some "model" nonalgebraic curves, namely

$$\Gamma = \{(x, e^x)\} \ \text{ or } \ \Gamma = \{(x, x^\alpha)\}, \qquad x \in [a,b]$$

where $\alpha > 1$ is not an integer.

First we consider the *analytic* case (that is, we assume that $a > 0$ if $\Gamma = \{(x, x^\alpha)\}$). Denote by $d_n(\Gamma)$ the dimension of P_n^2 on Γ. Clearly, $d_n(\Gamma) \sim n^2$ if $\Gamma = \{(x, e^x)\}$ or $\{(x, x^\alpha)\}$ with irrational α, and $d_n(\Gamma) \sim n$ if $\Gamma = \{(x, x^\alpha)\}$ and α is rational.

Theorem 4 (Kroó and Szabados [19]). *Let* $\Gamma = \{(x, e^x),\ x \in [a, b]\}$ *or* $\Gamma = \{(x, x^\alpha),\ x \in [a, b],\ \alpha > 1,\ a > 0\}$. *Then*

$$B_n^T(\Gamma, x) \sim \frac{d_n^2(\Gamma)}{1 + d_n(\Gamma)\sqrt{(x-a)(b-x)}}, \qquad x \in [a, b]\ .$$

The above general result yields that

$$B_n^T(\Gamma, x) \sim \frac{d_n(\Gamma)}{\sqrt{(x-a)(b-x)}}, \quad a + d_n^{-2}(\Gamma) \le x \le b - d_n^{-2}(\Gamma) \tag{6.1}$$

and

$$M_n^T(\Gamma) \sim d_n^2(\Gamma)\ , \tag{6.2}$$

for Γ as in the above theorem. In particular, $M_n^T(\Gamma) \sim n^4$ for $\Gamma = \{(x, e^x)\}$. This latter result also appears in Bos, Brudnyi, Levenberg, Totik [5]. It should be noted that relations (6.1)–(6.2) are analogous to (1.3)–(1.4) with the dimension $d_n(\Gamma)$ replacing n!

Now let us consider the nonanalytic case, that is, $\Gamma_\alpha := \{(x, x^\alpha),\ 0 \le x \le 1\}$ and $\alpha > 1$ not an integer. In this case the rates of Markov and Bernstein Factors will differ depending on α being rational or irrational. If $\alpha = \frac{p}{q}$ is rational, then we have $M_n^T(\Gamma_\alpha) \sim n^{2q}$ (see Bos, Levenberg, Milman, Taylor [4]). The rate of the Bernstein Factor is given in Kroó, Szabados [19]:

$$B_n^T(\Gamma_\alpha, x) \sim \frac{n}{x^{1-1/2q}\sqrt{1-x}}\ ,\ n^{-2q} \le x \le 1 - n^{-2}.$$

In case when $\alpha > 1$ is irrational the Markov Factors have exponential rate! That is, for some $A, B > 1$ we have

$$B^n \le M_n^T(\Gamma_\alpha) \le A^n\ .$$

This was verified independently in Bos, Brudnyi, Levenberg, Totik [5], and Erdélyi, Kroó [8].

In addition, Kroó, Sabados [19] found the rate of Bernstein Factors for Γ_α, α irrational:

$$B_n^T(\Gamma_\alpha, x) \sim \frac{n^{3/2}}{x\ \sqrt{\log 1/x}}\ , \qquad e^{-\frac{c\,n}{\log n}} \le x \le 1 - n^{-3}.$$

Thus one can observe that for Γ_α, $\alpha > 1$ irrational the regular pattern ((1.3)–(1.4) or (6.1)–(6.2)) of Markov–Bernstein Factors is not preserved anymore. Based on this observation one can predict that solving the problem for general curves must be rather hard. It is not even clear what properties of Γ yield polynomial or subexponential growth of Markov Factors $M_n^T(\Gamma)$. An example given in [3] shows that for $\Gamma = \{(x, f(x)\}$ with $f(x) = \sum\limits_{k=0}^{\infty} c_k\, x^{n_k}$ being a rather lacunary gap series the Markov Factors can grow arbitrarily fast. This means that analyticity by itself does not guarantee some regular behaviour of Markov Factors.

References

[1] M. Baran, Bernstein-type theorems for compact sets in $\mathbb{R}^n$ revisited, *J. Approx. Theory* **79** (1994), 190–198.

[2] M. Baran, Markov inequality on sets with polynomial parametrization, *Ann. Polon. Math.* **60** (1994), 69–79.

[3] L. Bos, N. Levenberg, P. Milman, and B. A. Taylor, Tangential Markov inequalities which characterize algebraic submanifolds of $\mathbb{R}^N$, *Indiana Univ. Math. J.* **44** (1995), 115–138.

[4] L. Bos, N. Levenberg, P. Milman, and B. A. Taylor, Tangential Markov inequalities on real algebraic varieties, *Indiana Univ. Math. J.* **47** (1998), 1257–1272.

[5] L. Bos, A. Brudnyi, N. Levenberg, and V. Totik, Tangential Markov inequalities on transcendental curves, submitted.

[6] Ju. Brudnyi and M. I. Ganzburg, On one exact inequality for polynomials of many variables, *in* "Proceedings of 7th Winter Meeting on Function Theory an Functional Analysis", pp. 118–123, Drogobich, 1974. [In Russian]

[7] L. Bialas–Cież and P. Goetgheluck, Constants in Markov inequality on convex sets, *East J. Approx.* **1** (1995), 379–389.

[8] T. Erdélyi and A. Kroó, Markov-type inequalities on certain irrational arcs and domains, submitted.

[9] L. A. Harris, A Bernstein-Markov theorem for normed spaces, *J. Math. Anal. Appl.* **208** (1997), 476–486.

[10] A. Kroó and D. Schmidt, Some extremal problems for multivariate polynomials on convex bodies, *J. Approx. Theory* **90** (1997), 415–434.

[11] A. Kroó and Sz. Révész, On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies, *J. Approx. Theory* **99** (1999), 134–152.

[12] A. Kroó, Chebyshev-type extremal problems for multivariate polynomials on convex bodies, *East J. Approx.* **5** (1999), 211–221.

[13] A. Kroó, On Remez–type inequalities for polynomials in $\mathbb{R}^m$ and $\mathbb{C}^m$, *Anal. Math.* **27** (2001), 50–70.

[14] A. Kroó, On multivariate polynomials with largest gradients on convex bodies, *J. Math. Anal. Appl.* **253** (2001), 322–333.

[15] A. Kroó, Extremal properties of multivariate polynomials on sets with analytic parametrization, *East J. Approx.* **7** (2001), 27–40.

[16] A. Kroó, Markov-type inequalities for surface gradients of multivariate polynomials, *J. Approx. Theory*, to appear.

[17] A. Kroó, Markov-type inequalities for homogeneous polynomials on smooth convex bodies, manuscript.

[18] A. Kroó and J. Szabados, Markov-Bernstein type inequalities for multivariate polynomials on sets with cusps, *J. Approx. Theory* **102** (2000), 72–95.

[19] A. Kroó and J. Szabados, Tangential Markov-Bernstein inequalities for bivariate polynomials on curves, *East J. Approx.* **8** (2002), 261–278.

[20] O. D. Kellogg, On bounded polynomials in several variables, *Math. Z.* **27** (1928), 55–64.

[21] W. Pawlucki and W. Plesniak, Markov's inequality and C^∞ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.

[22] T. J. Rivlin and H. S. Shapiro, A unified approach to certain problems of approximation and minimization, *J. Soc. Ind. Appl. Math.* **9** (1961), 670–699.

[23] Sz. Révész, Uniqueness of multivariate Chebyshev-type extremal polynomials for convex bodies, *East J. Approx.* **7** (2001), 205–240.

[24] Sz. Révész, Uniqueness of Markov extremal polynomials on symmetric convex bodies, *Constr. Approx.* **17** (2001), 465–478.

[25] Sz. Révész and Y. Sarantopoulos, Chebyshev's extremal problems of polynomial growth in real normed spaces, manuscript.

[26] Y. Sarantopoulos, Bounds on the derivatives of polynomials on Banach spaces, *Math. Proc. Cambridge Philos. Soc.* **110** (1991), 307–312.

[27] V. I. Skalyga, Estimates of the derivatives of polynomials on convex bodies, *Proc. Steklov Inst. Math.* **218** (1997), 372–383.

[28] Y. N. Subbotin and D. Nadzhmiddinov, Markov inequalities for polynomials on triangles, *Math. Notes* (1990), 627–631.

[29] V. Totik, On Markoff inequality, *Constr. Approx.*, to appear.

[30] D. R. Wilhelmsen, A Markov inequality in several dimensions, *J. Approx. Theory* **11** (1974), 216–220.

[31] P. Goetheluck, Inegalité de Markov dans ensembles effilés, *J. Approx. Theory* **30** (1980), 149–154.

András Kroó
Alfréd Rényi Mathematical Institute
Hungarian Academy of Sciences
Budapest, Reáltanoda u. 13-15
H-1053, HUNGARY
E-mail: kroo@renyi.hu