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Entire Functions of Exponential Type in Approximation Theory

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The classical theorem of K. Weierstrass about polynomial approximation says that any function g continuous on a compact interval I, and so bounded and uniformly continuous on I, can be approximated arbitrarily closely on I by polyomials. However, for any given non-constant function g that is continuous and bounded on the real axis, and a positive number ε, we cannot find a polynomial p such that $|g(x)-p(x)| < \varepsilon$ for all real x. We will see that *entire functions of exponential type bounded on the real axis* constitute the *smallest class* of entire functions which, for any g bounded and uniformly continuous on $\mathbb{R}$ and any $\varepsilon > 0$, contains a function f such that $\max\{|g(x) - f(x)| : -\infty < x < \infty\} < \varepsilon$. Problems of best approximation by polynomials whose degree does not exceed a fixed given integer n were concidered by P. L. Chebyshev much before Weierstrass proved his theorem. We shall discuss analogous questions for best approximation by entire functions of exponential type. In particular, we shall prove the existence of an entire function f_* of exponential type τ minimizing the quantity $\sup\{|g(x) - f(x)| : -\infty < x < \infty\}$ as f varies in the class of all entire functions of exponential type τ.

There is no loss of generality in supposing that the interval I in the theorem of Weierstrass is the unit interval $[-1\,,\,1]$. A theorem of L. Féjer says that if H_{2n-1} is the polynomial of degree $2n-1$, which interpolates g in the points $\cos\dfrac{(2\nu-1)\pi}{2n}$, $\nu = 1, \ldots, n$, and has a vanishing derivative at these points, then $H_{2n-1}(x) \to g(x)$ uniformly on $[-1\,,\,1]$. We shall present an analogous result about approximation, via Hermite interpolation, to a uniformly continuous function g on the whole real axis.

The famous theorem of Faber excludes Lagrange interpolation as a viable process for uniform approximation by polynomials. The situation is just as hopeless for uniform approximation on the whole real line via Lagrange interpolation by entire functions of exponential type. Like P. Turán and others who considered $(0\,,\,m)$- interpolation by polynomials and trigonometric polynomials, we consider $(0\,,\,m)$- interpolation by entire functions of exponential type, and look at their convergence properties of such interpolants.

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For any function g continuous on $[-1\,,\,1]$, let $L_{n-1}(.\,;g)$ be the unique polynomial of degree not exceeding $n-1$, which duplicates the function g in the zeros of T_n, the points $\cos\dfrac{(2\nu-1)\pi}{2n}$, $\nu = 1, \ldots, n$, then (see [31] ; [16])

$$\lim_{n\to\infty}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,|g(x)-L_{n-1}(x\,;g)|^p\,\mathrm{d}x \to 0 \qquad (p>0)\,.$$

Various extensions of this result have been obtained by P. Erdős and P. Turán, R. Askey, P. Nevai, Y. Xu, and others. Here, we discuss analogous results about the mean convergence of Lagrange interpolating entire functions of exponential type for appropriately chosen systems of interpolation points. Certain Gaussian quadrature formulae are known to play a crucial role in the study of mean convergence of sequences of interpolating polynomials. A similar role is played by certain Gaussian quadrature formulae for entire functions of exponential type. So, we shall discuss such formulae too.

1. Introduction

Any function g, defined and continuous on a compact interval can be approximated arbitrarily closely by polynomials. This is what the following famous theorem of Weierstrass [43] says.

Theorem A (Weierstrass' First Approximation Theorem). *Let $g \in C[a\,,\,b]$. Then, for any $\varepsilon > 0$, there exists a polynomial p such that*

$$\max_{a\le x\le b}|g(x)-p(x)| < \varepsilon\,.$$

□

We remark that the preceding theorem holds for functions in $C[a\,,\,b]$ if and only if it holds for those in $C[0\,,\,1]$. It is not obvious but the above theorem of Weierstrass is equivalent to the following result, which also bears his name.

Theorem B (Weierstrass' Second Approximation Theorem). *Let g be a continuous 2π-periodic function. Then, for any $\varepsilon > 0$, there exists a trigonometric polynomial t such that*

$$\max_{-\pi\le\theta\le\pi}|g(\theta)-t(\theta)| < \varepsilon\,.$$

□

For any non-constant g, continuous and bounded on the real axis, there exists a positive number ε_g such that $\sup\{|g(x)-p(x)| \,:\, -\infty < x < \infty\} > \varepsilon_g$, whatever the polynomial p may be, simply because a non-constant polynomial does not remain bounded on any line.

Since polynomials are entire functions of the simplest kind, the question arises: given any non-constant function g, continuous and bounded on the whole real axis, and any $\varepsilon > 0$, can we find a transcendental entire function

f such that $|g(x) - f(x)| < \varepsilon$ for all $x \in (-\infty, \infty)$? This was answered in the affirmative by Carleman [12]. We refer the reader to the Appendix for details. However, Carleman's result does not say anything about the "growth" of the approximating entire function f. In order to clarify this last statement we need to recall certain basic notions from the theory of entire functions.

1.1. The Notions of Order and Type

For a function f, holomorphic in $|z| < R$ (or entire), we denote by $M(r)$ the maximum of $|f(z)|$ for $|z| = r < R$ (or $< \infty$). We write $M_f(r)$, etc., when it is necessary to call attention to the particular function that is being considered. Except in the case where f is a constant, $M_f(r)$ is a strictly increasing function of r for $r < R$, tending to ∞ with r if f is entire. If f is a polynomial of degree n then $M_f(r) = O(r^n)$ as $r \to \infty$. Conversely, if $M_f(r) = O(r^\lambda)$ as $r \to \infty$, then f is a polynomial of degree at most $\lfloor \lambda \rfloor$. By the three-circles theorem of Hadamard, $\log M(r)$ is a convex function of $\log r$. For a transcendental entire function f, the quotient $\dfrac{\log M(r)}{\log r}$ is not only unbounded but also strictly increasing for all large values of r.

The quantity

$$\rho := \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \qquad (0 \le \rho \le \infty),$$

associated with a non-constant entire function f is said to be its order. A constant has order 0, by convention. The order of an entire function f defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ is given by the formula

$$\rho = \limsup_{n \to \infty} \frac{n \log n}{\log (1/|a_n|)},$$

where the quotient on the right is taken as 0 if $a_n = 0$.

An entire function f, of *finite positive* order ρ, is said to be of type T if

$$\limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} = T \qquad (0 \le T \le \infty).$$

The type of an entire function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ of order ρ $(0 < \rho < \infty)$ is given in terms of its coefficients by the formula

$$T = \limsup_{r \to \infty} \frac{1}{\rho \, \mathrm{e}} n \, |a_n|^{\rho/n}.$$

1.2. Functions of Exponential Type

A function is said to be "of growth (ρ, τ)", $\rho > 0$ if it is of order not exceeding ρ and of type not exceeding τ if of order ρ. We shall find this notion quite convenient because of the fact that it sets up an hierarchy in the class of all entire functions. In fact, if $\mathfrak{C}_{\rho,\tau}$ denotes the class of all functions of growth $(\rho\,,\,\tau)$, then

$$\mathfrak{C}_{\rho_1,\tau_1} \subset \mathfrak{C}_{\rho_2,\tau_2}$$

if $\rho_1 < \rho_2$, whatever τ_1 and τ_2 may be, and also when $\rho_1 = \rho_2$ provided that $\tau_1 < \tau_2$. The type is not defined for functions of order 0, but according to the above definition, they belong to $\mathfrak{C}_{\rho,\tau}$ for any $\rho > 0$ and any $\tau \geq 0$.

A function of growth $(1\,,\,\tau)$, $\tau < \infty$, is called a fuction of *exponential type* τ. Thus, f is an entire function of exponential type τ if, for any $\varepsilon > 0$, there exists a constant K_ε such that

$$|f(z)| \leq K_\varepsilon\, \mathrm{e}^{(\tau+\varepsilon)|z|} \qquad (z \in \mathbb{C})\,.$$

In particular, any entire function of order less than 1 is of exponential type τ, and so is any entire function of order 1 type $T \leq \tau$. Using Stirling's formula, we can check that *the power series* $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}\, a_k\, z^k$ *defines an entire function of exponential type* τ *if*

$$\limsup_{k\to\infty}\, |a_k|^{1/k} \leq \tau\,.$$

The indicator function of an entire function f of exponential type is defined as

$$h(\theta) = h_f(\theta) := \limsup_{r\to\infty} \frac{\log\left|f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right|}{r} \qquad (\theta \in \mathbb{R})\,.$$

The following result contains a fundamental property of entire functions of exponential type.

Proposition 1.1. *Let* f *be an entire function of exponential type such that* $h_f\,(\pm\pi/2) \leq \tau$. *Suppose, in addition, that* $|f(x)| \leq M$ *for all real* x. *Then,*

$$|f(z)| \leq M\, \mathrm{e}^{\tau|\Im z|} \qquad (z \in \mathbb{C})\;. \tag{1.1}$$

In particular, if f *belongs to* $\mathfrak{C}_{1,0}$, *then* f *must be a constant.*

Proof. For any $\varepsilon > 0$, let $F_\varepsilon(z) := f(z)\,\mathrm{e}^{\mathrm{i}\,(\tau+\varepsilon)z}$. Then, $|F_\varepsilon(x)| \leq M$ on the real axis. Note that $M' := \sup_{y\geq 0} |F_\varepsilon(\mathrm{i}\,y)|$ is finite and that $M' = |F_\varepsilon(\mathrm{i}\,y')|$ for some $y' \in [0\,,\,\infty)$. Thus, F_ε satisfies the conditions of Proposition 7.1 with $\alpha := 2$, any $\beta \in (1\,,\,2)$ and $\theta_0 := \pi/4$. Hence, $|F_\varepsilon(z)| \leq \max\{M\,,\,M'\}$ in the first quadrant. The same estimate for $|F_\varepsilon(z)|$ holds in the second quadrant since F_ε satisfies the conditions of Proposition 7.1 with $\alpha := 2$, any $\beta \in (1\,,\,2)$ and $\theta_0 := 3\pi/4$ also. It follows that $|F_\varepsilon(z)| \leq \max\{M\,,\,M'\}$ in the closed upper half-plane. Now, note that M' cannot be larger than M since, otherwise $|F_\varepsilon(z)|$ would have a local maximum at $\mathrm{i}\,y'$, without F_ε being a constant. We conclude that $|F_\varepsilon(z)| \leq M$ in the upper half-plane. In other words, $|f(z)| \leq M\,\mathrm{e}^{(\tau+\varepsilon)\Im z}$ if $\Im z \geq 0$. Letting $\varepsilon \to 0$, we see that $|f(z)| \leq M\,\mathrm{e}^{\tau|\Im z|}$

in the upper halp-plane. The same estimate holds in the lower half-plane since the function $\overline{f(\overline{z})}$ also satisfies the conditions of Proposition 1.1.

If f belongs to $\mathfrak{C}_{1,0}$, then $h_f(\pi/2) \le 0$ and (1.1) shows that $|f(z)| \le M$ in the upper half-plane. However, the function $\overline{f(\overline{z})}$ also belongs to $\mathfrak{C}_{1,0}$, which implies that $|f(z)| \le M$ in the lower half-plane too. Thus, $|f(z)| \le M$ in the entire complex plane and so must be a constant, by Liouville's theorem. □

If p is a polynomial of degree at most n such that $|p(z)| \le M$ on the unit circle, then $f(z) := p\left(\mathrm{e}^{-\mathrm{i}z}\right)$ satisfies the conditions of Proposition 1.1 with $\tau := n$ and is thus seen to contain the following familiar result.

Corollary 1.1. *Let p be a polynomial of degree at most n such that $|p(z)| \le M$ on the unit circle. Then,*

$$|p(z)| \le MR^n \qquad (|z| \le R\,,\ R > 1)\,.$$

□

As we shall see next, Proposition 1.1 also plays a crucial role in the proof of an important interpolation formula for the derivative of an entire function of exponential type.

Proposition 1.2. *Let f be an entire function of exponential type τ bounded on the real axis. Then,*

$$f'(x) = \frac{4\tau}{\pi^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)^2}\, f\left(x + \frac{(2k+1)\pi}{2\tau}\right) \qquad (x \in \mathbb{R})\,. \tag{1.2}$$

Proof. We shall first suppose that f is of exponential type 1 and that $|f(x)| \le 1$ on the real axis. Then, except for simple poles of the form

$$\zeta_k := \frac{(2k+1)\,\pi}{2} \qquad (k = 0, \pm 1, \pm 2, \dots)$$

and a double pole at 0, the function $F(z) := f(z)/\bigl(z^2 \cos z\bigr)$ is holomorphic in the complex plane. The residue of F at ζ_k is $(-1)^{k+1} f(\zeta_k)/\zeta_k^2$ and the residue at 0 is $f'(0)$. Hence, if Γ_n denotes the square contour whose corners lie at the points $(1+\mathrm{i})\pi n$, $(1-\mathrm{i})\pi n$, $(-1+\mathrm{i})\pi n$ and $(-1-\mathrm{i})\pi n$, then for $n = 1, 2, \dots,$

$$\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_n} F(z)\,\mathrm{d}z = f'(0) - \frac{4}{\pi^2} \sum_{k=-n}^{n-1} (-1)^k \frac{1}{(2k+1)^2}\, f\left(\frac{(2k+1)\pi}{2}\right).$$

Clearly,

$$|\cos(\pm n\pi + \mathrm{i}y)| = \frac{\mathrm{e}^y + \mathrm{e}^{-y}}{2} > \frac{1}{2}\,\mathrm{e}^{|y|} \qquad (-\infty < y < \infty)\,.$$

Besides, for any real x,

$$|\cos(x \pm \mathrm{i}n\pi)| \ge \frac{\mathrm{e}^{\pi n} - \mathrm{e}^{-\pi n}}{2} = \frac{1}{2}\,\mathrm{e}^{\pi n}\left(1 - \mathrm{e}^{-2\pi n}\right) > \frac{1}{3}\,\mathrm{e}^{\pi n} \qquad (n = 1, 2, \dots)\,.$$

Thus, $|\cos z| > (1/3)\,\mathrm{e}^{|\Im z|}$ for any $z \in \Gamma_n$. On the other hand, since $|f(x)| \le 1$ on the real axis, Proposition 1.1 implies that

$$|f(z)| \le \mathrm{e}^{|\Im z|} \qquad (z \in \mathbb{C})\,,$$

and so $|F(z)| < 3/|z|^2$ for any $z \in \Gamma_n$. Hence, $\int_{\Gamma_n} F(z)\,\mathrm{d}z \to 0$ as $n \to \infty$. Thus, if f is an entire function of exponential type 1 such that $|f(x)| \le 1$ on the real axis, then

$$f'(0) = \frac{4}{\pi^2}\sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{(2k+1)^2}\,f\left(\frac{(2k+1)\pi}{2}\right).$$

If t is a given real number, then the preceding formula may be aplied to $f(\cdot + t)$ to conclude that for any entire function f of exponential type 1 such that $|f(x)| \le 1$ on the real axis, we have

$$f'(t) = \frac{4}{\pi^2}\sum_{k=-\infty}^{\infty}(-1)^k\frac{1}{(2k+1)^2}\,f\left(t+\frac{(2k+1)\pi}{2}\right) \qquad (t \in \mathbb{R})\,.$$

If f is an entire function of exponential type $\tau > 0$ such that $|f(x)| \le M$ on the real axis, then $z \mapsto \dfrac{1}{M}\,f\left(\dfrac{z}{\tau}\right)$ is an entire function of exponential type to which the preceding formula applies. After some simplification, we obtain (1.2). $\square$

Remark 1.1. Applying Proposition 1.2, to the function $f(z) := \sin \tau z$ and taking $x = 0$ in (1.2), we obtain the well-known formula

$$\frac{4}{\pi^2}\sum_{\nu=-\infty}^{\infty}\frac{1}{(2k+1)^2} = 1\,.$$

Proposition 1.2 in conjunction with the preceding remark readily implies the following result known as Bernstein's inequality for entire functions of exponential type.

Theorem 1.1. *Let f be an entire function of exponential type τ such that $|f(x)| \le M$ for $-\infty < x < \infty$. Then $|f'(x)| \le M\tau$ for $-\infty < x < \infty$.* $\square$

If f is as in Theorem 1.1, then for any two points x_1 and x_2 on the real axis, we have

$$|f(x_2) - f(x_1)| = \left|\int_{x_1}^{x_2} f'(x)\,\mathrm{d}x\right| \le M\,\tau\,|x_2 - x_1|\,.$$

Thus, as a consequence of Theorem 1.1, we obtain the following property of entire functions of exponential type bounded on the real axis.

Corollary 1.2. *An entire function of exponential type bounded on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.* □

Replacing x by z in $t(x) := \sum_{\nu=-n}^{n} c_\nu e^{i\nu x}$ we see that a trigonometric polynomial of degree at most n is the restriction of an entire function of exponential type n to the real axis. Theorem 1.1 can be used to show that an entire function of exponential type which is 2π-periodic is necessarily a trigonometric polynomial.

Corollary 1.3. *Let f be a 2π-periodic entire function of exponential τ. Then, f has the form*

$$f(z) := \sum_{\nu=-n}^{n} c_\nu e^{i\nu z} \qquad (n = \lfloor \tau \rfloor)\ .$$

In other words, the restriction of f to the real axis is a trigonometric polynomial of degree not exceeding $\lfloor \tau \rfloor$.

Proof. Let $\sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu z}$ be the Fourier series of f. Then, for any $\nu \in \mathbb{Z}$,

$$\left|\nu^k c_\nu\right|^2 \le \sum_{\nu=-\infty}^{\infty} \left|\nu^k c_\nu\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f^{(k)}(t)|^2\, dt \qquad (k = 0, 1, 2, \dots)\,.$$

If $|f(x)| \le M$ on the real axis, then Theorem 1.1 implies that $|f^{(k)}(t)| \le M\,\tau^k$ for any real t, and so

$$|\nu|^{2k}|c_\nu|^2 \le \left(M\,\tau^k\right)^2 \qquad (k = 0, 1, 2, \dots)\,,$$

that is, $|c_\nu| \le M\ (\tau/|\nu|)^k$ for $k = 0, \pm1, \pm2, \dots$ Thus, by taking k sufficiently large, $|c_\nu|$ is seen to be smaller than any positive number provided that $|\nu| > \tau$. Hence, c_ν must be zero except possibly for $\nu = 0, \dots, \pm\lfloor \tau \rfloor$. □

Knowing fully well that a non-constant continuous function g bounded on the real axis cannot be approximated arbitrarily closely by polynomials it is natural to ask for the smallest class $\mathfrak{C}_{\rho,\tau}$ which, for any given $\varepsilon > 0$, contains a function f such that $\sup\{|g(x) - f(x)| :\ x \in \mathbb{R}\} < \varepsilon$? By Proposition 1.1, the class $\mathfrak{C}_{1,0}$ does not have this property.

Let us go one step further and take a number $\tau > 0$. Given any non-constant function g, continuous and bounded on the real axis, and any $\varepsilon > 0$, can we find a function $f \in \mathfrak{C}_{1,\tau}$ such that $|g(x) - f(x)| < \varepsilon$ for all real x? The answer is again "no" . To see this, let us consider the function $g(z) := \cos \sigma z$ with $\sigma > \tau$. It vanishes at the points $(2k+1)\pi/2\sigma$ for $k = 0, \pm1, \pm2, \dots$ Now, let us assume that f is an entire function of exponential type τ such that $\sup\{|g(x) - f(x)|\ :\ -\infty < x < \infty\} < \varepsilon$. Then, $\left|f\big((2k+1)\pi/2\sigma\big)\right| < \varepsilon$ for $k = 0, \pm1, \pm2, \dots$ Thus, the function $f\big((2z+1)\,\pi/2\sigma\big)$ is of exponential type

$b := \pi\tau/\sigma < \pi$, and its modulus is less than ε at the integers. By a theorem of M. L. Cartwright [8, p. 180],

$$\begin{aligned}\sup_{-\infty<x<\infty}|f(x)| &\le \left(4+2\,\mathrm{e}\,\log\frac{\pi}{\pi-b}\right)\varepsilon\\ &< \frac{1}{4}\qquad\left(0<\varepsilon<\left(16+8\,\mathrm{e}\,\log\frac{\pi}{\pi-b}\right)^{-1}<\frac{1}{16}\right).\end{aligned}$$

Hence, if $0<\varepsilon<\left(16+8\mathrm{e}\log\dfrac{\pi}{\pi-b}\right)^{-1}$, then $|g\left(k\pi/\sigma\right)-f\left(k\pi/\sigma\right)|>3/4>\varepsilon$ for $k=0,\pm1,\pm2,\ldots$, which is a contradiction. It is not without interest that the function $g(z):=\cos\sigma z$ is itself an entire function, though of order 1 type $\sigma>\tau$.

Now, let us go even further and consider the class $\bigcup_{\tau>0}\mathfrak{C}_\tau$ of all entire functions of exponential type. Given any non-constant function g, continuous and bounded on the real axis, and any $\varepsilon>0$, can we find a function f of exponential type such that $|g(x)-f(x)|<\varepsilon$ for all real x? The answer is still "no". To see this, let us consider the function $g(z):=\sin z^2$, which has the points $0,\pm\sqrt{\pi},\pm\sqrt{2\pi},\pm\sqrt{3\pi},\ldots$ amongst its zeros. Now, let $\varepsilon\in(0\,,\,1/10)$ and suppose that f_ε is an entire function of exponential type τ such that $\sup\limits_{-\infty<x<\infty}|g(x)-f_\varepsilon(x)|<\varepsilon$. Then,

$$\left|f_\varepsilon\left(\pm\sqrt{2k\pi}\right)\right|<\frac{1}{10}\qquad(k=0,1,2,\ldots)\,,\tag{1.3}$$

whereas

$$\left|f_\varepsilon\left(\pm\sqrt{\frac{4k+1}{2}\pi}\right)\right|>\frac{9}{10}\qquad(k=0,1,2,\ldots)\,.\tag{1.4}$$

However, $|f_\varepsilon(x)|<11/10$ for all real x, and so by Bernsteins's inequality, $\sup\limits_{-\infty<x<\infty}|f_\varepsilon'(x)|<(11/10)\,\tau$, and so

$$\left|f_\varepsilon\left(\sqrt{\frac{4k+1}{2}\pi}\right)-f_\varepsilon\left(\sqrt{2k\pi}\right)\right|=\left|\int_{\sqrt{2k\pi}}^{\sqrt{(4k+1)\pi/2}}f_\varepsilon'(t)\,\mathrm{d}t\right|<\frac{1}{\sqrt{k}}\,\frac{11\sqrt{2\pi}}{80}\,\tau\to0$$

as $k\to\infty$. This is, clearly, incompatible with (1.3) and (1.4).

Not only is the function $g(z):=\sin z^2$ continuous and bounded on the real axis, it is holomorphic throughout the complex plane. What could then be the reason that we still cannot make $\sup_{x\in\mathbb{R}}|g(x)-f(x)|$ arbitrarily small by varying f in the class $\bigcup_{\tau>0}\mathfrak{C}_{1\,,\,\tau}$ of all entire functions of exponential type? In view of the fact that the functions in $\bigcup_{\tau>0}\mathfrak{C}_{1\,,\,\tau}$, which are unbounded on the real axis, are irrelevant in the present situation, the following explanation comes to mind: "whereas, any entire function of exponential type bounded on $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$, the function $g(x):=\sin x^2$ is not". In fact, we shall see in the next section that a function g, continuous and bounded on $\mathbb{R}$, can be approximated arbitrarily closely by entire functions of exponential type *if and only if* it is uniformly continuous on $\mathbb{R}$.

2. Best Approximation by Functions of Exponential Type

To start with we shall recall some well-known results from the theory of *"best approximation by polynomials"*.

Let us denote by $\mathcal{P}_n$ the class of all polynomials of degree at most n, and for any given function $g \in C[a\,,\,b]$, let

$$\mathbb{E}_n = \mathbb{E}_n(g) := \inf_{p\in\mathcal{P}_n} \max_{a\le x\le b} |g(x) - p(x)|\,. \tag{2.1}$$

Clearly, $\mathbb{E}_n(g)$ is a non-increasing function of n. In addition, it is useful to know that if $|g(x)| \le M$ for $a \le x \le b$, then

$$\mathbb{E}_n(g) \le \max_{a\le x\le b} |g(x) - 0| = M \tag{2.2}$$

since the identically zero polynomial belongs to $\mathcal{P}_n$ for any n. Also, $\mathbb{E}_n(g)$ is attained, that is, the following theorem holds.

Proposition C. *For each n, there exists a polynomial p_n^* belonging to $\mathcal{P}_n$ such that* $\max\{|g(x) - p_n^*(x)| \,:\, a \le x \le b\} = \mathbb{E}_n(g)$. □

The following theorem of P. L. Chebyshev gives a useful set of necessary and sufficient conditions for a polynomial $p \in \mathcal{P}_n$ to be *a polynomial of best approximation* to a continuous function $g : [a\,,\,b] \to \mathbb{R}$.

Theorem D. *Let $g \in C[a\,,\,b]$ and $p \in \mathcal{P}_n$. Furthermore, let $\eta := g - p$. Then,* $\max\{|\eta(x)| : a \le x \le b\} = \mathbb{E}_n(g)$ *if and only if there exist $n+2$ points $x_0 < \ldots < x_{n+1}$ in $[a\,,\,b]$ such that $|\eta(x_\nu)| = \mathbb{E}_n(g)$ for $\nu = 0, \ldots, n+1$ and $\eta(x_\nu) \times \eta(x_{\nu+1}) < 0$ for $\nu = 0.\ldots, n$.* □

In this connection we must also mention the following result of de la Vallée Poussin, which gives a lower bound for $\mathbb{E}_n(g)$.

Theorem E. *Let $g \in C[a\,,\,b]$ and p a polynomial of degree at most n. Denote by η the function $g-p$, and let there exist $n+2$ points $x_0 < \ldots < x_{n+1}$ in $[a\,,\,b]$ such that*

1. $\eta(x_\nu) \ne 0$ *for* $\nu = 0, 1, \ldots, n+1$,
2. $\eta(x_\nu) \times \eta(x_{\nu+1}) < 0$ *for* $\nu = 0, 1, \ldots, n$.

Then, $\mathbb{E}_n(g) \ge \min\{|\eta(x_\nu)| \,:\, 0 \le \nu \le n\}$. □

The behaviour of $\mathbb{E}_n(g)$ for large n is closely related to the "smoothness" of g, which is often measured in terms of its *modulus of continuity* ω. For any function ϕ defined on an interval $I \subseteq \mathbb{R}$,

$$\omega(\delta) = \omega_\phi(\delta) := \sup_{|x_1 - x_2| \le \delta} |\phi(x_1) - \phi(x_2)| \qquad (\delta > 0\,,\ x_1 \in I,\ x_2 \in I)\ . \quad (2.3)$$

We refer the reader to ([34] or [35]) for a variety of results describing the relationship between $\mathbb{E}_n(g)$ and the structural properties of f. Here is one such result [35, p. 125].

Theorem F. *Let* $g \in C[a\,,\,b]$. *Then,*

$$\mathbb{E}_n(g) \le 12\,\omega\left(\frac{b-a}{2n}\right)\ . \qquad (2.4)$$

□

2.1. Best Approximation by Entire Functions of Exponential Type

Let B_τ denote the class of all entire functions of exponential type τ bounded on the real axis. This class bears a clear analogy with the class $\mathcal{P}_n$ of all polynomials of degree at most n. It plays the same role in the theory of best approximation of bounded, uniformly continuous functions on $\mathbb{R}$ by entire functions of exponential type, as $\mathcal{P}_n$ does in the theory of best approximation of continuous functions on a compact interval by polynomials.

Let $g : \mathbb{R} \to \mathbb{C}$ be bounded on the real axis, and denote by $A_\tau(g)$ the infimum of $\{|g(x) - f(x)| \,:\, -\infty < x < \infty\}$ as f varies in B_τ, that is

$$A_\tau(g) := \inf_{f \in B_\tau} \sup_{-\infty < x < \infty} |g(x) - f(x)|\,. \qquad (2.5)$$

By definition, the identically zero function is an entire function of exponential type τ for any τ. Hence, if $|g(x)| \le M$ on the real axis, then

$$A_\tau(g) \le \sup_{\infty < x < \infty} |g(x) - 0| \le M\,. \qquad (2.6)$$

The following result [6] may be compared with Proposition C.

Proposition 2.1. *Let* g *be bounded on the real axis. Then, there exists an entire function* $f^* \in B_\tau$ *such that*

$$\sup_{-\infty < x < \infty} |g(x) - f^*(x)| = A_\tau(g)\,. \qquad (2.7)$$

For a proof of this result, we need to introduce the notion of a *normal family of functions.*

Definition 2.1. *A family* $\mathfrak{F}$ *is said to be normal in a region* Ω *of the complex plane if every sequence* $\{f_n\}$ *of functions* $f_n \in \mathfrak{F}$ *contains a subsequence which converges uniformly on every compact subset of* Ω.

The following lemma [1, p. 216] contains a very useful criterion for a family of entire functions to be normal.

Lemma 2.1. *A family $\mathfrak{F}$ of entire functions is normal if and only if the functions in $\mathfrak{F}$ are uniformly bounded on every compact set.* □

Proof of Proposition 2.1. In view of the definition of $A_\tau(g)$ there exists, for each $n \in \mathbb{N}$, a function $f_n \in B_\tau$ such that

$$|g(x) - f_n(x)| < A_\tau(g) + \frac{1}{n} \qquad (x \in \mathbb{R},\ n \in \mathbb{N}).$$

If $|g(x)| \le M$ on the real axis then, taking (2.6) into account, we see that

$$|f_n(x)| < |g(x)| + A_\tau(g) + \frac{1}{n} \le 2M + 1 \qquad (x \in \mathbb{R},\ n \in \mathbb{N}).$$

By (1.1),

$$|f_n(x + \mathrm{i}y)| \le (2M + 1)\, \mathrm{e}^{\tau|y|} \qquad (y \in \mathbb{R},\ x \in \mathbb{R},\ n \in \mathbb{N}). \tag{2.8}$$

This implies that the functions in the family $\mathcal{F} := \{f_1,\ f_2,\ \dots\}$ are uniformly bounded on every compact set. Hence, $\mathcal{F}$ is a *normal family* of entire functions, that is, there exists a subsequence $f_{n_1},\ f_{n_2}, \dots, f_{n_k}, \dots$ of $\{f_n\}$ which converges uniformly on every compact subset of $\mathbb{C}$ to a function f^*. It is clear that f^* must be an entire function of exponential type τ and that

$$\sup_{-\infty<x<\infty} |g(x) - f^*(x)| = A_\tau(g).$$
□

Remark 2.1. From (2.6) and (2.7) it follows that if $|g(x)| \le M$ for all $x \in \mathbb{R}$, then

$$|f^*(x)| \le 2M \quad (x \in \mathbb{R}). \tag{2.9}$$

Theorem 1.1 implies that the kth derivative of f^* is bounded by $2M\tau^k$ for $k = 0,\ 1,\ 2, \dots$. Hence, if $f^*(z) := \sum_{k=0}^{\infty} b_k^* z^k$, then

$$|b_k^*| \le 2M \frac{\tau^k}{k!} \qquad (k = 0,\ 1,\ 2, \dots). \tag{2.10}$$

Definition 2.2. *For any $\tau > 0$, let C_τ denote the class (family) of all entire functions $f(z) := \sum_{k=0}^{\infty} b_k z^k$ of exponential type τ such that*

$$|b_k| \le 2M \frac{\tau^k}{k!} \qquad (k = 0,\ 1,\ 2, \dots). \tag{2.11}$$

In view of (2.10), the function f^* belongs to C_τ.

Remark 2.2. From (2.11) it follows that if f belongs to C_τ, then

$$|f(z)| \le 2\, M\, \mathrm{e}^{\tau R} \qquad (|z| \le R).$$

Hence, by Lemma 2.1, the family C_τ is normal.

Remark 2.3. As an addendum to Proposition 2.1 we wish to mention that *if* $g : \mathbb{R} \to \mathbb{C}$ *is* 2π- *periodic then, there exists a* 2π- *periodic entire function* $F \in B_\tau$ *such that*

$$\sup_{-\infty<x<\infty} |g(x) - F(x)| = A_\tau(g)\,.$$

To see this, it may be noted that if f^* is as in Proposition 2.1, then for any $\nu \in \mathbb{Z}$, we have

$$|g(x) - f^*(x + 2\nu\pi)| = |g(x + 2\nu\pi) - f^*(x + 2\nu\pi)| \le A_\tau(g) \qquad (x \in \mathbb{R})\,.$$

Consequently, if

$$F_n(x) := \frac{1}{2n+1} \sum_{\nu=-n}^{n} f^*(x + 2\nu\pi) \qquad (n = 1, 2, \dots)\,,$$

then

$$\sup_{-\infty<x<\infty} |g(x) - F_n(x)| \le A_\tau(f) \qquad (n = 1, 2, \dots)\,.$$

It is clear that we can find a subsequence $F_{n_1}, F_{n_2}, \dots, F_{n_j}, \dots$ which, as $j \to \infty$, converges to an entire function F of exponential type τ such that $|g(x) - F(x)| \le A_\tau(g)$ for all real x. Hence, in fact

$$\sup_{-\infty<x<\infty} |g(x) - F(x)| = A_\tau(g)\,.$$

The function F, so obtained, is 2π- periodic. To see this, let us suppose that $|g(x)| \le M$ on the real axis. Then, as indicated in Remark 2.1, $|f^*(x)| \le 2M$ for all real x, and so the difference

$$F_n(x + 2\pi) - F_n(x) = \frac{1}{2n+1} \left\{ f^*\big(x + (2n+2)\pi\big) - f^*\big(x - 2n\pi\big) \right\}$$

tends uniformly to 0 as $n \to 0$, from which it follows that $F(x+2\pi) - F(x) = 0$ for any real x. By Corollary 1.3, the restriction of F to the real axis must be a trigonometric polynomial of degree at most $\lfloor \tau \rfloor$. Hence, the following result [6] holds.

Proposition 2.2. *Let* g *be periodic with period* 2π. *Then*

$$A_\tau(g) = \mathbb{E}^*_{\lfloor \tau \rfloor}(g)\,,$$

where $\mathbb{E}^*_{\lfloor \tau \rfloor}(g)$ *denotes the error of a best approximation to* g *by trigonometric polynomials of degree at most* $\lfloor \tau \rfloor$. □

For a familiar reason, Theorem A may be formulated as follows: *A function* g *defined and bounded on a compact interval* $I := [a\,, b]$ *can be approximated* *arbitrarily closely in the uniform norm by polynomials if and only if it is uniformly continuous on* I. The following result [6] can then be seen as its analogue for uniform approximation of a bounded function $g : \mathbb{R} \to \mathbb{C}$ by entire functions of exponential type.

Theorem 2.1. *Let* $g : \mathbb{R} \to \mathbb{C}$ *be bounded on the real axis. Then* $A_\tau(g)$*, as defined in* (2.5)*, tends to zero as* $\tau \to \infty$ *if and only if* g *is uniformly continuous on* $\mathbb{R}$.

We start with the following auxiliary result.

Lemma 2.2. *Let* $T_m(x) := \sum_{\mu=0}^{m} t_{m,\mu}\, x^\mu$ *be the Chebyshev polynomial of the first kind of degree* m. *Then*

$$|t_{m,\mu}| \le \frac{m^\mu}{\mu!} \qquad (\mu = 0, 1, \ldots, m)\,.$$

Proof. We know that $T_m(0)$ is 0 or 1 according as m is odd or even, respectively. Since $T_m(\cos\theta) = \cos m\theta$, we also see easily that $|T_m'(0)|$ is equal to m if m is odd and 0 otherwise. So, our assertion certainly holds for $\mu = 0$ and $\mu = 1$. We shall use induction to prove it for other values of μ. For this we recall that T_m satisfies the differential equation

$$(1 - x^2)\, T_m''(x) - x\, T_m'(x) + m^2\, T_m(x) = 0\,,$$

from which we readily deduce that

$$T_m^{(j+2)}(0) + (m^2 - j^2)\, T_m^{(j)}(0) = 0 \qquad (j = 0, 1, \ldots)\,,$$

that is,

$$|t_{m,j+2}| = \frac{m^2 - j^2}{(j+2)(j+1)}\ |t_{m,j}| \qquad (j = 0, 1, \ldots)\,.$$

Hence,

$$\begin{aligned}
|t_{m,j+2}| &= \begin{cases} \frac{(m^2-j^2)\left(m^2-(j-2)^2\right)\cdots(m^2-0^2)}{(j+2)!}\,|t_{m,0}| & \text{if } j \text{ is even} \\ \frac{(m^2-j^2)\left(m^2-(j-2)^2\right)\cdots(m^2-1^2)}{(j+2)!}\,|t_{m,1}| & \text{if } j \text{ is odd} \end{cases} \\
&\le \begin{cases} \frac{(m^2)^{(j+2)/2}}{(j+2)!} & \text{if } j \text{ is even} \\ \frac{(m^2)^{(j+1)/2}}{(j+2)!} & \text{if } j \text{ is odd} \end{cases} \le \frac{m^{j+2}}{(j+2)!}\,.
\end{aligned}$$

□

Proof of Theorem 2.1. First we shall show that if $A_\tau(g) \to 0$ as $\tau \to \infty$ then g must be uniformly continuous on $\mathbb{R}$. Let $|g(x)| \le M$. Then, in view of (2.7), (2.9) and Theorem 1.1, we have

$$\begin{aligned}
|g(x+h) - g(x)| &\le 2A_\tau(g) + |f^*(x+h) - f^*(x)| = 2A_\tau(g) + \left|\int_x^{x+h} \frac{\mathrm{d}f^*}{\mathrm{d}t}\,\mathrm{d}t\right| \\
&\le 2A_\tau(g) + 2M\tau|h| \qquad (x \in \mathbb{R}\,,\ h \in \mathbb{R})\,.
\end{aligned}$$

Now, for any $\varepsilon > 0$ we may choose τ so large that $A_\tau(g) < \varepsilon/4$, and then

$$|g(x+h) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \left(x \in \mathbb{R}\,,\ \ h \in \mathbb{R}\,,\ |h| < \frac{\varepsilon}{4\,M\tau}\right).$$

Hence, g is uniformly continuous on $\mathbb{R}$

In order to prove the converse, we suppose that g is uniformly continuous, that is, its modulus of continuity

$$\omega(\delta) = \omega_g(\delta) := \sup_{|x_1 - x_2| \le \delta} |g(x_1) - g(x_2)| \qquad (\delta > 0\,,\ x_1 \in \mathbb{R},\ x_2 \in \mathbb{R})$$

tends to zero as δ tends to zero. As above, let f^* be a function of best (uniform) approximation to g amongst all functions in B_τ. We recall that if $|g(x)| \le M$ on the real axis, then $|f^*(x)| \le 2M$ for all real x and $f^* \in C_\tau$.

Now, let $\lambda_n := n/\tau$ for $n = 1, 2, \ldots$, and denote by $A_\tau(\lambda_n; g)$ the best approximation to g on the segment $[-\lambda_n\,,\,\lambda_n]$ by functions belonging to C_τ. Furthermore, let $p_n(x\,;\,\lambda_n) := \sum_{k=0}^n b_{k,\lambda_n} x^k$ be the polynomial of best approximation of degree at most n to g on the segment $[-\lambda_n\,,\,\lambda_n]$. Then, the polynomial $p_n(\lambda_n x; \lambda_n) = \sum_{k=0}^n \lambda_n^k\, b_{k,\lambda_n} x^k$ is the polynomial of best approximation of degree at most n to the function $G(x) := g(\lambda_n x)$ on the segment $[-1\,,\,1]$. Since $|g(x)| \le M$ on the real axis, it follows that for any $x \in [-1\,,\,1]$,

$$|p_n(\lambda_n x\,;\,\lambda_n)| \le 2M\,.$$

Setting $a_k := \lambda_n^k\, b_{k,\lambda_n}$ for $k = 0, 1, \ldots, n$, we see that $p(x) := \sum_{k=0}^n a_k\, x^k$ is a polynomial of degree at most n such that $|p(x)| \le 2M$ for $-1 \le x \le 1$. Hence, by a classical result of W. Markov [35, p. 56],

$$\lambda_n^k\,|b_{k,\lambda_n}| \;=\; |a_k| \le \begin{cases} 2M\,|t_{n,k}| & \text{if } n-k \text{ is even} \\ 2M\,|t_{n-1,k}| & \text{if } n-k \text{ is odd}\,. \end{cases}$$

Using Lemma 2.2 we conclude that

$$|b_{k,\lambda_n}| \le 2M \left(\frac{n}{\lambda_n}\right)^k \frac{1}{k!} = 2M\frac{\tau^k}{k!} \qquad (k = 0,\,1,\,\ldots)\,.$$

Thus, $p_n(\,.\,;\,\lambda_n)$ belongs to the class C_τ. It follows that if $\mathbb{E}_n(\lambda_n\,;\,g)$ denotes the best approximation to g on the segment $[-\lambda_n\,,\,\lambda_n]$ by polynomials of degree at most n, then

$$A_\tau(\lambda_n\,;\,g) \le \mathbb{E}_n(\lambda_n\,;\,g) \qquad \left(\lambda_n = \frac{n}{\tau}\right).$$

By Theorem F, if $G(x) := g(\lambda_n x)$, then $\mathbb{E}_n(1\,,\,G) \le 12\,\omega_G\left(\dfrac{1}{n}\right)$. Thus,

$$A_\tau(\lambda_n\,;\,g) \le \mathbb{E}_n(\lambda_n\,;\,g) = \mathbb{E}_n(1\,,\,G) \le 12\,\omega_G\left(\frac{1}{n}\right) = 12\,\omega_g\left(\frac{1}{\tau}\right),$$

that is, $A_\tau(\lambda_n\,;\,g) \le 12\,\omega_g(1/\tau)$ for $n = 1, 2, 3, \ldots$ Clearly,

$$A_\tau(\lambda_1\,;\,g) \le A_\tau(\lambda_2\,;\,g) \le \cdots \le A_\tau(\lambda_n\,;\,g) \le \cdots .$$

Hence, $A_\tau(\lambda_n\,;\,g)$ tends to a finite limit, say $A_\tau(\infty\,;\,g)$, as $n \to \infty$. Since $A_\tau(\lambda_n\,;\,g) \le A_\tau(g)$ for $n = 1, 2, \ldots$, the limit $A_\tau(\infty\,;\,g)$ cannot be larger than $A_\tau(g)$. We claim that it cannot be smaller than $A_\tau(g)$ either. For sake of argument let us assume that it is. For each $n \in \mathbb{N}$, let $f_n^* \in C_\tau$ be such that

$$\sup_{-\lambda_n \le x \le \lambda_n} |g(x) - f_n^*(x)| = A_\tau(\lambda_n\,;\,g)\,.$$

The family C_τ being normal in $\mathbb{C}$, there exists a subsequence of $\{f_n^*\}$, which converges uniformly on every compact subset of $\mathbb{C}$ to a function f^{**} such that

$$\sup_{-\infty < x < \infty} |g(x) - f^{**}(x)| = A_\tau(\infty\,;\,g) < A_\tau(g)\,.$$

This is a contradiction since f^{**} must belong to $C_\tau \cap B_\tau$. Thus, $A_\tau(\lambda_n\,;\,g) \ge A_\tau(g) - \varepsilon_n$, where $\varepsilon_n \to 0$ as $n = \lambda_n\tau \to \infty$. We see that

$$12\,\omega_g\left(\frac{1}{\tau}\right) \ge \mathbb{E}_n(\lambda_n\,;\,g) \ge A_\tau(\lambda_n\,;\,g) \ge A_\tau(g) - \varepsilon_n\,,$$

where ε_n can be smaller than any positive number we may think of. Hence,

$$A_\tau(g) \le 12\,\omega_g\left(\frac{1}{\tau}\right)\,. \tag{2.12}$$

However, $\omega_g\big(1/\tau\big) \to 0$ as $\tau \to \infty$ since, for this part of the proof, the function g is supposed to be uniformly continuous. Thus, we see that $A_\tau(g) \to 0$ as $\tau \to \infty$. $\square$

The next theorem [7] gives a set of *necessary and sufficient conditions* for an entire function f_* of exponential type τ to be a function of best approximation to a given function g on the whole real axis, in the sense that

$$L := \sup_{-\infty<x<\infty} |g(x) - f_*(x)| = A_\tau(g) := \inf_{f \in B_\tau}\ \sup_{-\infty<x<\infty} |g(x) - f_*(x)|\,.$$

Note the analogy with Theorem D which gives a set of necessary and sufficient conditions for a polynomial of degree at most n to be a polynomial of best approximation to a given function on a compact interval.

Theorem 2.2. *Let $g \notin B_\tau$ be defined and bounded on the real axis. In addition, let f_*, belong to B_τ, that is, we suppose f_* to be an entire function of exponential type τ bounded on the real axis. With*

$$L := \sup_{-\infty<x<\infty} |g(x) - f_*(x)| > 0\,,$$

and $\varepsilon \in (0\,,\,L)$*, let*

$$S_\varepsilon \;:=\; \big\{\xi\in\mathbb{R}: \quad L-\varepsilon \;\le\; |g(\xi)-f_*(\xi)| \;\le\; L\big\}\,.$$

Then, for f_* *to be a best approximation to* g *it is necessary and sufficient that any entire function* φ *of exponential type* τ *satisfying*

$$\varphi(\xi)\,\{g(\xi)-f_*(\xi)\}>0\ \ ,\ \ |\varphi(\xi)|\ge 1\ \ (\xi\in S_\varepsilon)\ , \tag{2.13}$$

is unbounded on the real axis, for all small ε*.*

Proof. We shall first prove the sufficiency of the condition. So, let us suppose that any entire function φ of exponential type τ for which (2.13) holds is necessarily unbounded on the real axis. We have to show that f_* must then be a best approximation to g. Assume not. Then, there exists $f^*\in B_\tau$ such that $\sup\limits_{-\infty<x<\infty} |g(x)-f^*(x)|\le L-2\varepsilon$ for some $\varepsilon>0$. Now, let us take, in particular, the function $\varphi^*(z):=\dfrac{1}{\varepsilon}\,\{f^*(z)-f_*(z)\}$ which, clearly, belongs to B_τ. However, at any point $\xi\in S_\varepsilon$, we have

$$\begin{aligned}\varphi^*(\xi)\,\{g(\xi)-f_*(\xi)\} \;&=\; \frac{1}{\varepsilon}\Big[\{g(\xi)-f_*(\xi)\}^2-\{g(\xi)-f^*(\xi)\}\,\{g(\xi)-f_*(\xi)\}\Big]\\ &\ge\; \frac{1}{\varepsilon}\,\Big\{(L-\varepsilon)^2-(L-2\varepsilon)L\Big\}=\varepsilon>0\end{aligned}$$

and also

$$|\varphi^*(\xi)|=\frac{1}{\varepsilon}\,|g(\xi)-f_*(\xi)-\{g(\xi)-f^*(\xi)\}|\ge\frac{1}{\varepsilon}\,\{(L-\varepsilon)-(L-2\varepsilon)\}=1\,.$$

Thus, $\varphi=\varphi^*$ is an entire function of exponential type τ for which (2.13) is satisfied although *it is not unbounded on the real axis.* We have been led to this contradiction since we had assumed that f_* was not a function of best approximation to g.

Now, let us turn to the necessity of the condition. This time we suppose that for some $\varepsilon>0$, say ε_0, there exists a function $\varphi\in B_\tau$ for which (2.13) is satisfied. Then, (2.13) holds for all $\varepsilon\in(0\,,\,\varepsilon_0)$. We have to show that in such a case the function f_* cannot be a function of best approximation to g.

Let $|\varphi(x)|\le H$ on the real axis. Then, for any $x\in S_\varepsilon$,

$$\frac{\varphi(x)}{g(x)-f_*(x)}=\frac{|\varphi(x)|}{|g(x)-f_*(x)|}\le\frac{H}{L-\varepsilon}\,.$$

Hence, for all sufficiently small $\lambda>0$,

$$\left|1-\lambda\frac{\varphi(x)}{g(x)-f_*(x)}\right|=1-\lambda\frac{|\varphi(x)|}{|g(x)-f_*(x)|}\qquad(x\in S_\varepsilon)\ ,$$

and so for any such λ and all $x\in S_\varepsilon$, we have

$$|g(x)-\{f_*(x)+\lambda\,\varphi(x)\}|=|g(x)-f_*(x)|-\lambda\,|\varphi(x)|\le L-\lambda<L\,.$$

Now, let $x \notin S_\varepsilon$. Then

$$\begin{aligned}|g(x)-\{f_*(x)+\lambda\,\varphi(x)\}| &\le |g(x)-f_*(x)|+\lambda\,|\varphi(x)|\le L-\varepsilon+\lambda H\\ &\le L-\varepsilon+\frac{\varepsilon\,H}{H+1}=L-\frac{\varepsilon}{H+1}\end{aligned}$$

if $0<\lambda\le\varepsilon/(H+1)$.

Thus, if there exists an entire function φ belonging to B_τ for which (2.13) is satisfied then, for all sufficiently small $\lambda>0$,

$$\sup_{-\infty<x<\infty}|g(x)-\{f_*(x)+\lambda\,\varphi(x)\}|<\sup_{-\infty<x<\infty}|g(x)-f_*(x)|\;,$$

that is, f_* cannot be a function of best approximation to g. □

We mention the following result [7] as an addendum to Theorem 2.2.

Theorem 2.2′. *Let g be defined and bounded on the real axis. Also, let f_* be an entire function of exponential type τ bounded on the real axis, and set*

$$L:=\sup_{-\infty<x<\infty}|g(x)-f_*(x)|\,.$$

Furthermore, let

$$S:=\{\xi\in\mathbb{R}:|g(\xi)-f_*(\xi)|=L\}\,,$$

and suppose that any entire function of exponential type τ, such that

$$\psi(\xi)\,\{g(\xi)-f_*(\xi)\}>0,\quad |\psi(\xi)|\ge 1\qquad (\xi\in S)\;,\tag{2.14}$$

is necessarily unbounded on the real axis. Then, the function f_* is a best approximation to g in the sense that

$$\sup_{-\infty<x<\infty}|g(x)-f_*(x)|=\inf_{f\in B_\tau}\ \sup_{-\infty<x<\infty}|g(x)-f(x)|\,.$$

Proof. For a proof by contradiction, let us assume that f_* is not a function of best approximation to g. Then, there must exist a function $f^*\in B_\tau$ such that $|g(x)-f^*(x)|\le L-\varepsilon<L$ for all $x\in\mathbb{R}$. Clearly, the function $\psi^*(z):=\{f^*(z)-f_*(z)\}/\varepsilon$ belongs to B_τ. Furthermore, for any $\xi\in S$,

$$\begin{aligned}\psi^*(\xi)\,\{g(\xi)-f_*(\xi)\} &= \frac{1}{\varepsilon}\Big[\{g(\xi)-f_*(\xi)\}^2-\{g(\xi)-f^*(\xi)\}\,\{g(\xi)-f_*(\xi)\}\Big]\\ &\ge \frac{1}{\varepsilon}\left\{L^2-(L-\varepsilon)L\right\}=L>0\,,\end{aligned}$$

and

$$|\psi^*(\xi)|=\frac{1}{\varepsilon}\,|g(\xi)-f_*(\xi)-\{g(\xi)-f^*(\xi)\}|\ge\frac{1}{\varepsilon}\,\{L-(L-\varepsilon)\}=1\,.$$

Thus, $\psi=\psi^*$ is an entire function of exponential type τ for which (2.14) is satisfied although *it is not unbounded on the real axis.* We have been led to this contradiction since we had assumed that f_* was not a function of best approximation to g. □

2.2. An Analogue of de la Vallée Poussin's Theorem

We start with a definition.

Definition 2.3. *We say that the set of points*

$$\cdots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 < \cdots$$

is a set of degree τ *if any entire function* φ *of exponential type* τ, *such that*

$$(-1)^k \varphi(\xi_k) > 0 \ , \ |\varphi(\xi_k)| \geq 1 \qquad (k = 0, \pm 1, \pm 2, \dots)\,, \tag{2.15}$$

is unbounded on the real axis.

Example 2.1. The points $0, \pm\pi, \pm 2\pi, \dots$ form a set of degree τ for any $\tau < 1$. For this, we note that if $\tau < 1$ then $\lfloor \tau \rfloor = 0$ and so by Proposition 2.2, the identically zero function minimizes $\sup\{|\cos x - f(x)| : -\infty < x < \infty\}$ as f varies in the class B_τ of all entire functions of exponential type τ bounded on the real axis. However, this could not be true if there was an entire function $\varphi \in B_\tau$ such that

$$(-1)^k \varphi(k\pi) > 0 \ , \ |\varphi(k\pi)| \geq 1 \qquad (k = 0, \pm 1, \pm 2, \dots) \tag{2.16}$$

since the quantity $\sup\{|\cos x - \varepsilon\, \varphi(x)| : -\infty < x < \infty\}$ would then be smaller than $1 = \sup\{|\cos x - 0| : -\infty < x < \infty\}$ for all sufficiently small $\varepsilon > 0$.

The following analogue [7] of Theorem E gives a lower bound for the quantity $A_\tau(g)$ defined in (2.5).

Theorem 2.3. *Let* g *be defined and bounded on the real axis. Furthermore, let* $\cdots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 < \cdots$ *be a set of degree* τ *and* $f_\circ$ *an entire function of exponential type* τ *bounded on the real axis such that*

$$|g(\xi_k) - f_\circ(\xi_k)| \geq L\ ,\ (-1)^k\big(g(\xi_k) - f_\circ(\xi_k)\big) > 0 \quad (k = 0, \pm 1, \pm 2, \dots)\,. \tag{2.17}$$

Then,

$$A_\tau(g) := \inf_{f \in B_\tau} \sup_{-\infty < x < \infty} |g(x) - f(x)| \ \geq\ L\,.$$

Proof. For a proof by contradiction let us assume that $A_\tau(g) = L - \varepsilon$, where $\varepsilon > 0$. Then, there must exist a function $f^\circ \in B_\tau$ such that

$$|g(\xi_k) - f^\circ(\xi_k)| \ \leq\ L - \varepsilon\,. \tag{2.18}$$

Hence, writing $f^\circ(\xi_k) - f_\circ(\xi_k) = g(\xi_k) - f_\circ(\xi_k) - \big(g(\xi_k) - f^\circ(\xi_k)\big)$ we see that

$$\begin{aligned}\big(g(\xi_k) - f_\circ(\xi_k)\big)\big(f^\circ(\xi_k) - f_\circ(\xi_k)\big) &= \big(g(\xi_k) - f_\circ(\xi_k)\big)^2 \left(1 - \frac{g(\xi_k) - f^\circ(\xi_k)}{g(\xi_k) - f_\circ(\xi_k)}\right) \\ &\geq L^2 - (L - \varepsilon)L > 0 \qquad (k = 0, \pm 1, \pm 2, \dots)\,,\end{aligned}$$

and so also

$$(-1)^k\big(g(\xi_k)-f_\circ(\xi_k)\big)\ (-1)^k\big(f^\circ(\xi_k)-f_\circ(\xi_k)\big)>0 \qquad (k=0,\,\pm1,\,\pm2,\,\dots)\,.$$

Taking note of the second inequality in (2.17), we conclude that

$$(-1)^k\big(f^\circ(\xi_k)-f_\circ(\xi_k)\big)>0 \qquad (k=0,\,\pm1,\,\pm2,\,\dots)\,. \tag{2.19}$$

Besides,

$$|f^\circ(\xi_k)-f_\circ(\xi_k)|\ge|g(\xi_k)-f_\circ(\xi_k)|-|g(\xi_k)-f^\circ(\xi_k)|\ ,$$

and so, using the first inequality in (2.17) to estimate $|g(\xi_k)-f_\circ(\xi_k)|$ from below by L, and (2.18) to estimate $|g(\xi_k)-f^\circ(\xi_k)|$ from above by $L-\varepsilon$, we see that

$$|f^\circ(\xi_k)-f_\circ(\xi_k)|\ge\varepsilon \qquad (k=0,\,\pm1,\,\pm2,\,\dots)\,. \tag{2.20}$$

Inequalities (2.19) and (2.20) together say that the entire function $\varphi:=f^\circ-f_\circ$ which is clearly of exponential type τ satisfies (2.15) at all the points of a set of degree τ in spite of being bounded on the real axis. This is a contradiction which proves the result. □

In the course of proving Theorem 2.1 we have also proved the following result analogous to Theorem F.

Proposition 2.3. *Let g be continuous and bounded on the real axis. Then,*

$$A_\tau(g):=\inf_{f\in B_\tau}\ \sup_{-\infty<x<\infty}|g(x)-f(x)|\ \le 12\,\omega_g\left(\frac{1}{\tau}\right)\,.$$

□

3. Hermite Interpolation by Functions of Exponential Type and Uniform Approximation on $\mathbb{R}$

As we have remarked earlier, Theorem A holds for functions in $C[a\,,\,b]$ if and only if it holds for those in $C[0\,,\,1]$. It was proved by S. Bernstein that if $g\in C[0\,,\,1]$ and

$$B_n(x):=\sum_{k=0}^{n} g\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\,,$$

then $\lim_{n\to\infty}B_n(x)=g(x)$ uniformly in $[0\,,\,1]$.

Theorem A has been proved in numerous other ways. It would be natural to think that if we were given an infinite triangular matrix $\mathfrak{A}$ whose nth row consisted of the points

$$x_{1,n}>x_{2,n}>\cdots>x_{n,n} \tag{3.1}$$

in $(-1\,,\,1)$ such that

$$\lim_{n\to\infty}\ \max_{0\le\nu\le n}\,(x_{\nu,n}-x_{\nu+1,n})=0\qquad (x_{0,n}=1\,,\ x_{n+1,n}=-1)\ ,$$

then, for any function $g\in C[-1\,,\,1]$ the associated sequence of Lagrange interpolating polynomials

$$L_{n-1}(x\,;\,\mathfrak{A}\,;\,g):=\sum_{\nu=1}^{n}\frac{\omega_n(x)}{(x-x_{\nu,n})\,\omega_n'(x_{\nu,n})}\,g(x_{\nu,n})\qquad (n=1,2,\dots)\,,\quad (3.2)$$

where $\omega_n(x):=\prod_{\nu=1}^{n}(x-x_{\nu,n})$, would converge uniformly to g on $[-1\,,\,1]$. However, this is not true. In fact, Faber [18] proved the following surprising result.

Theorem G. *Given any matrix $\mathfrak{A}$ as above, there exists a continuous function $g_{\mathfrak{A}}\ :\ [-1\,,\,1]\to\mathbb{R}$ such that the associated sequence of Lagrange interpolating polynomials $\{L_{n-1}(x\,;\,\mathfrak{A}\,;\,g_{\mathfrak{A}})\}$ does not converge uniformly to $g_{\mathfrak{A}}$ on $[-1\,,\,1]$.* □

The situation changes significantly if we consider Hermite interpolation and take for $\mathfrak{A}$ the triangular matrix whose n-th row consists of the zeros of T_n. In fact, Fejér [19] made the following remarkable discovery.

Theorem H-1. *Let*

$$x_\nu=x_{\nu,n}=\cos\frac{2\nu-1}{2n}\pi\qquad (\nu=1,2,\dots,n)\,,$$

and g a function continuous on $[-1\,,\,1]$. Furthermore, let

$$A_{\nu,n}:=\left\{\frac{T_n(x)}{n(x-x_\nu)}\right\}^2(1-xx_\nu)\qquad (\nu=1,2,\dots,n)\,.$$

Then,

$$H_{2n-1}(x):=\sum_{\nu=1}^{n}g(x_\nu)A_{\nu,n}(x)$$

is a polynomial of degree at most $2n-1$ such that

$$H_{2n-1}(x_\nu)=g(x_\nu),\ \ H_{2n-1}'(x_\nu)=0\qquad (\nu=1,2,\dots,n)\,,$$

and $\lim_{n\to\infty}H_{2n-1}(x)=g(x)$ uniformly on $[-1\,,\,1]$. □

An analogous result about 2π- periodic functions was proved much earlier. The Fejér kernel

$$k_n(u):=\frac{2}{n+1}\left\{\frac{\sin\frac12(n+1)u}{2\sin\frac12 u}\right\}^2$$

vanishes at the points $2\pi\nu/(n+1)$ for $\nu = 1, 2, \ldots, n$ and equals $(n+1)/2$ at $u = 0$. Thus, if $t_0, t_1, \ldots, t_n$ are any $n+1$ points equally spaced over $[0\,,\,2\pi)$, for example

$$t_\nu := t_0 + \frac{2\pi\nu}{n+1} \qquad (\nu = 0, 1, \ldots, n)\,,$$

then the trigonometric polynomial

$$J_n(x) = J_n(x\,;\,g) := \frac{2}{n+1}\sum_{\nu=0}^{n} g(t_\nu)\, k_n(x - t_\nu)\,,$$

which was introduced by Jackson [28], is of degree at most n and coincides with g at these points. Since $k_n'(u) = 0$ for $u = 2\pi\nu/(n+1)$, the derivative $J_n'(x\,;\,g)$ vanishes at the points t_ν. Thus, as observed by Bernstein [5], $J_n(.\,;\,g)$ is a trigonometric polynomial of degree at most n coinciding with g at the points t_ν and having a vanishing derivative there.

Note the analogy between Theorem G and the following result about the uniform convergence of the sequence $\{J_n(.\,;\,g)\}$ of Hermite interpolating trigonometric polynomials in equally spaced points to a 2π-periodic function.

Theorem H-2. *Let g be bounded and 2π-periodic. Then* (i) $J_n(.\,;\,g)$ *remains within the same bounds as g;* (ii) $J_n(x\,;\,g)$ *converges to $g(x)$ at every point x of continuity of g as $n \to \infty$. The convergence is uniform on every closed interval $[\alpha\,,\,\beta]$ of continuity.* □

Now, we turn to uniform approximation of non-periodic functions on $\mathbb{R}$ via Hermite interpolation by entire functions of exponential type. As in the periodic case, the simplest interpolation points

$$\lambda_{\nu,\tau} := \frac{\nu\pi}{\tau} \qquad (\nu = 0, \pm 1, \pm 2, \ldots) \tag{3.3}$$

turn out to be the most convenient. The functions

$$h_{n,\tau}(z) := \left\{\frac{\sin \tau z}{\tau z - n\pi}\right\}^2 \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

play a fundamental role. It is clear that $h_{n,\tau}$ is of exponential type 2τ, vanishes at all the points in (3.3) except $\lambda_{n,\tau}$, assumes the value 1 at $\lambda_{n,\tau}$, has a vanishing derivative at the points (3.3), and is bounded on the real axis.

Let $g : \mathbb{R} \to \mathbb{C}$ be non-periodic but uniformly continuous and bounded. Then

$$\mathbf{H}_\tau(z\,;\,g) := \sum_{n=-\infty}^{\infty} g(\lambda_{n,\tau}) h_{n,\tau}(z) \qquad (\tau > 0) \tag{3.4}$$

is an entire function of exponential type 2τ having the four properties:

(a) it is bounded on the real axis;
(b) it interpolates g in the points $\lambda_{n,\tau}$;

(c) its derivative vanishes at these points;

(d) $|\mathbf{H}_\tau(\mathrm{i}y\,;\,g)| = O\left(|y|^{-1}\mathrm{e}^{2\tau|y|}\right)$ as $y \to \pm\infty$.

The following proposition [23, p. 148] shows that $\mathbf{H}_\tau(.\,;\,g)$ is the only entire function of exponential type 2τ for which (a), (b), (c) and (d) are satisfied.

Proposition 3.1. *Let f be an entire function of exponential type 2τ such that $f\,(k\,\pi/\tau) = f'\,(k\,\pi/\tau) = 0$ for $k = 0, \pm 1, \pm 2, \ldots,$ and $|f(x)|$ is bounded on the real axis, then $f(z) := c\sin^2 \tau z$, where c is a constant.*

Proof. Without loss of generality we may suppose that τ is equal to 1 and that $|f(x)| \le 1$ on the real axis.

Consider the entire function $F(z) := f(z)/(\sin^2 z)$. Clearly,

$$\left|\sin\left(\pm(n - \frac{1}{2}\right)\pi + \mathrm{i}y)\right| = \frac{\mathrm{e}^y + \mathrm{e}^{-y}}{2} > \frac{1}{2}\,\mathrm{e}^{|y|} \qquad (n \in \mathbb{N}\,,\ y \in \mathbb{R})\,.$$

Besides,

$$\begin{aligned}\left|\sin\left(x \pm \mathrm{i}\left(n - \frac{1}{2}\right)\pi\right)\right| &\ge \frac{\mathrm{e}^{\left(n-\frac{1}{2}\right)\pi} - \mathrm{e}^{-\left(n-\frac{1}{2}\right)\pi}}{2} = \frac{1}{2}\,\mathrm{e}^{\left(n-\frac{1}{2}\right)\pi}\left(1 - \mathrm{e}^{-(2n-1)\pi}\right)\\ &> \frac{1}{3}\,\mathrm{e}^{\left(n-\frac{1}{2}\right)\pi} \qquad (x \in \mathbb{R}\,,\ n \in \mathbb{N})\,.\end{aligned}$$

Thus, $|\sin z|^2 > (1/9)\,\mathrm{e}^{2\,|\Im z|}$ for any z lying on the square contour Γ_n^*, whose corners lie at the points

$$(\pm 1 \pm \mathrm{i})\left(n - \frac{1}{2}\right)\pi \qquad (n \in \mathbb{N})\,.$$

Now, note that by Proposition 1.1,

$$|f(z)| \le \mathrm{e}^{2\,|\Im z|} \qquad (z \in \mathbb{C})\,.$$

Hence, $|F(z)| < 9$ for any $z \in \Gamma_n^*$, where n is an arbitrary positive integer. Thus, $|F|$ is bounded throughout the complex plane. By Liouville's theorem, the entire function F must be a constant, that is, $f(z) \equiv c\,\sin^2 z$, where c is a constant. $\square$

In analogy with Theorems H-1 and H-2 we have the following result about the uniform convergence of Hermite interpolating entire functions of exponential type ([23]; also see [14]).

Theorem 3.1. *Let $g : \mathbb{R} \to \mathbb{C}$ be uniformly continuous and bounded. Furthermore, let $\mathbf{H}_\tau(.\,;\,g)$ be as in* (3.4). *Then*

$$\sup_{-\infty<x<\infty} |g(x) - \mathbf{H}_\tau(x\,;\,g)| \to 0 \quad \text{as} \quad \tau \to \infty\,.$$

Proof. Applying Proposition 3.1 to the function $\mathfrak{H}(z) := 1 - \sum_{n=-\infty}^{\infty} h_{n,\tau}(z)$ and observing that $\mathfrak{H}(\mathrm{i}y) = O\left(|y|^{-1}\mathrm{e}^{2\tau|y|}\right)$ as $y \to \pm\infty$, we conclude that

$$\sum_{n=-\infty}^{\infty} h_{n,\tau}(z) \equiv 1\,. \tag{3.5}$$

Let $|g(x)| \leq M$ on the real axis. If $\omega(.\,;\,g)$ is the modulus of continuity of g, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(\delta\,;\,g) < \varepsilon/2$, and so for any real x, we have

$$\begin{aligned}
|g(x) - \mathbf{H}_\tau(x\,;\,g)| &\leq \Big(\sum_{\left|\frac{n\pi}{\tau}-x\right|<\delta} + \sum_{\left|\frac{n\pi}{\tau}-x\right|\geq\delta}\Big) \left|g(x) - g\left(\frac{n\pi}{\tau}\right)\right| h_{n,\tau}(x) \\
&\leq \omega(\delta\,;\,g) \sum_{n=-\infty}^{\infty} h_{n,\tau}(x) + \frac{2M}{\tau^2} \sum_{\left|\frac{n\pi}{\tau}-x\right|\geq\delta} \frac{1}{\left(x - \frac{n\pi}{\tau}\right)^2} \\
&\leq \omega(\delta\,;\,g) + \frac{2M}{\tau^2}\,\frac{2}{\delta^2}\left\{1 + \sum_{n=1}^{\infty} \frac{1}{\left(1 + \frac{n\pi}{\delta\tau}\right)^2}\right\} \\
&\leq \omega(\delta\,;\,g) + \frac{2M}{\tau^2}\,\frac{2}{\delta^2}\left\{1 + \frac{\delta\tau}{\pi}\int_0^\infty \frac{1}{(1+x)^2}\,\mathrm{d}x\right\} \\
&= \omega(\delta\,;\,g) + \frac{4M}{\tau\delta}\left(\frac{1}{\tau\delta} + \frac{1}{\pi}\right) < \varepsilon
\end{aligned}$$

for all sufficiently large τ. □

4. $(0\,,\,m)$ - interpolation by Functions of Exponential Type and Approximation on $\mathbb{R}$

P. Turán and others investigated the behaviour of $(0\,,\,m)$ - interpolating polynomials, that is, polynomials $R_n(.\,;\,g)$ of degree $\leq 2n-1$, which duplicate the function g at the n points $x_{n,1} > \cdots > x_{n,n}$ in $[-1, 1]$ and whose m-th derivative assumes prescribed values $y_{\nu,n}^{(m)}$ at these points. Even in the case where $m = 2$ the polynomials $R_n(.\,;\,g)$ do not necessarily exist and may not be unique if they exist. *If* n *is even* and the points $x_{\nu,n}$ are taken to be the zeros of the polynomial $\pi_n(x) := (1-x^2)P'_{n-1}(x)$, where P_{n-1} is the Legendre polynomial of degree $n-1$, the $(0\,,\,2)$ - interpolating polynomials $R_n(.\,;\,g)$ exist and are unique. Furthermore, if g is 'continuously differentiable' in $[-1\,,\,1]$ with the modulus of continuity $\omega(t\,,\,g')$ of g' such that $\int_0 t^{-1}\,\omega(t\,,\,g')\,\mathrm{d}t$ exists and the numbers $y_{\nu,n}^{(2)}$ satisfy $\max_{1\leq\nu\leq n} |y_{\nu,n}^{(2)}| = o(n)$, then the sequence $\{R_n(.\,;\,g)\}$ converges uniformly to g on $[-1\,,\,1]$ as n tends to ∞ taking even integral values.

O. Kiš, A. Sharma, A. K. Varma, and others obtained similar results for $(0\,,\,m)$- interpolation of 2π- periodic functions, by trigonometric polynomials.

Consideration of $(0\,,\,m)$ - interpolation by entire functions of exponential type raises serious questions of uniqueness which need to be answered first. For example, to what extent is an entire function of exponential type $< \tau$ determined by its values and those of its second derivative at the points $0,\ \pm 2\pi/\tau,\ \pm 4\pi/\tau, \ldots$? Here, without loss of generality, we may take $\tau = 2\pi$ and the points $0,\ \pm 1,\ \pm 2, \ldots$ as nodes. Then the 'question of uniqueness' just asked is answered by the following theorem [22].

Theorem 4.1. *Let f be an entire function of exponential type $< 2\pi$ such that*

$$f(n) = f''(n) = 0 \qquad (n = 0,\ \pm 1,\ \pm 2, \ldots)\,.$$

Then $f(z) = c \sin \pi z$, where c is a constant. Here, functions of order 1 type 2π are not admissible as the example $\sin 2\pi z$ shows. □

Gervais and Rahman [22] also proved the next three uniqueness theorems.

Theorem 4.2. *Let m be an even integer ≥ 4. Furthermore, let f be an entire function of exponential type $\tau < \pi\, \sec(\pi/m)$ such that*

$$f(n) = f^{(m)}(n) = 0 \qquad (n = 0,\ \pm 1,\ \pm 2, \ldots)\,.$$

Then, $f(z) = c\, \sin \pi z$, where c is a constant. Here, $\tau = \pi\, \sec(\pi/m)$ is inadmissible as the example $f(z) := \exp\bigl(z\pi \tan(\pi/m)\bigr) \sin \pi z$ shows. □

Theorem 4.3. *Let m be an odd integer ≥ 3. Furthermore, let f be an entire function of exponential type $\tau < \pi \sec\bigl(\pi/2m\bigr)$ such that*

$$f(n) = f^{(m)}(n) = 0 \qquad (n = 0,\ \pm 1,\ \pm 2, \ldots)\,.$$

Then $f(z) \equiv 0$. The example $f(z) := \exp\bigl(z\pi \tan(\pi/2m)\bigr) \sin \pi z$ shows that here $\tau = \pi \sec(\pi/2m)$ is inadmissible. □

Since the function g, which we wish to interpolate and approximate, is supposed to be bounded on the real axis, we should see what more can be said in Theorems 4.1–4.3 if f is bounded on the real axis.

Theorem 4.4. *Let m be an integer ≥ 2 and λ an arbitrary number in $[0\,,\,1)$. Furthermore, let f be an entire function of exponential type 2π such that $|f(x)| \leq \alpha + \beta\, |x|^{\lambda}$ on the real axis for certain constants α and β, and suppose that*

$$f(n) = f^{(m)}(n) = 0 \qquad (n = 0,\ \pm 1,\ \pm 2, \ldots)\,.$$

Then,

$$f(z) := \begin{cases} c_1 \sin \pi z + c_2 \sin 2\pi z & \text{if m is even} \\ c \sin^2 \pi z & \text{if m is odd}\,. \end{cases}$$ □

Now, we wish to mention some contributions of Liu Yongping [30], which we find very interesting and noteworthy.

Theorem 4.5. *Let P be an odd polynomial with only real coefficients, and suppose that $P(\mathrm{i}\,t) \neq 0$ for $t \in (0\,,\,2\tau]$. Then, there exists a unique entire function A_τ of exponential type 2τ belonging to $L^2(\mathbb{R})$ such that*

$$A_\tau\left(\frac{k\pi}{\tau}\right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = \pm 1, \pm 2, \dots \end{cases} \tag{4.1}$$

and

$$(P(D)A_\tau)\left(\frac{k\pi}{\tau}\right) = 0 \qquad (k = 0, \pm 1, \pm 2, \dots)\,. \tag{4.2}$$

It is given by the formula

$$A_\tau(x) = \frac{1}{\tau}\int_0^{2\tau} \frac{P\bigl(\mathrm{i}\,(2\tau - t)\bigr)}{P(\mathrm{i}\,t) + P\bigl(\mathrm{i}\,(2\tau - t)\bigr)}\,\cos tx\,\mathrm{d}t\,. \tag{4.3}$$

Also, there exists an entire function B_τ of exponential type 2τ belonging to $L^2(\mathbb{R})$ such that

$$(P(D)B_\tau)\left(\frac{k\pi}{\tau}\right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = \pm 1, \pm 2, \dots \end{cases} \tag{4.4}$$

and

$$B_\tau\left(\frac{k\pi}{\tau}\right) = 0 \qquad (k = 0, \pm 1, \pm 2, \dots)\,. \tag{4.5}$$

It is given by the formula

$$B_\tau(x) = \frac{1}{\tau}\int_0^{2\tau} \frac{\mathrm{i}}{P(\mathrm{i}\,t)) + P\bigl(\mathrm{i}\,(2\tau - t)\bigr)}\,\sin tx\,\mathrm{d}t\,. \tag{4.6}$$

□

Remark 4.1. Using the restrictions imposed upon P, we readily see that $P(\mathrm{i}\,t) + P\bigl(\mathrm{i}\,(2\tau - t)\bigr) \neq 0$ on $[0\,,\,2\tau]$.

In the case where $P(t) := t^{2j+1}$, Theorem 4.5 combined with Theorem 4.4 gives the following result.

Corollary 4.1. (i) *Let U_τ be an entire function of exponential type 2τ such that*

$$U_\tau\left(\frac{k\pi}{\tau}\right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = \pm 1,\, \pm 2, \dots\,, \end{cases} \quad \text{and} \quad U_\tau^{(2j+1)}\left(\frac{k\pi}{\tau}\right) = 0 \ \text{ for all } \ k \in \mathbb{Z}\,.$$

Then, there exists a constant γ_1 such that

$$U_\tau(z) = \frac{1}{\tau}\int_0^{2\tau} \frac{(2\tau - t)^{2j+1}}{t^{2j+1} + (2\tau - t)^{2j+1}}\,\cos tz\,\mathrm{d}t + \gamma_1\,\sin^2 \tau z\,.$$

(ii) *Let V_τ be an entire function of exponential type 2τ such that*

$$V_\tau\left(\frac{k\pi}{\tau}\right) = 0 \text{ for all } k \in \mathbb{Z}, \quad \text{and} \quad V_\tau^{(2j+1)}\left(\frac{k\pi}{\tau}\right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = \pm 1, \pm 2, \ldots \end{cases}$$

Then, there exists a constant γ_2 such that

$$V_\tau(z) = \frac{1}{\tau}\int_0^{2\tau} \frac{(-1)^j}{t^{2j+1} + (2\tau - t)^{2j+1}} \sin tz \,\mathrm{d}t + \gamma_2 \sin^2 \tau z \,.$$

□

The proof of Theorem 4.5 uses the Poisson Summation Formula (PSF for short) and some other results from the theory of Fourier integral. It also uses certain results about entire functions of exponential type belonging to $L^p(\mathbb{R})$, like the Paley–Wiener theorem [8, Theorem 6.8.1]. In view of its relevance here and also in the next §, we wish to include the statement of PSF, starting with some preparatory remarks.

The Fourier transform $g^\wedge$ of $g \in L^1(\mathbb{R})$ is defined by

$$g^\wedge(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)\,\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t \qquad (x \in \mathbb{R})\,,$$

and of $g \in L^2(\mathbb{R})$ by the limit in the $L^2(\mathbb{R})$-norm of $\displaystyle\frac{1}{\sqrt{2\pi}}\int_{-\rho}^{\rho} g(t)\,\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t$ as $\rho \to \infty$. If g belonging to $L^1(\mathbb{R})$ or to $L^2(\mathbb{R})$ is such that $g^\wedge$ belongs to $L^1(\mathbb{R})$, then at each point of continuity of g,

$$g(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g^\wedge(t)\,\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t\,.$$

Lemma 4.1 (Poisson Summation Formula). *Let $g \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then* [11, p. 202],

$$\sum_{k=-\infty}^{\infty} g(\beta k) = \frac{\sqrt{2\pi}}{\beta}\sum_{k=-\infty}^{\infty} g^\wedge\left(\frac{2k\pi}{\beta}\right) \qquad (\beta > 0)\,.$$

In particular, this formula holds if $g \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$ or if "$g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ and $g^\wedge \in L^1(\mathbb{R}) \cap AC(\mathbb{R})$". □

In order to prove Theorem 4.5, Liu Yongping first made the following useful observation for which he used the Paley–Wiener representation of an entire function of exponential type belonging to $L^2(\mathbb{R})$, and the Poisson Summation Formula.

Lemma 4.2. *Let U be an entire function of exponential type 2τ belonging to $L^2(\mathbb{R})$. Then*

$$U\left(\frac{k\pi}{\tau}\right) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k = \pm 1, \pm 2, \ldots \end{cases}$$

if and only if $U^\wedge(t) + U^\wedge(2\tau + t) = \left(\sqrt{\pi/2}\right)/\tau$ *a.e. in* $(-2\tau\,,\,0)$. □

The next result due to Liu Yongping contains some useful estimates for the functions A_τ and B_τ of Theorem 4.5.

Lemma 4.3. *Let* P *be an odd polynomial with only real coefficients, and suppose that* $P(\mathrm{i}\,t) \neq 0$ *for* $t \in (0\,,\,\infty)$. *Then,* $\displaystyle\sum_{k=-\infty}^{\infty} A_\tau\left(x - \frac{k\pi}{\tau}\right) \equiv 1$, *and for any* $\delta > 0$,

$$\sum_{\left|x - \frac{k\pi}{\tau}\right| > \delta} \left|A_\tau\left(x - \frac{k\pi}{\tau}\right)\right| \leq c_1\,\frac{1+\delta}{\delta^2\,\tau}\,,$$

Furthermore, $\int_{-\infty}^{\infty} |A_\tau(x)|\,\mathrm{d}x \leq c_2/\tau$ *for some constant* c_2 *independent of* τ.
Also, there exists a constant c_3 *independent of* τ *such that for any* $p > 1$,

$$\left(\int_{-\infty}^{\infty} |B_\tau(x)|^p\,\mathrm{d}x\right)^{1/p} \leq c_3\,\frac{1}{|P(\mathrm{i}\tau)|}\,\left(\frac{1}{\tau}\int_{-\infty}^{\infty}\left|\frac{1-\cos u}{u}\right|^p\,\mathrm{d}u\right)^{1/p}.$$

□

The estimates for A_τ and B_τ contained in Lemma 4.3 allowed Liu Yongping to obtain the following convergence theorem.

Theorem 4.6. *Let* P *be an odd polynomial with only real coefficients, and suppose that* $P(\mathrm{i}\,t) \neq 0$ *for* $t \in (0\,,\,\infty)$. *Also, let* A_τ *and* B_τ *be as in* (4.3) *and* (4.6), *respectively. Furthermore, for any bounded function* $g : \mathbb{R} \to \mathbb{C}$, *let*

$$R_\tau(x) := \sum_{k=-\infty}^{\infty} g\left(\frac{k\pi}{\tau}\right)\,A_\tau\left(x - \frac{k\pi}{\tau}\right) + \sum_{k=-\infty}^{\infty} \beta_k\,B_\tau\left(x - \frac{k\pi}{\tau}\right)\,,$$

where, for some $p > 1$,

$$\left(\tau\sum_{k=-\infty}^{\infty}|\beta_k|^q\right)^{1/q} = o\big(P(\mathrm{i}\tau)\big) \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Then $R_\tau(x)$ *converges to* $g(x)$ *at each point of continuity of* g. *The convergence is uniform in the case where* g *is uniformly continuous and bounded on the real axis.* □

5. Gaussian Quadrature Formulae for Functions of Exponential Type and an Analogue of Turán's Formula

A polynomial $f(x) := \sum_{\nu=0}^{n-1} a_\nu x^\nu$ of degree $< n$ is completely determined by its values at any set of n distinct points. However, the integral over $[-1\,,\,1]$ of any polynomial of degree $< 2n$, can be correctly evaluated by a formula of Gauss if we know the values of the polynomial at a special set of n points, namely the n zeros of the Legendre polynomial of degree n.

It is known [8, Corollary 9.4.4] that if f_1 and f_2 are two entire functions of exponential type, both $o\left(\mathrm{e}^{\sigma|z|}\right)$ as $|z| \to \infty$, agree at the points

$$0,\ \pm\frac{\pi}{\sigma},\ \pm\frac{2\pi}{\sigma},\ \dots\,,$$

then they must be identical. This means that an entire function f of exponential type is completely determined by its values at these points provided that $f(z) = o\left(\mathrm{e}^{\sigma|z|}\right)$ as $|z| \to \infty$. As explained in the second paragraph of the proof of Theorem 1 in [26], if f is entire function of exponential type σ belonging to $L^1(-\infty\,,\,\infty)$, then $f(z) = o\left(\mathrm{e}^{\tau|z|}\right)$ as $|z| \to \infty$. Thus, an entire function of exponential type τ belonging to $L^1(-\infty\,,\,\infty)$ is completely determined by its values at the points $0, \pm\pi/\sigma, \pm 2\pi/\sigma, \dots$ if $\tau \le \sigma$. The same cannot be said for any $\tau > \sigma$. To see this, let $\tau > \sigma$. Then, the functions

$$f_\varepsilon(z) := \left(\frac{\sin \varepsilon z}{z}\right)^2 \sin \sigma z \qquad \left(0 < \varepsilon < \frac{\tau - \sigma}{2}\right)$$

are all of exponenential type τ, belong to $L^1(-\infty\,,\,\infty)$ and agree with each other at all the points $0, \pm\pi/\sigma, \pm 2\pi/\sigma, \dots$ without being identical.

Using the Paley–Wiener Theorem [8, Theorem 6.8.1] in conjunction with the Poisson Summation Formula (Lemma 4.1), Boas [9] concluded that the integral of a function f of exponential type 2π belonging to $L^1(-\infty\,,\,\infty)$ can be correctly evaluated if we know the values of f at the integers. In fact, the conclusion can be formulated in the following more general form.

Theorem 5.1. *Let f be an entire function of exponential type 2σ belonging to $L^1(-\infty\,,\,\infty)$. Then,*

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi}{\sigma}\sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right) = \frac{\pi}{\sigma}\sum_{\nu=-\infty}^{\infty} f\left(\frac{(2\nu-1)\pi}{2\sigma}\right). \tag{5.1}$$

□

It may be note that (5.1) uses the values of the function only at the points $0, \pm\pi/\sigma, \pm 2\pi/\sigma, \dots$, but still it correctly evaluates the integral of any entire function of exponential type 2 times σ belonging to $L^1(-\infty\,,\,\infty)$. We therefore see it as a Gaussian quadrature formula.

The utility of formula (5.1) is compromised by the requirement that f belong to $L^1(-\infty\,,\,\infty)$ since it excludes the possibility of evaluating many familiar integrals like $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} \frac{\sin x}{x}\,\mathrm{d}x$. With this in mind Frappier and Rahman [21] looked for conditions under which (5.1) would remain true if the integral was taken in the sense of Cauchy. Let us recall that a function f is said to be integrable in the sense of Cauchy on $(-\infty\,,\,\infty)$ if it is integrable on $(0\,,\,R)$ and $(-R\,,\,0)$ for every $R > 0$, and if $I_1 := \lim_{R\to\infty}\int_0^R f(x)\,\mathrm{d}x$ and $I_2 := \lim_{R\to\infty}\int_{-R}^0 f(x)\,\mathrm{d}x$ exist. The sum of I_1 and I_2 is called the integral of f in the sense of Cauchy and is usually denoted by $\int_{\to -\infty}^{\to\infty} f(x)\,\mathrm{d}x$. Frappier and Rahman [21, Theorem 1] proved the following companion to Theorem 5.1.

Theorem 5.1′. *Let f be an entire function of exponential type $\tau < 2\sigma$. Then,*

$$\int_{\to -\infty}^{\to\infty} f(x)\,\mathrm{d}x = \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right) \tag{5.1′}$$

provided that the integral and the series in (5.1′) *are convergent.* □

Example 5.1. Applying (5.1′) to the entire function $(\sin\sigma z)/z$, which is of exponential type σ, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin\sigma x}{x}\,\mathrm{d}x = \frac{\pi}{\sigma} \lim_{z\to 0} \frac{\sin\sigma z}{z} = \pi\,.$$

Example 5.2. Now, let $f(z) := (\sin 2\sigma z)/z$. Then,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi \quad \text{whereas} \quad \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right) = \frac{\pi}{\sigma} f(0) = 2\pi\,,$$

that is, $\int_{\to -\infty}^{\to\infty} f(x)\,\mathrm{d}x \neq \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right)$. This shows that $\tau = 2\sigma$ is not admissible in (5.1′).

Remark 5.1. Writing

$$R_\sigma[f] := \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \frac{\pi}{\sigma} \sum_{\nu=-\infty}^{\infty} f\left(\frac{\nu\pi}{\sigma}\right)$$

we see that, by Theorem 5.1, $R_\sigma[f]$ is zero for any entire function of exponential type 2σ belonging to $L^2(\mathbb{R})$. If $f_h(z) := f(z+h)$, then also $R_\sigma[f_h]$ is zero for any real h. For a converse of this result see [15, Theorem 3.1].

5.1. Turán's Formula and its Analogue for Functions of Exponential Type

Generalizing the quadrature formula of Gauss, it was proved by Turán [41] that the formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} \sum_{\mu=0}^{m-1} \lambda_\nu^{(\mu)} f^{(\mu)}(x_\nu)$$

holds for every polynomial of degree $< (m+1)n$ if m is odd and the nodes $x_1, \ldots, x_n$ are the zeros of the (monic) polynomial $\pi_{n,\,m+1}$ which minimizes the integral $\int_{-1}^{1} |\pi(x)|^{m+1}\,\mathrm{d}x$ amongst all monic polynomials of degree n. In the case $m = 1$, this agrees with the fact that the zeros of the Legendre polynomial of degree n minimize the integral $\int_{-1}^{1} \left|\prod_{\nu=1}^{n} (x - x_\nu)\right|^2 \mathrm{d}x$ over all vectors $(x_1, \dots, x_n) \in \mathbb{R}^n$.

The following result (see [36]; also see [29] and [33]) contains a quadrature formula for entire functions of exponential type that is completely analogous to the preceding formula of Turán.

Theorem 5.2. *Let m be an odd positive integer and $\sigma > 0$. Furthermore. let $a_{0,0} = 1$ and for $m > 1, 0 \le \mu \le m-1$ let $a_{\mu,m-1}$ be defined by*

$$Q_m(z) := \prod_{\mu=1}^{(m-1)/2} \left(1 + \frac{z^2}{\mu^2}\right) = \sum_{\mu=0}^{m-1} a_{\mu,m-1}\, z^\mu\,.$$

Then,

$$\int_{\to -\infty}^{\to \infty} f(x)\,\mathrm{d}x = \frac{\pi}{\sigma} \sum_{\mu=0}^{(m-1)/2} \frac{1}{(2\sigma)^{2\mu}}\, a_{2\mu,m-1} \sum_{\nu=-\infty}^{\infty} f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right) \tag{5.2}$$

holds for any entire function of exponential type τ less than $(m+1)\sigma$ if the integral on the left (taken in the sense of Cauchy) and the $(m+1)/2$ series on the right are convergent. □

Example 5.3. If $f(z) := \dfrac{\sin^m \sigma z}{z}\,(1 - \cos \sigma z)$, then

$$\int_{\to -\infty}^{\to \infty} f(x)\,\mathrm{d}x = \binom{m}{\frac{m+1}{2}} \quad \text{whereas} \sum_{\mu=0}^{(m-1)/2} \frac{1}{(2\sigma)^{2\mu}}\, a_{2\mu,m-1} \sum_{\nu=-\infty}^{\infty} f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right) = 0\,.$$

Thus, $\tau = (m+1)\sigma$ is inadmissible in Theorem 5.2.

Remark 5.2. Olivier and Rahman [36, Theorem 2] also proved that if $f \in L^1(-\infty\,, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \frac{\pi}{\sigma} \sum_{\mu=0}^{(m-1)/2} \frac{1}{(2\sigma)^{2\mu}}\, a_{2\mu,m-1} \sum_{\nu=-\infty}^{\infty} f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right) \tag{5.2$'$}$$

not only for entire functions of exponential type less than $(m+1)\sigma$ but also for those of order 1 type $(m+1)\sigma$.

Since we are dealing with entire functions of exponential type, it is natural to wonder if the requirement about the convergence of the $(m+1)/2$ series in (5.2) was really necessary. The following result of Boas and Schaeffer [10], applied to $\int_{1/2}^{z} f(\zeta)\,\mathrm{d}\zeta$, "suggests" that it might indeed be superfluous.

Proposition 5.1 *Let f be an entire function of exponential type π. Then, $f(x)$ approaches a limit as $x \to \infty$ if and only if $\sum_{k=1}^{\infty} f'(k+x)$ converges uniformly for $0 \le x \le 1$.* □

It was shown by Rahman and Schmeisser [37] that the conclusion of Proposition 5.1 remains true for any entire function of exponential type less than 2π. Thus, they proved that it was redundant to require the series in (5.1′) to be convergent. As regards (5.2) they established [37, Theorem 4] that if f is of exponential type less than $(m+1)\sigma$ and $\lim_{x\to\infty}\int_{1/2}^{x} f(t)\,\mathrm{d}t$ exists, then $\frac{\pi}{\sigma}\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{(m-1)/2}\frac{1}{(2\sigma)^{2\mu}}\,a_{2\mu,m-1}f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right)$ is necessarily convergent. In addition, they observed that the formula remains true even if the integral exists only as a Cauchy principal value. Let us recall that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists as a Cauchy principal value if $\int_0^{\infty}\{f(x)+f(-x)\}\,\mathrm{d}x$ exists in the sense of Cauchy, and that $\sum_{n=-\infty}^{\infty} a_n$ exists as a *Cauchy principal value* if $\sum_{n=1}^{\infty}(a_n+a_{-n})$ converges.

Rahman and Schmeisser [37, Corollary 4] proved that if f is of exponential type less than $(m+1)\sigma$ and $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists as a Cauchy principal value, then $\frac{\pi}{\sigma}\sum_{\nu=-\infty}^{\infty}\sum_{\mu=0}^{(m-1)/2}\frac{1}{(2\sigma)^{2\mu}}\,a_{2\mu,m-1}f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right)$ exists as a *Cauchy principal value*. Furthermore,

$$\int_0^{\to\infty}(f(x)+f(-x))\,\mathrm{d}x=\frac{\pi}{\sigma}\sum_{\nu=-\infty}^{\infty}\sum_{\mu=0}^{(m-1)/2}\frac{1}{(2\sigma)^{2\mu}}\,a_{2\mu,m-1}f^{(2\mu)}\left(\frac{\nu\pi}{\sigma}\right). \tag{5.3}$$

5.2. A Characterization of the Nodes $\left\{\frac{\nu\pi}{\sigma}\right\}_{\nu\in\mathbb{Z}}$ Appearing in (5.2) and (5.2′)

For any odd $m\in\mathbb{N}$, let $C_{\sigma,m}$ denote the class of all entire functions of exponential type σ such that (i) $g(0)=g'(0)-1=0$, that is, the Maclaurin series of g has the form $z+\sum_{\nu=2}^{\infty}c_\nu z^\nu$, and (ii) $\int_{-\infty}^{\infty}x^{-2}\,|g(x)|^{m+1}\,\mathrm{d}x$ exists. We shall prove that, analogously to the characterization of the nodes in the formula of Turán, *the nodes* $\{\nu\pi/\sigma\}_{\nu\in\mathbb{Z}}$, *appearing in* (5.2) *and* (5.2′), *are the zeros of that function in* $C_{\sigma,m}$, *which minimizes the integral* $\int_{-\infty}^{\infty}x^{-2}\,|g(x)|^{m+1}\,\mathrm{d}x$.

It suffices to show that if $g(z)\not\equiv g_*(z):=\left(\sin\sigma z\right)/\sigma$, then

$$\int_{-\infty}^{\infty}x^{-2}|g(x)|^{m+1}\,\mathrm{d}x>\int_{-\infty}^{\infty}x^{-2}|g_*(x)|^{m+1}\,\mathrm{d}x=\int_{-\infty}^{\infty}x^{-2}\left|\frac{\sin\sigma z}{\sigma}\right|^{m+1}\mathrm{d}x\,. \tag{5.4}$$

The function $\varphi(z):=g(z)-g_*(z)$ is of exponential type σ and is not identically zero. Furthermore, $\int_{-\infty}^{\infty}x^{-2}|\varphi(x)|^{m+1}\,\mathrm{d}x<\infty$ since

$$\varphi(0)=\varphi'(0)=0\,. \tag{5.5}$$

Using Hölder's inequality we conclude that the entire function

$$f(z):=\frac{1}{z^2}\left(g_*(z)\right)^m\varphi(z)\,, \tag{5.6}$$

which is clearly of exponential type $(m+1)\sigma$, belongs to $L^1(\mathbb{R})$.

Now, let us suppose that $g(x)$ is real for real x, consider the entire function

$$F(z) := \frac{1}{z^2}\,(g(z))^{m+1} - \frac{1}{z^2}\,(g_*(z))^{m+1} - (m+1)\frac{1}{z^2}\,(g_*(z))^m\,\varphi(z)\,.$$

It is of exponential type $(m+1)\sigma$, and belongs to $L^1(\mathbb{R})$. Bernoulli's inequality may be used to see that $F(x) > 0$ except at the zeros of g_* and φ. Hence $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x > 0$, that is,

$$\int_{-\infty}^{\infty} x^{-2}|g(x)|^{m+1}\,\mathrm{d}x > \int_{-\infty}^{\infty} x^{-2}|g_*(x)|^{m+1}\,\mathrm{d}x + (m+1)\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\,,$$

where f is as in (5.6). Taking (5.5) into account we see that

$$f\left(\frac{\nu\pi}{\sigma}\right) = \cdots = f^{(m-1)}\left(\frac{\nu\pi}{\sigma}\right) = 0 \qquad (\nu = 0,\,\pm 1.\,\pm 2,\,\ldots)\,.$$

Hence, by formula (5.2′), $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 0$, and so (5.4) holds.

5.3. Another Extension of Formula (5.1)

The Bessel function of the first kind of order α is defined by [42, p. 40]

$$J_\alpha(z) := \left(\frac{1}{2}z\right)^{\alpha}\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{(z/2)^{2\nu}}{\nu!\,\Gamma(\nu+\alpha+1)}\,, \tag{5.7}$$

Here, it may be added that by ζ^α, $\zeta \neq 0$ we mean $\exp(\alpha\log\zeta)$, where the logarithm has its principal value. From the coefficients in the expansion (5.7) for $J_\alpha(z)$ it is easily seen that the function $G_\alpha(z) := z^{-\alpha}J_\alpha(z)$ is an even entire function of order 1 type 1 and so is of exponential type 1. It is relevant to mention that

$$G_\alpha(z) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\,\cos z & \text{if } \alpha = -\dfrac{1}{2} \\ \sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin z}{z} & \text{if } \alpha = \dfrac{1}{2}\,. \end{cases}$$

According to a theorem of Lommel [42, p. 482], the zeros of J_α, for any $\alpha > -1$, are all real. They are all simple with the possible exception of $z = 0$. Arranging the positive zeros of J_α in increasing order of magnitude, we denote the kth zeros by $j_{k,\alpha}$ or by j_k for short. For any $k \in \mathbb{N}$, the zero $-j_{k,\alpha}$ of J_α by $j_{-k,\alpha}$ or simply by j_{-k}.

The next extension of (5.1) is suggested by the following well-known generalization [39] of the Gauss' quadrature formula.

Theorem I. *Let $x_{1,n} > \cdots > x_{n,n}$ be the zeros the Jacobi polynomial $P_n^{(\alpha,\beta)}$. Then, for any polynomial p of degree at most $2n-1$, we have*

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,p(x)\,\mathrm{d}x = \sum_{\nu=1}^{n}\lambda_\nu\,p(x_{\nu,n}) \qquad (\alpha > -1\,,\,\beta > -1)\,, \tag{5.8}$$

where the numbers $\lambda_1, \ldots, \lambda_n$ *are all positive.* □

According to a classical formula [40, Theorem 8.1.1],

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = \left(\frac{1}{2}\,z\right)^{-\alpha} J_\alpha(z) \qquad (\alpha > -1\,,\ \beta > -1)\,, \quad (5.9)$$

uniformly in every bounded region of the complex plane. This implies that if $x_{1,n} > \cdots, > x_{n,n}$ are the zeros of $P_n^{(\alpha,\beta)}$ and if we write $x_{k,n} = \cos\theta_{k,n}$, where $0 < \theta_{k,n} < \pi$, then for a fixed $k \geq 1$,

$$\lim_{n\to\infty} n\,\theta_{k,n} = j_k\,.$$

The following formula, which is to be compared with (5.2′) and (5.3), was proved by Frappier and Olivier [20].

Theorem 5.3. *Let* $\alpha > -1$*. Furthermore, let* f *be an entire function of exponential type* 2σ *such that for some* $\delta > 2\alpha + 2$,

$$f(x) = O\left(|x|^{-\delta}\right) \quad \text{as} \quad x \to \pm\infty\,.$$

Then,

$$\int_0^\infty x^{2\alpha+1}\big(f(x)+f(-x)\big)\,\mathrm{d}x = \frac{2}{\sigma^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{\big(J_k'(j_k)\big)^2}\left(f\left(\frac{j_k}{\sigma}\right)+f\left(-\frac{j_k}{\sigma}\right)\right). \quad (5.10)$$

□

Motivated by Theorem 5.1 and 5.1′, Grozev and Rahman sought to relax the restriction on the growth of f along the real axis and proved [25, Theorem 1, Theorem 2] the following results.

Theorem 5.3′. *Let* $\alpha > -1$*. Furthermore, let* f *be an entire function of exponential type* 2σ *such that* $x^{2\alpha+1}\big(f(x)+f(-x)\big)$ *belongs to* $L^1[0\,,\infty)$*. Then* (5.10) *holds.* □

Theorem 5.3″. *Let* $\alpha > -1$*. Furthermore, let* f *be an entire function of exponential type* $\tau < 2\sigma$ *such that* $x^{2\alpha+1}\big(f(x)+f(-x)\big)$ *is integrable in the sense of Cauchy on* $[0\,,\infty)$*. Then, formula* (5.10) *with* $\int_0^\infty x^{2\alpha+1}\big(f(x)+f(-x)\big)\,\mathrm{d}x$ *replaced by* $\int_0^{\to\infty} x^{2\alpha+1}\big(f(x)+f(-x)\big)\,\mathrm{d}x$*, holds if the series on the right is convergent.* □

Remark 5.3. It was proved by Ben Ghanem [4] that in Theorem 5.3″ the assumption about the convergence of the series on the right-hand side of (5.10) was superfluous. The reader will find some other extensions of Theorems 5.3′ and 5.3″ in [4].

Theorem 5.3′ contains the following result. It is a direct generalization of Theorem 5.1 to which it reduces in the case where $\alpha = -1/2$.

Theorem 5.1*. *Let* f *be an entire function of exponential type* 2σ *such that* $|x|^{2\alpha+1}f(x) \in L^1(\mathbb{R})$ *for some* $\alpha > -1$*. Then*

$$\int_{-\infty}^{\infty} |x|^{2\alpha+1} f(x)\,\mathrm{d}x = \frac{2}{\sigma^{2\alpha+2}} \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} \left|\frac{1}{G_\alpha'\left(|j_{\alpha,\nu}|\right)}\right|^2 f\left(\frac{j_{\alpha,\nu}}{\sigma}\right). \quad (5.1^*)$$

□

6. Lagrange Interpolation and Mean Convergence

According to Theorem G, no triangular matrix $\mathfrak{A}$ has the property that, for any continuous function $g : [-1, 1] \to \mathbb{R}$, the associated sequence of Lagrange interpolating polynomials $\{L_{n-1}(.\,; \mathfrak{A}\,; g)\}$, converges uniformly to g on $[-1, 1]$. However, it was shown by Marcinkiewicz [31] that if the points $x_{1,n}, \ldots, x_{n,n}$ appearing in the n-th row of $\mathfrak{A}$ are the zeros of T_n the Chebyshev polynomial of the first kind of degree n, then for each $g \in C[-1, 1]$, we have

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, |g(x) - L_{n-1}(x\,; \mathfrak{A}\,; g)|^p \, \mathrm{d}x = 0 \qquad (p > 0)\,; \tag{6.1}$$

so did Erdős and Feldheim [16] , at about the same time. The polynomials $T_0, T_1, \ldots$ are orthogonal with respect to the weight function $\dfrac{1}{\sqrt{1-x^2}}$. Given any weight function w on $[-1, 1]$ let $\{p_n(.\,; w)\}$ be the corresponding sequence of orthonormal polynomials. It was proved by Erdős and Turán [17] that if the points $x_{1,n}, \ldots, x_{nn}$ in the nth row of $\mathfrak{A}$ are the zeros of $p_n(.\,; w)$, then for each $g \in C[-1, 1]$,

$$\lim_{n\to\infty} \int_{-1}^{1} w(x) \, |g(x) - L_{n-1}(x\,; \mathfrak{A}\,; g)|^p \, \mathrm{d}x = 0 \qquad (0 < p \le 2)\,. \tag{6.2}$$

As regards the restriction on p in (6.2), it was noted by Askey [2, p. 77] that, for any given $p > 2$ there exists a weight function of the form $(1-x)^\alpha(1+x)^\beta$ such that $\int_{-1}^{1} w(x)\, |g(x) - L_{n-1}(x\,; \mathfrak{A}\,; g)|^p \, \mathrm{d}x$ does not tend to 0 as $n \to \infty$. In the positive direction he proved [3, Theorem 10] the following result.

Theorem J. *Let $w(x) := (1-x)^\alpha(1+x)^\beta$ and $P_n^{\alpha,\beta}$ be the corresponding orthogonal (Jacobi) polynomial of degree n, and let the points in* (3.1) *be the zeros of $P_n^{\alpha,\beta}$. Furthermore, let $0 < p < 4(\alpha+1)/(2\alpha+1)$. Then, for each continuous function $g : [-1, 1] \to \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{-1}^{1} (1-x)^\alpha(1+x)^\beta \, |g(x) - L_{n-1}(x\,; \mathfrak{A}\,; g)|^p \, \mathrm{d}x = 0 \tag{6.3}$$

when $\alpha, \beta \ge -1/2$ or when

$$|\alpha - k| \le 1 + \beta, \quad -1 < \beta < -\frac{1}{2}, \quad 2k = 2, 3, \ldots$$

□

In the special case where $\alpha = \beta = -1/2$, this result agrees with that of Marcinkiewicz and of Erdős and Feldheim cited above.

Remark 6.1. Whatever the triangular matrix $\mathfrak{A}$ of points in $(-1, 1)$ may be, there exists a continuous function g such that the sequence $\{L_{n-1}(.\,; \mathfrak{A}\,; g)\}$ of Lagrange interpolating polynomials does not converge uniformly to g as

$n \to \infty$. Why then the matrix whose n-th row consists of the zeros of $P_n^{(\alpha,\beta)}$ is fine for "weighted" L^p convergence, at least for some values of p? The secret seems to lie in the fundamental property of Gaussian quadrature illustrated in Theorem I. This hypothesis is not simply speculative but has been put forward by other authors (see for example [2], [44, Chapter X, p. 29]). Notice the form in which the weight $(1-x)^\alpha(1+x)^\beta$ appears in (6.3) and in the quadrature formula (5.8).

6.1. Mean Convergence of Lagrange Interpolating Functions of Exponential Type

We start out by mentioning a result of Marcinkiewicz [31] about the mean convergence of Lagrange interpolation trigonometric polynomials.

Theorem K . *For any* $n \in \mathbb{N}$, *let*

$$\theta_{k,n} := \frac{2k\pi}{2n+1} \qquad (k = 0, \pm 1, \ldots, \pm n).$$

In addition, let $g : \mathbb{R} \to \mathbb{C}$ *be continuous and* 2π*- periodic. Furthermore, denote by* $t_n(.\,; g)$ *the trigonometric interpolatory polynomial of degree not exceeding* n *with* $t_n(\theta_{k,n}; g) = g(\theta_{k,n})$ *for* $k = 0, \pm 1, \ldots, \pm n$. *Then,*

$$\lim_{n\to\infty} \int_0^{2\pi} |g(\theta) - t_n(\theta\,;\, g)|^p \, \mathrm{d}\theta = 0 \qquad (p > 0). \tag{6.4}$$

□

If g is continuous on $[-1, 1]$, then $g(\cos\theta)$ is defined for all real θ; it is continuous as well as 2π- periodic, and so Theorem J applies giving (6.1) as a corollary. This result is particulary interesting since $\sup_{n\to\infty} |t_n(\theta\,;\, g) = \infty$ for every θ if the continuous and 2π- periodic function g is suitably chosen (see [27] , [32]).

Now, we shall discuss Lagrange interpolation of *non-periodic* functions in an infinite set of points on $\mathbb{R}$.

A priori it is not clear what kind of points on $\mathbb{R}$ would be suitable for interpolation by entire functions of exponential type so as to obtain a convergence theorem like Theorem J of Askey and Theorem K of Marcinkiewicz. For $\mu \in \mathbb{Z}$,

$$\ell_{\mu,\tau}(z) := \begin{cases} \dfrac{\sin(\tau z - \mu\pi)}{\tau z - \mu\pi} & \text{if } z \neq \dfrac{\mu\pi}{\tau} \\ 1 & \text{if } z = \dfrac{\mu\pi}{\tau} \end{cases}$$

is an entire function of exponential type τ such that

$$\ell_{\mu,\tau}\left(\frac{\nu\pi}{\tau}\right) = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu\,. \end{cases}$$

To an arbitrary $g : \mathbb{R} \to \mathbb{C}$, let us formally associate

$$\mathfrak{L}_\tau(z\,;\,g) := \sum_{\nu=-\infty}^{\infty} g\left(\frac{\nu\pi}{\tau}\right) \ell_{\nu,\tau}(z)\,,$$

which interpolates g in the points $\nu\pi/\tau$. However, this associated function may not be defined at other points even if g is unformly continuous and bounded on the real axis. If g is continuous and vanishes outside some compact interval I, then $\mathfrak{L}_\tau(.\,;\,g)$ does define an entire function of exponential type bounded on the real axis. It can be shown [38, pp. 303–304] that given I there exists a continuous function g_I with support in I such that $\max_{x\in I}|g_I(x) - \mathfrak{L}_\tau(x\,;\,g_I)|$ does not remain bounded as $\tau \to \infty$. This can be seen as an analogue of the result of Grünwald [27] and Marcinkiewicz [32] in the periodic case. Is there also an analogue of (6.4) in this case, and if so what it is? We would like that for any $p > 0$ and a large class of functions $g \in L^p(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |g(x) - \mathfrak{L}_\tau(x\,;\,g)|^p \,\mathrm{d}x \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{6.5}$$

Definition 6.1. *For any* $p > 1$ *let* $\mathcal{F}^p(\delta)$ *denote the set of all measurable functions* $g : \mathbb{R} \to \mathbb{C}$ *such that* $(|x| + 1)^{\frac{1}{p}+\delta} g(x)$ *is bounded on the real axis for some* $\delta > 0$*, and let* $\mathcal{F}^p := \cup_{\delta>0}\mathcal{F}^p(\delta)$.

Clearly, $\mathcal{F}^p \subset L^p(\mathbb{R})$.

Definition 6.2. *Let* $\mathcal{R}$ *denote the set of all functions* $g : \mathbb{R} \to \mathbb{C}$ *which are Riemman integrable on every finite interval.*

The following result, which may be seen as an analogue of Theorem J, was proved by Rahman and Vértesi [38].

Theorem 6.1. *For any* $p > 1$*, let* $\mathcal{F}^p$ *and* $\mathcal{R}$ *be as in Definition 6.1 and Definition 6.2, respectively. Then,* (6.5) *holds if* $g \in \mathcal{F}^p \cap \mathcal{R}$. □

The conditions imposed on g in Theorem 6.1 do not seem to us to be the weakest possible. If $g \in \mathcal{F}^p \cap \mathcal{R}$ for some $p > 1$, then [38, Lemma 13],

$$\lim_{\tau\to\infty} \frac{\pi}{\tau} \sum_{\nu=-\infty}^{\infty} \left|g\left(\frac{\nu\pi}{\tau}\right)\right| = \int_{-\infty}^{\infty} |g(x)|^p \,\mathrm{d}x\,. \tag{6.6}$$

The proof of Theorem 6.1 required several auxiliary resluts, some known and others new at the time. In it, a central role is played by the function

$$S_\tau(z\,;\,g) := \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\,\frac{\sin\tau(z-t)}{z-t}\,\mathrm{d}t \qquad (\tau > 0)\,,$$

which certainly exists if $g : \mathbb{R} \to \mathbb{C}$ belongs to $L^p(\mathbb{R})$ for some $p > 1$. It is easily seen that if g belongs to $L^p(\mathbb{R})$ for some $p > 1$, then $S_\tau(z\,;\,g)$ is an entire function of exponential type τ bounded on $\mathbb{R}$. Also [38, Lemma 9],

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty}|S_\tau(x\,;\,g)-g(x)|^p\;\mathrm{d}x=0\,.$$

The following result was mentioned there as Lemma 5 but its proof, though non-trivial, was inadvertantly omitted. We take this opportunity to include the details here.

Proposition 6.1. *Let $g\in L^p(\mathbb{R})$ for some $p>1$. Then,*

$$\int_{-\infty}^{\infty}|S_\tau(x\,;\,g)|^p\;\mathrm{d}x\le K_p\int_{-\infty}^{\infty}|g(x)|^p\;\mathrm{d}x\,,$$

where K_p depends on p only.

Proof. Since $g\in L^p(\mathbb{R})$, $p>1$, the Hilbert transform

$$g^{\sim}(x):=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{x-t}\,\mathrm{d}t=-\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{g(x+t)-g(x-t)}{t}\,\mathrm{d}t$$

exists almost everywhere, and [44, p. 256]

$$\int_{-\infty}^{\infty}|g^{\sim}(x)|^p\;\mathrm{d}x\le c_p\int_{-\infty}^{\infty}|g(x)|^p\;\mathrm{d}x\,,$$

where c_p depends on p only. If $g_1(x):=g(x)\,\cos\tau x$ and $g_2(x):=g(x)\,\sin\tau x$, then

$$|S_\tau(x\,;\,g)|=|(\sin\tau x)g_1^{\sim}(x)-\cos\tau x)g_2^{\sim}(x)|\le|g_1^{\sim}(x)|+|g_1^{\sim}(x)|$$

and so

$$\begin{aligned}\left(\int_{-\infty}^{\infty}|S_\tau(x\,;\,g)|^p\;\mathrm{d}x\right)^{1/p}&\le\left(\int_{-\infty}^{\infty}|g_1^{\sim}(x)|^p\;\mathrm{d}x\right)^{1/p}+\left(\int_{-\infty}^{\infty}|g_2^{\sim}(x)|^p\;\mathrm{d}x\right)^{1/p}\\&\le c_p\left(\int_{-\infty}^{\infty}|g_1(x)|^p\;\mathrm{d}x\right)^{1/p}+c_p\left(\int_{-\infty}^{\infty}|g_2(x)|^p\;\mathrm{d}x\right)^{1/p}\\&\le 2\,c_p\left(\int_{-\infty}^{\infty}|g(x)|^p\;\mathrm{d}x\right)^{1/p}.\end{aligned}$$

□

6.2. An Analogue of Theorem J

Let J_α be the Bessel function of order $\alpha>-1$. Arranging the positive zeros of J_α in increasing order as before, we use $j_{\alpha,k}$ or simply j_k to denote the k-th one. Recall that $G_\alpha(z):=z^{-\alpha}J_\alpha(z)$ is an entire function of order 1 type 1. Let $\tau>0$. To any $g:\mathbb{R}\to\mathbb{C}$, let us formally associate

$$\mathfrak{L}_{\tau,\alpha}(z\,;\,g):=\sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty}\frac{G_\alpha(\tau z)}{G'_\alpha(j_\nu)\,(\sigma z-j_\nu)}\,g\left(\frac{j_\nu}{\tau}\right),\tag{6.7}$$

which interpolates g in the points j_ν/τ for $\nu \in \mathbb{Z}\backslash\{0\}$. The following result [26, Theorem 1] about the mean convergence of $\{\mathfrak{L}_{\sigma,\alpha}(.\,;g)\}$, as $\tau \to \infty$, can be seen as an analogue of Theorem J. In the special case where $\alpha = -1/2$, it is equivalent to Theorem 6.1.

Theorem 6.2. *Let* $\alpha \geq -1/2$ *and* $p > 1$, *or let* $-1 < \alpha < -1/2$ *and* $1 < p < 2/|2\alpha+1|$. *Furthermore, let* $g : \mathbb{R} \to \mathbb{C}$ *be Riemann integrable on every finite interval and satisfy*

$$g(x) = O\left(\frac{1}{(|x|+1)^{\alpha+\frac{1}{2}+\frac{1}{p}+\delta}}\right) \qquad (x \in \mathbb{R})$$

for some $\delta > 0$. *Then*

$$\int_{-\infty}^{\infty} \left|x^{\alpha+\frac{1}{2}}\big(g(x) - \mathfrak{L}_{\tau,\alpha}(x\,;g)\big)\right|^p \mathrm{d}x \to 0 \quad \text{as } \sigma \to \infty\,. \tag{6.8}$$

□

It may be added that if $\alpha \geq -1/2$, then $\sup_{x\in\mathbb{R}}\left|x^{\alpha+\frac{1}{2}}\big(g(x) - \mathfrak{L}_{\tau,\alpha}(x\,;g)\big)\right|$ may not tend to zero as $\tau \to \infty$ even if the support of g lies in $[0\,,1]$.

Remark 6.2. We know [26, §3.4] that if $\alpha \geq -1/2$, then (6.8) can be replaced by

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty} |x|^{2\alpha+1}\, |g(x) - \mathfrak{L}_{\tau,\alpha}(x\,;g)|^p \,\mathrm{d}x = 0 \tag{6.8$'$}$$

provided that $p \geq 2$.

Remark 6.3. Refer to Remark 6.1, and then notice the form in which the weight $|x|^{2\alpha+1}$ appears in (6.8′) and in the quadrature formula (5.1*). This may help understand the *raison d'être* of the weight $|x|^{2\alpha+1}$ in (6.8′).

7. Appendix

In this section, the reader will find with proof a theorem of Phragmén and Lindelöf, which is of fundamental importance in the theory of entire functions of exponential type, and so may be known to most of the potential readers. We also present a complete proof of a result of Carleman furnishing certain nontrivial details he apparently had no time for.

7.1. A Theorem of Phragmén and Lindelöf

The following result due to Phragmén and Lindelöf (see [40, pp. 176–178]) is of great significance in the study of entire funtions of exponential type. It can be seen as a generalization of the maximum modulus plinciple.

Proposition 7.1. *Let* f *be holomorphic in the angle*

$$S_{\theta_0} := \left\{z = r\mathrm{e}^{\mathrm{i}\theta} : 0 < r < \infty,\ |\theta - \theta_0| < \frac{\pi}{2\alpha}\right\},$$

continuous in the closed angle, $|f(z)| \leq M$ *on the boundary,* $f\left(re^{i\theta}\right) = O\left(e^{r^\beta}\right)$, $\beta < \alpha$*, uniformly in* θ*, for* $r = r_n \to \infty$*. Then* $|f(z)| \leq M$ *throughout* S_{θ_0}.

Proof. We take $\theta_0 = 0$ since there is no loss of generality in doing so. Let $F(z) := e^{-\varepsilon z^\gamma} f(z)$, where $\beta < \gamma < \alpha$. Then

$$\left|F\left(re^{i\theta}\right)\right| = e^{-r^\gamma \cos\gamma\theta} \, |f(z)| \, .$$

Hence,

$$\left|F\left(re^{\pm i\pi/2\alpha}\right)\right| \leq e^{-\varepsilon r^\gamma \cos(\gamma\pi/2\alpha)} M \leq M \, .$$

Also on the arc $|\theta| < \pi/2\alpha$ of the circle $|z| = r_n$,

$$|F(z)| \leq e^{-r_n^\gamma \cos\gamma\theta} \, |f(z)| = O\left(e^{r_n^\beta - \varepsilon r_n^\gamma \cos(\gamma\pi/2\alpha)}\right) ,$$

and the right-hand side tends to 0 as $n \to \infty$. Hence, if n is sufficiently large, $|F(z)| \leq M$ on the arc $z = r_n \, e^{i\theta}$, $-\pi/2\alpha \leq \theta \leq \pi/2\alpha$. By the maximum modulus principle, $|f(z)| \leq M$ throughout

$$\overline{S}_{0,n} := \left\{ z = re^{i\theta} : 0 \leq r \leq r_n \, , \, -\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha} \right\} .$$

Letting n tend to ∞, we conclude that $|F(z)| \leq M$ in the closed angle

$$\overline{S}_0 := \left\{ z = re^{i\theta} : 0 \leq r < \infty \, , \, -\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha} \right\} .$$

Finally, making ε tend to 0 the desired result follows. □

7.2. A Theorem of Carleman

In order to make the presentation self-contained we include a proof of the fact that *any* continuous function can be approximated arbitrarily closely on $(-\infty\,,\,\infty)$ by entire functions, though not necessarily by those of exponential type.

Theorem 7.1. *Let* g *be continuous on the whole real axis. Then, for any* $\varepsilon > 0$*, there exists an entire function* f *such that*

$$\sup_{-\infty < x < \infty} |g(x) - f(x)| < \varepsilon \, .$$

This result is due to Carleman [12] and so is the proof we present. We do add certain details he had chosen to omit but without which the reader might loose a good deal of time scratching his/her head.

The proof makes use of the fact that for any prescribed set of points $x_1, \ldots, x_n$ in $[a\,,\,b]$ and given $\varepsilon > 0$, we can find a polynomial p such that $|g(x) - p(x)| < \varepsilon$ for $a \leq x \leq b$ and $p(x_\nu) = g(x_\nu)$ for $\nu = 1, \ldots, n$. In fact, the following result (see [13, pp. 121–122]) holds.

Proposition 7.2. *Let* S *be a compact subset of the complex plane* $\mathbb{C}$*. Furthermore, let* $z_1, \ldots, z_n$ *be* n *distinct points in* $\mathbb{C}$*. Suppose that* g *is defined on* S *and is uniformly approximable by polynomials on that set. Then, for any given* $\varepsilon > 0$*, there* *exists a polynomial* p *such that* $|g(z) - p(z)| < \varepsilon$ *for any* $z \in S$, *and* $p(z_\nu) = g(z_\nu)$ *for* $\nu = 1, \ldots, n$.

Proof. Let $\omega(z) := (z - z_1) \cdots (z - z_n)$ and

$$\ell_\nu(z) := \frac{\omega(z)}{(z - z_\nu)\,\omega'(z_\nu)} \qquad (\nu = 1, \ldots, n)\,.$$

Furthermore, let

$$\mathbf{m} := \max_{z \in S} \sum_{\nu=1}^{n} |\ell_\nu(z)|\,.$$

According to the hypothesis, we can find a polynomial p_ε such that

$$|g(z) - p_\varepsilon(z)| < \frac{1}{1+\mathbf{m}}\,\varepsilon \qquad (z \in S)\,.$$

Then, $q(z) := \sum_{\nu=1}^{n} \big(g(z_\nu) - p_\varepsilon(z_\nu)\big)\,\ell_\nu(z)$ is the unique polynomial of degree at most $n-1$ such that $q(z_\nu) = g(z_\nu) - p_\varepsilon(z_\nu)$ for $\nu = 1, \ldots, n$, and

$$|q(z)| \le \max_{1 \le \nu \le n} |g(z_\nu) - p_\varepsilon(z_\nu)| \max_{z \in S} \sum_{\nu=1}^{n} |\ell_\nu(z)| < \frac{\mathbf{m}}{1+\mathbf{m}}\varepsilon \qquad (z \in S)\,.$$

Hence, setting $p := p_\varepsilon + q$ we see that

$$|g(z) - p(z)| \le |g(z) - p_\varepsilon(z)| + |q(z)| < \frac{1}{1+\mathbf{m}}\,\varepsilon + \frac{\mathbf{m}}{1+\mathbf{m}}\varepsilon = \varepsilon \qquad (z \in S)\,,$$

and $p(z_\nu) = g(z_\nu)$ for $\nu = 1, \ldots, n$. □

The proof of Theorem 7.1 also uses the following result due to C. Runge, for whose proof we refer the reader to [13, pp. 273 - 277].

Proposition 7.3. *Let* F *be holomorphic in a bounded simply-connected domain* D. *Then, there exists a sequence of polynomials* $p_1, p_2, \ldots$, *which converges to* F *in* D, *uniformly on compact sets.* □

The basic idea of the proof of Theorem 7.1 is contained in the following lemma.

Lemma 7.1. *For any two positive numbers* R *and* d *let* g_1 *and* g_2 *be continuous on* $[R\,, R+d]$ *and* $[-R-d\,, -R]$, *respectively. In addition, let* F *be an entire function such that* $F(R) = g_1(R)$ *and* $F(-R) = g_2(-R)$. *Then, to any* $\varepsilon > 0$ *there corresponds a polynomial* G_ε *such that*

$$\begin{aligned} |F(z) - G_\varepsilon(z)| &\le \varepsilon \qquad (|z| \le R)\,,\\ |g_1(x) - G_\varepsilon(x)| &\le \varepsilon \qquad (R \le x \le R+d)\,,\\ |g_2(x) - G_\varepsilon(x)| &\le \varepsilon \qquad (-R-d \le x \le -R)\,, \end{aligned}$$

and

$$\begin{aligned} G_\varepsilon(R) = g_1(R)\,, &\qquad G_\varepsilon(R+d) = g_1(R+d)\,,\\ G_\varepsilon(-R) = g_2(-R)\,, &\qquad G_\varepsilon(-R-d) = g_2(-R-d)\,. \end{aligned}$$

Proof. In view of Proposition 7.2, we can find a polynomial p_1 such that

$$p_1(R) = g_1(R)\,, \qquad p_1(R+d) = g_1(R+d) \tag{7.1}$$

and

$$\max_{R\le x\le R+d} |p_1(x) - g_1(x)| < \frac{\varepsilon}{3}\,. \tag{7.2}$$

We can also find a polynomial p_2 such that

$$p_2(-R) = g_2(-R)\,, \qquad p_2(-R-d) = g_2(-R-d) \tag{7.3}$$

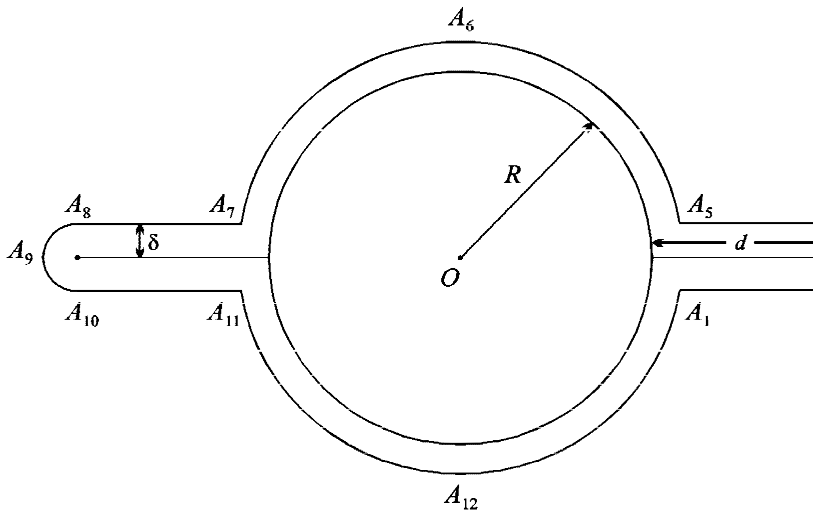
and

$$\max_{-R-d\le x\le -R} |p_2(x) - g_2(x)| < \frac{\varepsilon}{3}\,. \tag{7.4}$$

Denote by U_δ the union of the open sets $\{z\,:\,|z| < R+\delta\}$,

$$O_1 := \bigcup_{R\le t\le R+d} \{z\,:\,|z-t| < \delta\} \quad \text{and} \quad O_2 := \bigcup_{-R-d\le t\le -R} \{z\,:\,|z-t| < \delta\}\,.$$

The boundary of U_δ is a Jordan curve which we denote by Γ_δ. We specify twelve different points $A_1, A_2, \dots, A_{12}$, lying on Γ_δ, as indicated in the figure.



For sake of clarity, we wish to specify the complex numbers these points correspond to. The point A_1 corresponds to the complex number $\zeta_1 = (R+\delta)e^{-i\theta}$, where $\theta \in (0\,,\pi/2)$ and $\sin\theta = \delta/(R+\delta)$; A_5 corresponds to $\overline{\zeta}_1$; A_7 to $-\zeta_1$ and A_{11} to $-\overline{\zeta}_1$. The point A_2 corresponds to $\zeta_2 = R+d-i\delta$; A_4 corresponds to $\overline{\zeta}_2$; A_8 to $-\zeta_2$ and a_{10} to $-\overline{\zeta}_2$. The point A_3 corresponds to $\zeta_3 = R+d+\delta$ and A_9 to $-\zeta_3$. The point A_6 corresponds to $\zeta_4 = i\,(R+\delta)$ and A_{12} to $\overline{\zeta}_4$. We shall find it convenient to denote by γ_1 the arc comprised of the directed line segment $\overrightarrow{A_1A_2}$ followed by the semi-circular arc joining A_2 to A_4 via A_3 and continued to A_5 by the directed line segment $\overrightarrow{A_4A_5}$ We use γ_2 to denote the circular arc of radius $R+\delta$ having A_5 as initial point, A_7 as its final point and A_6 as mid-point. By γ_3 we mean the arc consisting of the directed line segment $\overrightarrow{A_7A_8}$ followed first by the semi-circular arc joining A_8 to A_{10} via A_9 and then by the line segment $\overrightarrow{A_{10}A_{11}}$.

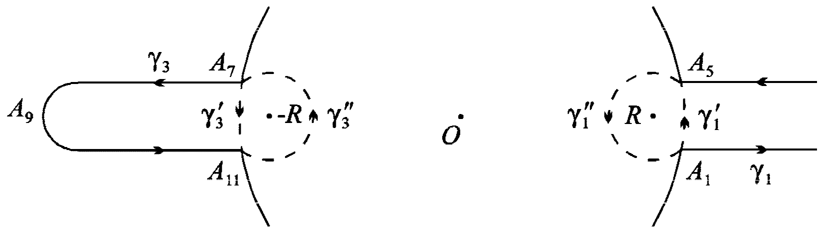
Finally, we use γ_4 to denote the circular arc of radius $R+\delta$ with A_{11} as initial point, A_1 as final point and A_{12} as mid-point.

Now, let

$$I_1(z) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{p_1(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \quad (z \notin \gamma_1) \quad , \quad I_2(z) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_2} \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \quad (z \notin \gamma_2)\,,$$

$$I_3(z) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3} \frac{p_2(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \quad (z \notin \gamma_3) \quad , \quad I_4(z) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma_4} \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \quad (z \notin \gamma_4)\,,$$

and $\mathcal{I}_\delta(z) := \sum_{j=1}^4 I_j(z)$. For any $j \in \{1,2,3,4\}$, the function I_j is holomorphic in $\mathbb{C}\backslash\gamma_j$, and so $\mathcal{I}_\delta$ is holomorphic inside the Jordan curve $\Gamma_\delta := \gamma_1+\gamma_2+\gamma_3+\gamma_4$.



Using Cauchy's contour integration theorem we see that if γ_1' is the circular arc of radius $R+\delta$ going from A_1 to A_5 via the point $R+\delta$, then

$$I_1(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{p_1(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{p_1(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \qquad (|z| < R+\delta)\,.$$

Similarly, if γ_3' is the circular arc of radius $R+\delta$ going from A_7 to A_{11} via the point $-R-\delta$, then

$$I_3(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{p_2(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{p_2(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \qquad (|z| < R+\delta)\,.$$

Thus, for any z in the open disk of radius $R+\delta$ centred at the origin, we have

$$\mathcal{I}_\delta(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'+\gamma_2+\gamma_3'+\gamma_4} \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{p_1(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{p_2(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta\,,$$

that is,

$$\mathcal{I}_\delta(z) - F(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{p_1(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{p_2(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \qquad (|z| < R+\delta)\,.$$

The function $p_1 - F$ is holomorphic in $|\zeta| < R+\delta$ and vanishes at R. Hence, $p_1(\zeta) - F(\zeta) = O(\zeta - R)$ for $\zeta \in \gamma_1'$ as $\delta \to 0$. Since the length of γ_1' tends to zero as $\delta \to 0$, we conclude that as $\delta \to 0$,

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma_1'} \frac{p_1(\zeta) - F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \to 0$$

uniformly for $|z| \le R$. The same can be said about the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma_3'} \frac{p_2(\zeta)-F(\zeta)}{\zeta-z}\, \mathrm{d}\zeta\,.$$

Thus, $\max_{|z|\le R} |\mathcal{I}_\delta(z) - F(z)| \to 0$ as $\delta \to 0$.

We shall now show that

$$\max_{x\in[R\,,\,R+d]\cup[-R-d\,,\,-R]} |\mathcal{I}_\delta(x) - p_1(x)| \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Let γ_1'' denote that part of the circle centred at R which originates at A_5, terminates at A_1 and does not intersect with the line segment $[R\,,\,R+d]$. Furthermore, let γ_3'' denote that part of the circle, centred at the point $-R$, which originates at A_{11}, terminates at A_7 and does not intersect with the line segment $[-R-d\,,\,-R]$. Then, for $x \in [R\,,\,R+d]$, we have

$$\begin{aligned} I_2(x) + I_4(x) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{F(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{F(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{F(\zeta)-p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \\ &\quad + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{F(\zeta)-p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta\,, \end{aligned}$$

and so

$$\begin{aligned} \mathcal{I}_\delta(x) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{F(\zeta)-p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{F(\zeta)-p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \\ &\quad + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1+\gamma_1''} \frac{p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3+\gamma_3''} \frac{p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta\,. \end{aligned}$$

By Cauchy's integral formula, the value of the third integral is $p_1(x)$, and that of the fourth is 0 by Cauchy's contour integration theorem. Hence,

$$\mathcal{I}_\delta(x) - p_1(x) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{F(\zeta)-p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{F(\zeta)-p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \qquad (R \le x \le R+d)\,.$$

Since $\left|\dfrac{F(\zeta)-p_1(\zeta)}{\zeta-x}\right|$ is uniformly bounded on γ_1'' for all $x \in [R\,,\,R+d]$, and the length of the arc γ_1'' tends to 0 as δ tends to zero, we see that

$$\max_{R\le x\le R+d} \left| \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1''} \frac{F(\zeta)-p_1(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \right| \to 0 \quad \text{as} \quad \delta \to 0\,.$$

For a similar reason

$$\max_{R\le x\le R+d} \left| \frac{1}{2\pi \mathrm{i}} \int_{\gamma_3''} \frac{F(\zeta)-p_2(\zeta)}{\zeta-x}\, \mathrm{d}\zeta \right| \to 0 \quad \text{as} \quad \delta \to 0\,.$$

Hence, $\max_{R\le x\le R+d} |\mathcal{I}_\delta(x) - p_1(x)| \to 0$ as $\delta \to 0$.

We omit the proof of the fact that $\max_{-R\le x\le -R-d} |\mathcal{I}_\delta(x) - p_1(x)| \to 0$ as $\delta \to 0$.

Thus, $\max_{|z|\le R}|F(z)-\mathcal{I}_\delta(z)|$ together with

$$\max_{R\le x\le R+d}|p_1(x)-\mathcal{I}_\delta(x)| \quad \text{and} \quad \max_{-R-d\le x\le -R}|p_2(x)-\mathcal{I}_\delta(x)|$$

can be made as small as we like by taking δ sufficiently small. Now, note that

$$q_1(z) := \frac{(z-R-d)(z+R)(z+R+d)}{-d(2R)(2R+d)} \tag{7.5}$$

is the trinomial which takes the value 1 at R and vanishes at $R+d$, $-R$, $-R-d$;

$$q_2(z) := \frac{(z-R)(z+R)(z+R+d)}{d(2R+d))(2R+2d)} \tag{7.6}$$

is the trinomial which takes the value 1 at $R+d$ and vanishes at R, $-R$, $-R-d$;

$$q_3(z) := \frac{(z-R)(z-R-d)(z+R+d)}{-2R(-2R-d)(d)} \tag{7.7}$$

is the trinomial which takes the value 1 at $-R$ and vanishes at R, $R+d$, $-R-d$; and

$$q_4(z) := \frac{(z+R)(z-R)(z-R-d)}{-d(-2R-d))(-2R-2d)} \tag{7.8}$$

is the trinomial which takes the value 1 at $-R-d$ and vanishes at R, $R+d$, $-R$. Hence,

$$\begin{aligned} Q_\delta(z) \ :=\ & (p_1(R)-\mathcal{I}_\delta(R))\, q_1(z) + (p_1(R+d)-\mathcal{I}_\delta(R+d))\, q_2(z) \\ & + (p_2(-R)-\mathcal{I}_\delta(-R))\, q_3(z) + (p_2(-R-d)-\mathcal{I}_\delta(-R-d))\, q_4(z) \end{aligned}$$

is the polynomial of degree 3, which agrees with $p_1-\mathcal{I}_\delta$ at the points R, $R+\delta$ and with $p_2-\mathcal{I}_\delta$ at the points $-R$, $-R-d$. It is clear that $\max_{|z|\le R}|Q_\delta(z)|$ together with $\max_{R\le x\le R+d}|Q_\delta(x)|$ and $\max_{-R-d\le x\le -R}|Q_\delta(x)|$ can be made as small as we like by taking δ sufficiently small. Thus, if $\mathfrak{I}_\delta := \mathcal{I}_\delta + Q_\delta$, then there exists a positive number Δ such that

$$\max_{|z|\le R}|F(z)-\mathfrak{I}_\delta(z)|\,, \quad \max_{R\le x\le R+d}|p_1(x)-\mathfrak{I}_\delta(x)| \quad \text{and} \quad \max_{-R-d\le x\le -R}|p_2(x)-\mathfrak{I}_\delta(x)|$$

are all less than $\varepsilon/3$ for $0<\delta<4\Delta$. In addition,

$$\mathfrak{I}_\delta(x) = p_1(x) \ \text{ for } x\in\{R\,,\, R+d\} \quad \text{and} \quad \mathfrak{I}_\delta(x) = p_2(x) \ \text{ for } x\in\{-R\,,\,-R-d\}\,.$$

Hence, taking (7.5) and (7.7) into account, we see that

$$\max_{|z|\le R}|F(z)-\mathfrak{I}_\delta(z)| < \frac{\varepsilon}{3}\,,$$

$$\max_{R\le x\le R+d}|p_1(x)-\mathfrak{I}_\delta(x)| < \frac{2\varepsilon}{3} \quad \text{and} \quad \max_{-R-d\le x\le -R}|p_2(x)-\mathfrak{I}_\delta(x)| < \frac{2\varepsilon}{3}$$

for $0<\delta<4\Delta$. Besides, in view of (7.1) and (7.3),

$$\mathfrak{I}_\delta(x) = g_1(x) \ \text{ for } x\in\{R\,,\, R+d\} \quad \text{and} \quad \mathfrak{I}_\delta(x) = g_2(x) \ \text{ for } x\in\{-R\,,\,-R-d\}\,.$$

In order to complete the proof of the lemma, it is now enough to prove the existence of a polynomial G_ε such that

$$|\mathfrak{I}_{3\Delta}(z) - G_\varepsilon(z)| < \frac{\varepsilon}{3} \qquad (z \in U_\Delta) \tag{7.9}$$

and

$$G_\varepsilon(x) = \mathfrak{I}_{3\Delta}(x) \qquad (x \in \{-R,\ -R-d,\ R,\ R+d\})\ . \tag{7.10}$$

The existence of such a polynomial G_ε is clear from Propositions 7.2 and 7.3. However, in the case at hand, the simply-connected domain to which Runge's theorem is applied happens to be "starlike" with respect to the origin. This helps us to simplify the argument, a little. So, for sake of completeness and as a service to the reader we are going to present the details which remain difficult to grasp. We shall present it in two steps.

Step I. For the trinomials q_1, q_2, q_3 and q_4 introduced in (7.5), (7.6), (7.7) and (7.8), respectively, let

$$\mathfrak{m}_j := \sup_{z\in\Gamma_\Delta} |q_j(z)| \quad (j = 1,\ 2,\ 3,\ 4) \qquad \text{and} \qquad \mathfrak{M} := \sum_{j=1}^{4} \mathfrak{m}_j\ . \tag{7.11}$$

For sake of simplicity, we shall write $\mathfrak{I}$ for $\mathfrak{I}_{3\Delta}$. The function $\mathfrak{I}$ is holomorphic inside $\Gamma_{3\Delta}$ and so, in particular,

$$\mathfrak{I}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{3\Delta}} \frac{\mathfrak{I}(w)}{w-z}\,\mathrm{d}w \qquad (z \in U_\Delta)\ . \tag{7.12}$$

Let $\ell(\Gamma_{3\Delta})$ denote the length of $\Gamma_{3\Delta}$. For any z in U_Δ, and w', w'' on $\Gamma_{3\Delta}$, we clearly have

$$\begin{aligned} \left|\frac{\mathfrak{I}(w'')}{w''-z} - \frac{\mathfrak{I}(w')}{w'-z}\right| &= \left|\frac{\mathfrak{I}(w'')-\mathfrak{I}(w')}{w''-z} + \mathfrak{I}(w')\left(\frac{1}{w''-z} - \frac{1}{w'-z}\right)\right| \\ &\le \frac{1}{\Delta}\left|\mathfrak{I}(w'')-\mathfrak{I}(w')\right| + |\mathfrak{I}(w')|\,\frac{1}{\Delta^2}\,|w''-w'|\,. \end{aligned}$$

Hence, there exists a positive number δ' such that

$$\left|\frac{\mathfrak{I}(w'')}{w''-z} - \frac{\mathfrak{I}(w')}{w'-z}\right| < \frac{\pi}{3\,\ell(\Gamma_{3\Delta})\,(1+\mathfrak{M})}\varepsilon$$

if $|w''-w'| < \delta'$. Determine sufficiently many points $w_1, w_2, \ldots, w_N = w_1$ on $\Gamma_{3\Delta}$ so that $|w - w_\nu| < \delta'$ for w in the arc $w_\nu w_{\nu+1}$, where $\nu = 1, \ldots, N$. Then clearly,

$$\begin{aligned} \left|\int_{\Gamma_{3\Delta}} \frac{\mathfrak{I}(w)}{w-z}\,\mathrm{d}w - \sum_{\nu=1}^{N} \frac{\mathfrak{I}(w_\nu)}{w_\nu - z}\,(w_{\nu+1}-w_\nu)\right| &= \left|\sum_{\nu=1}^{N}\int_{w_\nu}^{w_{\nu+1}} \left(\frac{\mathfrak{I}(w)}{w-z} - \frac{\mathfrak{I}(w_\nu)}{w_\nu-z}\right)\mathrm{d}w\right| \\ &\le \sum_{\nu=1}^{N}\int_{w_\nu}^{w_{\nu+1}} \left|\frac{\mathfrak{I}(w)}{w-z} - \frac{\mathfrak{I}(w_\nu)}{w_\nu-z}\right|\,|\mathrm{d}w| \\ &< \frac{\pi}{3\,\ell(\Gamma_{3\Delta})\,(1+\mathfrak{M})}\varepsilon \sum_{\nu=1}^{N}\int_{w_\nu}^{w_{\nu+1}} |\mathrm{d}w| \\ &= \frac{\pi}{3\,(1+\mathfrak{M})}\,\varepsilon\,. \end{aligned}$$

Hence, in view of (7.12), we obtain

$$\left| \mathfrak{I}(z) - \sum_{\nu=1}^{N} \frac{\mathfrak{I}(w_\nu)\ (w_{\nu+1} - w_\nu)}{2\pi \mathrm{i}} \frac{1}{w_\nu - z} \right| < \frac{1}{6\,(1+\mathfrak{M})}\,\varepsilon \qquad (z \in U_\Delta)\,. \tag{7.13}$$

Step II. Let $m := \lfloor \frac{d}{\Delta} \rfloor + 1$. Furthermore, for $\nu = 1, \dots, N$, let

$$w_{\nu\mu} := w_\nu + \mu\,\Delta\,\frac{w_\nu}{|w_\nu|} \qquad (\mu = 0, \dots, m)\,.$$

Note that $w_{\nu 0} = w_\nu$, and that

$$\frac{1}{w_\nu - z} = \frac{1}{w_{\nu 1} - z} - \frac{w_{\nu 0} - w_{\nu 1}}{(w_{\nu 1} - z)^2} + \frac{(w_{\nu 0} - w_{\nu 1})^2}{(w_{\nu 1} - z)^3} - \cdots\,,$$

where the series converges uniformly for all z inside and on Γ_Δ since $\left|\frac{w_{\nu 0} - w_{\nu 1}}{w_{\nu 1} - z}\right| \le \frac{1}{2}$. We can therefore find a rational function $R_{\nu 1}(z) := \sum_{0 \le \kappa \le k_{\nu 1}} \frac{\alpha_{\nu 1,\kappa}}{(w_{\nu 1} - z)^\kappa}$ such that

$$\left| \frac{\mathfrak{I}(w_\nu)\ (w_{\nu 2} - w_1)}{2\pi \mathrm{i}} \frac{1}{w_\nu - z} - R_{\nu 1}(z) \right| < \frac{1}{6\,(1+\mathfrak{M})Nm}\,\varepsilon \qquad (z \in \Gamma_\Delta \cup U_\Delta)\,.$$

Working with $\frac{1}{(w_{\nu 1} - z)^\kappa}$ for any given κ, we can similarly find a rational function $R_{\nu 2}(z) := \sum_{0 \le \kappa \le k_{\nu 2}} \frac{\alpha_{\nu 2,\kappa}}{(w_{\nu 2} - z)^\kappa}$ such that

$$|R_{\nu 1}(z) - R_{\nu 2}(z)| < \frac{1}{6\,(1+\mathfrak{M})Nm}\,\varepsilon \qquad (z \in \Gamma_\Delta \cup U_\Delta)\,.$$

Thus,

$$\left| \frac{\mathfrak{I}(w_\nu)\ (w_{\nu 2} - w_1)}{2\pi \mathrm{i}} \frac{1}{w_\nu - z} - R_{\nu 2}(z) \right| < 2\,\frac{1}{6\,(1+\mathfrak{M})Nm}\,\varepsilon \qquad (z \in \Gamma_\Delta \cup U_\Delta)\,.$$

This procedure can be continued leading us to a rational function

$$R_{\nu m}(z) := \sum_{0 \le \kappa \le k_{\nu m}} \frac{\alpha_{\nu m,\kappa}}{(w_{\nu m} - z)^\kappa}$$

such that

$$\left| \frac{\mathfrak{I}(w_\nu)\ (w_{\nu 2} - w_1)}{2\pi \mathrm{i}} \frac{1}{w_\nu - z} - R_{\nu m}(z) \right| < \frac{1}{6\,(1+\mathfrak{M})N}\,\varepsilon \qquad (z \in \Gamma_\Delta \cup U_\Delta)\,.$$

Now, note that $|w_{\nu m}| \ge R + d + 2\Delta$, and so $R_{\nu m}$ is holomorphic in the disc $|z| < R + d + 2\delta$. There exists therefore a polynomial $\wp_\nu$ such that

$$|R_{\nu m}(z) - \wp_\nu(z)| < \frac{1}{6\,(1+\mathfrak{M})N}\,\varepsilon \qquad (|z| < R + d + 2\delta)\,.$$

Thus,

$$\left| \frac{\mathfrak{I}(w_\nu)\ (w_{\nu 2} - w_1)}{2\pi\mathrm{i}} \, \frac{1}{w_\nu - z} - \wp_\nu(z) \right| < \frac{1}{3\,(1+\mathfrak{M})N}\, \varepsilon \qquad (z \in \Gamma_\Delta \cup U_\Delta)\ ,$$

and from (7.16) we see that $\wp := \sum_{\nu=1}^{N} \wp_\nu$ is a polynomial such that

$$|\mathfrak{I}(z) - \wp(z)| < \frac{1}{3\,(1+\mathfrak{M})}\, \varepsilon \qquad (z \in U_\Delta)\ . \tag{7.14}$$

If q_1, q_2, q_3 and q_4 are as in (7.5), (7.6), (7.7), and (7.8), respectively, then the trinomial

$$\begin{aligned} \mathfrak{Q}(z) \ :=\ & (\mathfrak{I}(R) - \wp(R))\ q_1(z) + (\mathfrak{I}(R+d) - \wp(R+d))\ q_2(z) \\ & + (\mathfrak{I}(-R) - \wp(-R))\ q_3(z) + (\mathfrak{I}(-R-d) - \wp(-R-d))\ q_4(z) \end{aligned}$$

takes the same values as the function $\mathfrak{I} - \wp$ in the points R, $R+d$, $-R$ and $-R-d$, and in view of (7.11) and (7.14),

$$|\mathfrak{Q}(z)| < \frac{\mathfrak{M}}{3\,(1+\mathfrak{M})}\, \varepsilon \qquad (z \in U_\Delta)\ . \tag{7.15}$$

Hence, $G_\varepsilon := \wp + \mathfrak{Q}$ has the properties required in (7.9) and (7.10). □

Proof of Theorem 7.1. Let $\alpha_1, \alpha_2, \dots$ be an infinite sequence of positive numbers with $\sum_{n=1}^{\infty} \alpha_n < \varepsilon$. Proposition 7.2 allows us to choose a polynomial P_1 such that $P_1(\pm 1) = g(\pm 1)$ and $|g(x) - P_1(x)| < \alpha_1$ for $-1 \le x \le 1$. Using the preceding lemma we may inductively find a sequence of polynomials $P_2, P_3, \dots$ satisfying the conditions

$$\max_{|z| \le n-1} |P_n(z) - P_{n-1}(z)| < \alpha_n\ , \tag{7.16}$$

$$|P_n(x) - g(x)| < \alpha_n \qquad \big(x \in [-n\,,\, -n+1] \cup [n-1\,,\, n]\big)\ , \tag{7.17}$$

and

$$P_n\big(\pm(n-1)\big) = g\big(\pm(n-1)\big)\ \ ,\ \ P_n(\pm n) = g(\pm n)\,.$$

From (7.16) it follows that for any given R and any $\varepsilon' > 0$, there exists a positive integer N such that

$$\left| \sum_{\nu=N}^{N+k} \big(P_{\nu+1}(z) - P_\nu(z)\big) \right| < \varepsilon' \qquad (k = 0, 1, 2, \dots)\,.$$

Hence, the series $P_1(z) + \sum_{\nu=1}^{\infty} \big(P_{\nu+1}(z) - P_\nu(z)\big)$ is uniformly convergent on every compact subset of $\mathbb{C}$, and so defines an entire function f. It may be noted that

$$f(z) = P_m(z) + \sum_{\nu=m}^{\infty} \big(P_{\nu+1}(z) - P_\nu(z)\big) \qquad (m = 1, 2, \dots)\,. \tag{7.18}$$

In view of (7.16) and (7.17), we have

$$\begin{aligned} |g(x) - f(x)| \ &\le\ |f(x) - P_m(x)| + \sum_{\nu=m}^{\infty} \big|\big(P_{\nu+1}(z) - P_\nu(z)\big)\big| \\ &<\ \alpha_m + \alpha_{m+1} + \cdots < \varepsilon \qquad \big(x \in [m-1\,,\, m] \cup [-m\,,\, -m+1]\big)\,. \end{aligned}$$

Hence, (7.18) allows us to conclude that $|g(x) - f(x)| < \varepsilon$ for all real values of x. □

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