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Orthogonal Expansions and Weighted Approximation on the Unit Sphere

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We survey recent progress on weighted approximation and orthogonal expansions on the unit sphere for a family of weight functions that are invariant under reflection groups. The orthogonal expansions are given with respect to h-harmonics, which are homogeneous orthogonal polynomials with respect to a specified weight function on the sphere. We discuss various means of h-harmonic expansions, define weighted modulus of smoothness and K-functional on the sphere, show that they are equivalent and use them to give direct and inverse theorems for weighted best approximation by polynomials on the sphere.

1. Introduction

Let $S^{d-1} = \{x : \|x\| = 1\}$ be the unit sphere of $\mathbb{R}^d$, where $\|x\|$ denotes the usual Euclidean norm. The theory of spherical harmonics is classical and well documented; see, for example, [12, 24, 26]. Harmonic polynomials are homogeneous polynomials that satisfy the Laplace equation $\Delta P = 0$, where

$$\Delta = \partial_1^2 + \cdots + \partial_d^2, \qquad \partial_i = \partial/\partial x_i,$$

the spherical harmonics are the restrictions of harmonic polynomials on S^{d-1}. They are orthogonal with respect to the surface (Lebesgue) measure $d\omega$ on S^{d-1}, which is the unique measure on the sphere that is invariant under the rotation group. Approximation on S^{d-1} is usually studied with the help of spherical harmonics and approximation polynomials are often constructed using harmonic expansions. The work in the literature is almost exclusively in the setting of L^p space defined with respect to $d\omega$. We refer to [4, 5, 7, 15, 18, 16, 19, 20, 23] and the references therein.

Only rather recently has the theory of sphere harmonics been extended beyond the setting of the Lebesgue measure. It was discovered by Dunkl ([10, 11], see [12] and the references therein) that there is an analogous theory for measures invariant under reflection groups. The homogeneous orthogonal

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polynomials are called h-harmonics, they enjoy a theory comparable to the theory of ordinary harmonics. Furthermore, as shown in more recent work ([17, 28, 34, 36, 37]), one can establish a theory of h-harmonic expansions and weighted best approximation for the reflection invariant measures on the sphere. The purpose of this paper is to report recent progress in this direction.

We will concentrate on the results of h-harmonic expansions and weighted best approximation on the sphere. It should be mentioned, however, that there is a close relation between h-spherical harmonics on S^d and orthogonal polynomials on the unit ball $B^d = \{x \in \mathbb{R}^d : \|x\| \le 1\}$ of $\mathbb{R}^d$ ([29, 34, 35]) and a further connection to orthogonal polynomials on the simplex $T^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0,\ 1 - x_1 - \dots - x_d \ge 0\}$ ([30, 35]). Because of these relations, the investigation on h-harmonics and the best approximation on the sphere also lead to results about orthogonal expansions and the best approximation on the ball and on the simplex. For results on B^d and T^d, see [17, 31, 32, 37] and the discussions there.

The paper is organized as follows: the next section contains preliminary and background; Section 3 deals with summability of h-harmonic expansions, including examples of Cesàro means and de la Vallée Poussin means; Section 4 discusses spherical means and weighted modulus of smoothness; Section 5 is devoted to the relation between modulus of smoothness, K-functional and best approximation.

2. Background

Throughout this paper we use the notation $\mathbb{N}_0$ to denote the set of natural integers. For $n \in \mathbb{N}_0$, denote by $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n in d variables and Π_n^d the space of polynomials of degree n in d variables. It is known that

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{d} \qquad \text{and} \qquad \dim \Pi_n^d = \binom{n+d}{d}.$$

For $x, y \in \mathbb{R}^d$, we denote by $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ the usual inner product of x and y, and by $\|x\| = \sqrt{\langle x, x\rangle}$ the usual Euclidean norm of x.

2.1. Ordinary Spherical Harmonics

Harmonic polynomials are homogeneous polynomials P that satisfy the equation $\Delta P = 0$, where Δ is the usual Laplace operator. Their restriction on the unit sphere S^{d-1} is called the (ordinary) spherical harmonics. We use $\mathcal{H}_n^d$ to denote the space of ordinary harmonic polynomials of degree n.

The theory of spherical harmonics is well-known (see, for example, [12, 24, 26]) and much of it will appear below in the discussion of h-harmonics. We only mention that spherical harmonics are orthogonal with respect to the Lebesgue measure $d\omega$ on the sphere, $\int_{S^{d-1}} PQ\,d\omega = 0$, $P \in \mathcal{H}_n^d$, $Q \in \Pi_{n-1}^d$; furthermore $L^2(S^{d-1}) = \sum_{n=0}^{\infty} \bigoplus \mathcal{H}_n^d$. The reproducing kernel $Y_n(x,y)$ of the space $\mathcal{H}_n^d$ is the so-called zonal harmonic, given in terms of the Gegenbauer polynomials C_n^λ as

$$Y_n(x,y) = \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x,y\rangle), \qquad \lambda = \frac{d-2}{2}. \tag{2.1}$$

The Gegenbauer polynomials are orthogonal with respect to the weight function

$$w_\lambda(t) = (1-t^2)^{\lambda-1/2}, \qquad -1 \le t \le 1. \tag{2.2}$$

The fact that the orthogonal group acts transitively on the sphere plays an important role in deriving the formula of Y_n and in the theory of spherical harmonics.

2.2. *h*-harmonics

The h-harmonics are defined by Dunkl recently ([10, 11], see [12] and the references therein). They are associated with reflection groups instead of the orthogonal group.

For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to v,

$$x\sigma_v := x - 2(\langle x,v\rangle/\|v\|^2)v, \qquad x \in \mathbb{R}^d.$$

Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system R_+, normalized so that $\langle v,v\rangle = 2$ for all $v \in R_+$. The group G is a subgroup of the orthogonal group, it is generated by the reflections $\{\sigma_v : v \in R_+\}$. A function f defined on $\mathbb{R}^d$ is called invariant under G if $f(xw) = f(x)$ for all $w \in G$. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G (that is, $ug = v$ for some $g \in G$). This defines a G-invariant function.

The essential ingredient of the theory of h-harmonics is a family of first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), defined by ([10])

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v\in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x,v\rangle}\langle v,\varepsilon_i\rangle, \qquad 1 \le i \le d,$$

where $\{\varepsilon_1, \dots, \varepsilon_d\}$ is the standard unit basis of $\mathbb{R}^d$. These operators generate a commutative algebra; that is, they satisfy $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2$. Keeping the notation Δ for the ordinary Laplacian, then Δ_h is equal to

$$\Delta_h f(x) = \Delta f(x) + \sum_{v\in R_+} \kappa_v \left[\frac{2\langle \nabla f(x), v\rangle}{\langle x,v\rangle} - \frac{f(x) - f(x\sigma_v)}{\langle x,v\rangle^2}\right].$$

The h-Laplacian plays the role similar to that of the ordinary Laplacian. An h-harmonic P is a homogeneous polynomial satisfying the equation $\Delta_h P = 0$. It turns out that

$$\Delta_h P = 0 \quad \text{if and only if} \quad \int_{S^{d-1}} PQh_\kappa^2 \, d\omega = 0, \quad P \in \mathcal{P}_n^d, \quad Q \in \Pi_{n-1}^d,$$

in which h_κ is a positive homogeneous function defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d. \tag{2.3}$$

Let us denote by $\mathcal{H}_n^d(h_\kappa^2)$ the space of h-harmonic polynomials of degree n. It is known that $\dim \mathcal{H}_n^d(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. Throughout this paper, we denote

$$\lambda_\lambda := \gamma_\kappa + \frac{d-2}{2} \qquad \text{with} \qquad \gamma_\kappa = \sum_{v \in R_+} \kappa_v. \tag{2.4}$$

The number γ_κ is the homogeneous degree of h_κ.

The spherical h-harmonics are the restriction of h-harmonics on the unit sphere. In terms of the polar coordinates $y = ry'$, $r = \|y\|$, the operator Δ_h takes the form

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{d + 2\gamma_\kappa}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{h,0}, \tag{2.5}$$

where $\Delta_{h,0}$ is the (Laplace-Beltrami) operator on the sphere. An application of $\Delta_{h,0}$ to the h-harmonics $Y \in \mathcal{H}_n(h_\kappa^2)$ with $Y(y) = r^n Y(y')$ shows that the spherical h-harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0} Y_n(x) = -n(n + 2\lambda_\kappa) Y_n(x), \qquad x \in S^{d-1}, \quad Y_n \in \mathcal{H}_n^d(h_\kappa^2).$$

2.3. Examples of Weight Functions

We give several examples of weight functions h_κ in (2.3). The simplest case is $G = \mathbb{Z}_2^d$, the group of sign changes, for which

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0. \tag{2.6}$$

For symmetric group of d objects,

$$h_\kappa(x) = \prod_{1 \le i,j \le d} |x_i - x_j|^\kappa, \qquad \kappa \ge 0.$$

For hyperoctahedral group, the group generated by the reflections in the hyperplanes $x_i = 0$, $1 \le i \le d$ and $x_i \pm x_j = 0$, $1 \le i, j \le d$,

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_0} \prod_{1 \le i,j \le d} |x_i^2 - x_j^2|^{\kappa_1}, \qquad \kappa_0, \kappa_1 \ge 0. \tag{2.7}$$

2.4. L^p Space

Let a_κ denote the normalization constant of h_κ, $a_\kappa^{-1} = \int_{S^{d-1}} h_\kappa^2\, d\omega$. Let us denote by $L^p(h_\kappa^2)$, $1 \le p \le \infty$, the space of functions defined on S^{d-1} with the finite norm

$$\|f\|_{\kappa,p} = \Big(a_\kappa \int_{S^{d-1}} |f(y)|^p h_\kappa^2(y)\, d\omega(y)\Big)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(S^{d-1})$ with the usual uniform norm $\|f\|_\infty$. The surface measure $d\omega$ has a normalization constant $\sigma_{d-1} = \int_{S^{d-1}} d\omega = 2\pi^{d/2}/\Gamma(d/2)$, which is the surface area. We also define the inner product

$$\langle f, g\rangle_\kappa = a_\kappa \int_{S^{d-1}} f(y)g(y)h_\kappa^2(y)\, d\omega.$$

We will also need the weighted L^p space on the real line with respect to w_λ in (2.2), which we denote by $L^p(w_\lambda)$. Let $\|g\|_{w_\lambda,p}$ denote the weighted L^p norm for functions defined on $[-1,1]$,

$$\|g\|_{w_\lambda,p} = \Big(c_\lambda \int_{-1}^{1} |g(t)|^p w_\lambda(t)\, dt\Big)^{1/p}$$

for $1 \le p < \infty$ and $\|g\|_{w_\lambda,\infty} = \|g\|_\infty$ being the usual uniform norm on $[-1,1]$.

2.5. Intertwining Operator

Some properties of h-harmonics can be derived using the intertwining operator between the commutative algebra generated by the partial derivatives and the one generated by Dunkl's operators. This operator, V_κ, is a linear operator determined uniquely by

$$V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d, \qquad V_\kappa 1 = 1, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad 1 \le i \le d.$$

Note that V_κ maps $\mathcal{H}_n^d$ onto $\mathcal{H}_n^d(h_\kappa^2)$ by definition. An explicit formula of V_κ is known only in the case of symmetric group S_3 for three variables and in the case of the Abelian group $\mathbb{Z}_2^d$. In the latter case, V_κ is an integral operator,

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1t_1, \dots, x_dt_d) \prod_{i=1}^{d} (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \tag{2.8}$$

where c_κ denotes the constant

$$c_\kappa = b_{\kappa_1} \dots b_{\kappa_d}, \qquad \text{and} \qquad b_r = \Big(\int_{-1}^{1} (1-t^2)^{r-1} dt\Big)^{-1}.$$

If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} b_\lambda \int_{-1}^{1} f(t)(1-t)^{\lambda-1} dt = [f(1) + f(-1)]/2.$$

One important property of the intertwining operator is that it is positive ([21]), that is, $V_\kappa p \ge 0$ if $p \ge 0$. Another important property shows that the integral of $V_\kappa f$ can be explicitly evaluated, even though a compact formula of V_κ itself is unknown in general. The following theorem is proved in [28].

Proposition 1. *Let V_κ be the intertwining operator. Then*

$$\int_{S^{d-1}} V_\kappa f(x) h_\kappa^2(x)\, d\omega = A_\kappa \int_{B^d} f(x)(1-\|x\|^2)^{\gamma_\kappa - 1} dt,$$

for $f \in L^2(h_\kappa^2; S^{d-1})$ such that both integrals are finite; in particular, if $g : \mathbb{R} \mapsto \mathbb{R}$ is a function such that all integrals below are defined, then

$$\int_{S^{d-1}} V_\kappa g(\langle x, \cdot \rangle)(y) h_\kappa^2(y)\, d\omega(y) = B_\kappa \int_{-1}^{1} g(t\|x\|)(1-t^2)^{\gamma_\kappa - 1} dt,$$

where A_κ and B_κ are constants whose values can be determined by setting $f(x) = 1$ and $g(t) = 1$, respectively.

This proposition plays an important role in studying h-harmonic expansions and weighted best approximation. We emphasize that it is non-trivial even in the case of V_κ given in (2.8) for the product weight function (2.6).

3. Summability of h-harmonic Expansions

Let h_κ be the reflection invariant weight function defined in (2.3). Recall the definition of λ_κ in (2.4). We often write $\lambda = \lambda_\kappa$ in this section.

3.1. h-harmonic Expansions

The standard Hilbert space theory shows that

$$L^2(h_\kappa^2) = \sum_{n=0}^{\infty} \bigoplus \mathcal{H}_n^d(h_\kappa^2).$$

That is, with each $f \in L^2(h_\kappa^2)$ we can associate its h-harmonic expansion

$$f(x) = \sum_{n=0}^{\infty} Y_n(h_\kappa^2; f, x), \qquad x \in S^{d-1},$$

in $L^2(h_\kappa^2)$ norm. For the surface measure ($\kappa = 0$), such a series is called the Laplace series (cf. [13, Chapt. 12]). The orthogonal projection $Y_n(h_\kappa^2) : L^2(h_\kappa^2) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ takes the form

$$Y_n(h_\kappa^2; f, x) := \int_{S^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y)\, d\omega(y). \tag{3.1}$$

The kernel $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of h-harmonics $\mathcal{H}_n^d(h_\kappa^2)$ in $L^2(h_\kappa^2)$. The reproducing kernel $P_n(h_\kappa^2; x, y)$ enjoys a compact formula in terms of the intertwining operator V_κ ([28])

$$P_n(h_\kappa^2; x, y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} V[C_n^{\lambda_\kappa}(\langle \cdot, y\rangle)](x).$$

If all $\kappa_v = 0$, V_κ becomes the identity operator and the above formula becomes (2.1). Note that in the case of $G = \mathbb{Z}_2^d$, the explicit formula of V_κ in (2.7) gives a compact formula

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n+\lambda}{\lambda} \times \int_{[-1,1]^d} C_n^\lambda(x_1y_1t_1 + \cdots + x_dy_dt_d) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt,$$

where $\lambda = \kappa_1 + \cdots + \kappa_d + (d-2)/2$. The first proof of this formula ([27]) used an explicit orthonormal basis of $\mathcal{H}_n^d(h_\kappa^2)$ and the addition formula of the Gegenbauer polynomials.

As seen from (2.1), the kernel functions for the ordinary harmonic expansions are of the form $\phi(\langle x, y\rangle)$, where $\phi : [-1,1] \mapsto \mathbb{R}$, a fact that plays an essential role in the theory of ordinary harmonics. The function $G(x, y) = V_\kappa[\phi(\langle x, \cdot\rangle)](y)$ plays a similar role for the h-harmonics. We define an integral operator for functions on S^{d-1} whose kernel has the form of $G(x, y)$.

Definition 1. *For $f \in L^p(h_\kappa^2)$ and $g \in L^1(w_\lambda; [-1,1])$ with $\lambda = \lambda_\kappa$,*

$$(f \star_\kappa g)(x) := a_\kappa \int_{S^{d-1}} f(y) V_\kappa[g(\langle x, \cdot\rangle)](y) h_\kappa^2(y)\, d\omega.$$

The operation $f \star_\kappa g$ defines a sort of convolution of the functions f on S^{d-1} and g on $[-1,1]$. For the surface measure ($V_\kappa = id$), it is called spherical convolution in [8]. Using this notion, we can write

$$Y_n(h_\kappa^2; f) = f \star_\kappa p(w_\lambda) \quad \text{with} \quad p(w_\lambda; t) = \frac{n+\lambda}{\lambda} C_n^\lambda(t). \tag{3.2}$$

Here and in the following, we write $\lambda = \lambda_\kappa$. The operation $f \star_\kappa g$ satisfies many properties of the usual convolution in $\mathbb{R}^d$. In particular, using the second equation in Proposition 1, one can prove the familiar Young's inequality:

Proposition 2. *Let $p, q, r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(h_\kappa^2)$ and $g \in L^r(w_\lambda; [-1,1])$,*

$$\|f \star_\kappa g\|_{\kappa,p} \le \|f\|_{\kappa,q} \|g\|_{w_\lambda, r}.$$

The expression of $Y_n(h_\kappa^2; f)$ in terms of $\star_\kappa$ shows that the h-harmonic expansions are closely related to the Gegenbauer expansions. For a function $g \in L^2(w_\lambda)$, its Gegenbauer expansion takes the form

$$g(t) = \sum_{n=0}^{\infty} P_n(w_\lambda; g) \quad \text{with} \quad P_n(w_\lambda; g, t) = c_\lambda \int_{-1}^{1} g(s) p_n(w_\lambda; s, t) w_\lambda(s)\, ds,$$

in which the kernel is defined by

$$p_n(w_\lambda; s, t) = \frac{n+\lambda}{\lambda} \cdot \frac{C_n^\lambda(s) C_n^\lambda(t)}{C_n^\lambda(1)},$$

using the fact that $\|C_n^\lambda\|_{w_\lambda,2}^2 = C_n^\lambda(1)\lambda/(n+\lambda)$ (cf. [25, p. 80]). Note that $p_n(w_\lambda; t) = p_n(w_\lambda; 1, t)$.

Hence, h-harmonic expansions are related to the Gegenbauer expansions at the point $t = 1$, just as in the case of ordinary harmonic expansions. This fact has been used to prove a number of results about summability of h-harmonic expansions. We start with a general result.

Let $b_n := \{b_{k,n}\}$ be a sequence of real numbers such that $\sum_{k=0}^{\infty} b_{k,n} = 1$ for each n. It induces a summability method for the Gegenbauer expansions defined by

$$M_n(\{b_n\}; g, t) = \sum_{k=0}^{\infty} b_{k,n} P_k(w_\lambda; g, t) = c_\lambda \int_{-1}^{1} g(s) m_n(\{b_n\}; s, t) w_\lambda(s)\, ds. \tag{3.3}$$

The kernel of the expansion is $m_n(\{b_n\}; s, t) = \sum_{k=0}^{\infty} b_{k,n} p_k(w_\lambda; s, t)$. Again, we write $m_n(\{b_n\}; t) = m_n(\{b_n\}; 1, t)$. The sequence $\{b_{k,n}\}$ also induces a similar summability method for h-harmonic expansions,

$$M_n(h_\kappa^2; f) := \sum_{k=0}^{\infty} b_{k,n} Y_k(h_\kappa^2; f) = f \star_\kappa m_n(\{b_n\}), \tag{3.4}$$

in which the second equal sign follows from (3.2). As a consequence of Proposition 2 with $r = 1$, we have the following theorem:

Theorem 1. *Let $M_n(\{b_n\}; g)$, defined in (3.3), converge to g at $t = 1$. Assume that the means $M_n(h_\kappa^2; f)$ in (3.4) converge to f in $L^p(h_\kappa^2)$ norm for polynomials f. Then for $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$, the means $M_n(h_\kappa^2; f)$ converge to f in $L^p(h_\kappa^2)$ norm.*

Proof. The assumption on the convergence of $M_n(\{b_n\}; g, 1)$ shows that

$$\|m_n(\{b_n\})\|_{w_\lambda,1} = \int_{-1}^{1} |m_n(\{b_n\}; s, 1)| w_\lambda(s)\, ds < \infty.$$

Hence, using Proposition 2, it follows that $\|M_n(h_\kappa^2; f)\|_{\kappa,p}$ is bounded for $p = 1$ and $p = \infty$. The usual Riesz interpolation theorem shows that the same holds for $1 < p < \infty$. The stated result is a consequence of the triangle inequality and the fact that $M_n(h_\kappa^2; f)$ converges to polynomials f in $L^p(h_\kappa^2)$.

As examples of the means, we consider the Cesàro means and the de la Vallée Poussin means in the following two subsections. As we shall see, the above theorem is very useful, but it may hide some difficulties in the process since the reflection group does not act transitively on the sphere.

3.2. Cesàro (C, δ) Means

For $\delta > 0$, the Cesàro (C, δ) means s_n^δ of a sequence $\{c_n\}$ are defined by

$$s_n^\delta = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta c_k, \qquad A_{n-k}^\delta = \binom{n-k+\delta}{n-k}.$$

We say that $\{c_n\}$ is Cesàro (C, δ) summable to s if s_n^δ converges to s as $n \to \infty$. For the Gegenbauer expansion with respect to w_λ, the (C, δ) means of the Gegenbauer expansion, denoted by $S_n^\delta(w_\lambda; f)$, can be written as an integral operator,

$$S_n^\delta(w_\lambda; f, t) = c_\lambda \int_{-1}^{1} f(s) p_n^\delta(w_\lambda; s, t) w_\lambda(s)\, ds,$$

where $p_n^\delta(w_\lambda; s, t)$ is the (C, δ) means of the sequence $\{p_n(w_\lambda; s, t)\}$. It is known that $S_n^\delta(w_\lambda; f)$ converges to f in $L^p(w_\lambda)$ norm if $\delta > \lambda$.

We denote the n-th (C, δ) means of the h-harmonic expansion by $S_n^\delta(h_\kappa^2; f)$. They are studied in [28, 34, 17]; we recount the main results below. These means can be written as

$$S_n^\delta(h_\kappa^2; f) = f \star_\kappa p_n^\delta(w_\lambda), \qquad p_n^\delta(w_\lambda, t) = p_n^\delta(w_\lambda, 1, t).$$

The fact that $S_n^\delta(h_\kappa^2; f)$ converges to f for f being polynomials can be established easily since for $f \in \mathcal{P}_m^d$,

$$S_n^\delta(h_\kappa^2; f) - f = (A_n^\delta)^{-1} \sum_{k=n-m}^{n} A_{n-k}^{\delta-1} \Big(\sum_{j=0}^{k} Y_j(h_\kappa^2; f) - f \Big),$$

where we have used the fact that $\sum_{j=0}^{k} Y_j(h_\kappa^2; f)$ is the projection operator of Π_k^d, which preserves polynomials of degree $\le k$. Hence, as a consequence of Theorem 1, we get the following result:

Theorem 2. *Let $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^{d-1})$. The Cesàro (C, δ) means of the h-harmonic expansion of f converge to f in norm provided $\delta > \lambda_\kappa$. Furthermore, the (C, δ) means of the h-harmonic expansion define a positive linear operator if $\delta \ge 2\lambda_\kappa + 1$.*

Recall that λ_κ is defined in (2.4). For the fact that (C,δ) means of the Gegenbauer expansion with respect to w_λ converge at $t=1$ if and only if $\delta>\lambda$, see [25, p. 246]. The positivity of $(C, 2\lambda_\kappa+1)$ means follows from the fact that V_κ is a positive operator and the (C,δ) kernel of the Gegenbauer expansion with respect to w_λ is positive if $\delta \ge 2\lambda+1$ (see [1, p. 71]).

The proof of this theorem reduces the convergence of the h-harmonic expansions to that of the Gegenbauer expansions. With the help of Proposition 2, or Proposition 1, we eliminated the action of the reflection group hidden in V_κ and the proof is similar to the usual one for the ordinary harmonics in the sense that the convergence is reduced to the convergence at just one point. For the ordinary harmonics, the underline group is the orthogonal group which acts transitively on S^{d-1}, so the reduction to one point is to be expected. For the weight function h_κ, however, the underline reflection group is a subgroup of the orthogonal group, which no longer acts transitively on S^{d-1}. In this case, the reduction to one point seems to be artificial. In fact, doing so indeed costs us something: it requires stronger condition than it is necessary; that is, the condition on δ is not sharp, at least in the case of $G=\mathbb{Z}_2^d$. Indeed, for the weight function h_κ in (2.6), we can use the explicit formula of the reproducing kernel in (2.8) to conduct a more detailed analysis. The result is the following sharp theorem proved in [17]:

Theorem 3. *For h_κ given in* (2.6)*, the (C,δ) means of the h-harmonic expansion of a function f converge to f in the $L^p(h_\kappa^2; S^{d-1})$ norm, $1\le p<\infty$, or $C(S^{d-1})$ norm for $p=\infty$, provided*

$$\delta > \lambda_\kappa - \min_{1\le i\le d} \kappa_i, \qquad \lambda_\kappa = |\kappa| + \frac{d-2}{2}.$$

Moreover, for $p=1$ and $p=\infty$, the condition is also necessary.

Instead of the simple proof of Theorem 2, the proof of this theorem involves complicated estimate of the (C,δ) kernel $K_n^\delta(h_\kappa^2;x,y)$, which is made possible by the explicit formula of $P_n(h_\kappa^2;x,y)$ in (3.1). The central estimate is as follows: for $x,y\in S^{d-1}$ and $\delta>(d-2)/2$,

$$|K_n^\delta(h_\kappa^2;x,y)| \le c\Big[\frac{\prod_{j=1}^d (|x_jy_j| + n^{-1}|\bar x - \bar y| + n^{-2})^{-\kappa_j}}{n^{\delta-(d-2)/2}(|\bar x-\bar y| + n^{-1})^{\delta+\frac{d}{2}}} + \frac{\prod_{j=1}^d (|x_jy_j| + |\bar x - \bar y|^2 + n^{-2})^{-\kappa_j}}{n(|\bar x-\bar y| + n^{-1})^{d}}\Big],$$

where $\bar x = (|x_1|,\dots,|x_d|)$ and $\bar y = (|y_1|,\dots,|y_d|)$. The right hand side of this estimate is invariant under $\mathbb{Z}_2^d$. For the ordinary harmonics, the estimate of this kernel is essentially the estimate of the kernel for the Gegenbauer expansions, which is a function of a single variable; see [7], for example. For the h-harmonics, the estimate is much more difficult, as can be seen from the formula (2.8).

The proof of the necessity of the theorem follows from evaluating $I_n(x)$ at the points of intersection of the great circles defined by the intersection of S^{d-1} and the coordinate planes. In fact, these great circles are like boundaries on S^{d-1} and the proof of necessity shows that $I_n(x)$ attains its maximum on this boundary. Furthermore, let us define

$$S_{int}^{d-1} := S^{d-1} \setminus \bigcup_{i=1}^{d} \{x \in S^{d-1} : x_i = 0\},$$

which is the interior region bounded by these boundaries on S^{d-1}. The points on the planes $\{x : x_i = 0\}$ are exactly where the weight function h_κ^2 in (2.3) has singularity. We have the following result ([17]):

Theorem 4. *Let f be continuous on S^{d-1}. If $\delta > (d-2)/2$, then the (C,δ) means of the h-harmonic expansion of f for h_κ^2 in* (2.6) *converge to f for every $x \in S_{int}^{d-1}$. Moreover, the convergence is uniform over each compact set contained inside S_{int}^{d-1}.*

In other words, for the pointwise convergence away from the singularity of h_κ, the convergence holds if $\delta > (d-2)/2$, which is the same as the critical index for the ordinary harmonics. This phenomenon does not show up when we deal with the ordinary harmonics, for which there is no difference in critical index between uniform and pointwise convergence. According to this theorem, the convergence of the (C,δ) means of the h-harmonic expansions is the same as the ordinary harmonic expansions away from the great circles $\{x \in S^{d-1} : x_i = 0\}$ on the sphere.

One naturally expect that similar phenomenon should appear for h_κ^2 associated with other reflection groups. However, further study is hindered by the lack of explicit formula for V_κ.

3.3. de la Vallée Poussin Means

For h_κ defined in (2.3) and $f \in L^1(h_\kappa^2)$, these means are defined by

$$M_n(h_\kappa^2; f, x) := \sum_{k=0}^{n} \mu_{k,n}^\lambda Y_n(h_\kappa^2; f, x) = f \star_\kappa m_n(w_\lambda),$$

where $m_n(w_\lambda)$ is the de la Vallée Poussin means of the Gegenbauer polynomials,

$$m_n(w_\lambda; 1, t) = \sum_{k=0}^{n} \mu_{k,n}^\lambda \frac{k+\lambda}{\lambda} C_k^\lambda(t), \qquad \lambda = \lambda_\kappa,$$

in which the multipliers $\mu_{k,n}^\lambda$ are given by

$$\mu_{k,n}^\lambda = \frac{n!}{(n-k)!} \cdot \frac{\Gamma(n+2\lambda+1)}{\Gamma(n+k+2\lambda+1)}.$$

The classical de la Vallée Poussin sum for the cosine functions has an extension for the Gegenbauer polynomials (see, for example, [1, p. 11]),

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{\Gamma(n+\lambda+1/2)}{\Gamma(n+k+2\lambda+1)} \frac{k+\lambda}{\lambda} C_k^{\lambda}(t) = \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)} \left(\frac{1+t}{2}\right)^n .$$

These means were introduced by de la Vallèe Poussin for Fourier series. For the Gegenbauer expansions (more generally, Jacobi expansions), they were studied in [3, 6] and were also studied as the Bernstein-Durrmeyer polynomials by other authors. For the ordinary harmonic expansions, they were studied in [15, 5]. Several their properties are common in all cases. For example, we know that the means $M_n(h_\kappa^2; f)$ are positive linear operators, self-adjoint with respect to the inner product $\langle f, g\rangle_\kappa$, and they converge for f being polynomials. Hence, the convergence of $M_n(h_\kappa^2; f)$ to f in $L^p(h_\kappa^2)$ norm follows as a consequence of Theorem 1.

What makes these means special is the following property satisfied by the multiplier sequence $\mu_{k,n}^{\lambda}$:

$$\mu_{k,n}^{\lambda} - \mu_{k,n-1}^{\lambda} = \frac{k(k+2\lambda)}{n(n+2\lambda)} \mu_{k,n}^{\lambda}, \qquad 0 \le k \le n.$$

From these properties, the approximation behavior of the de la Vallée Poussin means can be easily described using a K-functional. Define $D^p(h_\kappa^2)$ as the set

$$\big\{ f \in L^p(h_\kappa^2) : -k(k+2\lambda_\kappa) Y_k(h_\kappa^2; f) = Y_k(h_\kappa^2; g) \text{ for some } g \in L^p(h_\kappa^2) \big\}.$$

Recall the definition of spherical Laplacian $\Delta_{h,0}$ defined in (2.5). Since h-spherical harmonics are eigenfunctions of $\Delta_{h,0}$, it follows that if $\Delta_{h,0} f \in L^p(h_\kappa^2)$, then $f \in D^p(h_\kappa^2)$, as we can take $g = \Delta_{h,0} f$. The Peetre K-functional between $L^p(h_\kappa^2)$ and $D^p(h_\kappa^2)$ is defined by

$$K(f,t)_{\kappa,p} := \inf \big\{ \|f-g\|_{\kappa,p} + t\|\Delta_{h,0}\, g\|_{\kappa,p},\ g \in D^p(h_\kappa^2) \big\}.$$

Theorem 5. *Let $f \in L^p(h_\kappa^2)$. Then there are two constants c_1 and c_2 such that*

$$c_1 \max_{k \ge n} \|M_k(h_\kappa^2, f) - f\|_{\kappa,p} \le K(f, n^{-1})_{\kappa,p} \le c_2 \max_{k \ge n} \|M_k(h_\kappa^2, f) - f\|_{\kappa,p}.$$

The proof of Theorem 5 is essentially the same as in the case of ordinary spherical harmonics (see [5]). The similarity ends as soon as we start to depart from these formal properties. For example, for ordinary harmonics, it is easy to prove that for $f \in L^1$, $M_n(f,x)$ converges to $f(x)$ for almost all $x \in S^{d-1}$ ([5]), using the fact that the kernel is of the form $\phi(\langle x, y\rangle)$ so that the estimate of the kernel is essentially of one variable. For h-harmonics, however, such a result has been proved only for f invariant under the group G and only for the case of $G = \mathbb{Z}_2^d$ ([36]).

We note that the K-functional appears naturally in the above result. Furthermore, it is equivalent to a modulus of smoothness which is defined in the following section.

4. Spherical Means and Modulus of Smoothness

For the Lebesgue measure on S^{d-1}, there is a modulus of smoothness defined via spherical means. The spherical means of f is defined by

$$T_\theta f(x) = \frac{1}{\sigma_{d-2}(\sin\theta)^{2\lambda}} \int_{\langle x,y\rangle=\cos\theta} f(y)\, d\omega(y), \qquad 0 \le \theta \le \pi.$$

The properties of the spherical means are well-known; see [4, 19], for example. Another expression for this means is

$$T_\theta f(x) = \frac{1}{\sigma_{d-2}} \int_{S_x^\perp} f(x\cos\theta + \sin\theta u)\, d\omega(u),$$

in which $S_x^\perp$ denotes the equator in S^{d-1} with respect to x, $S_x^\perp = \{y \in S^{d-1} : \langle x, y\rangle = 0\}$, it is isomorphic to the sphere S^{d-2}. For $r > 0$, the spherical modulus of smoothness of order r is defined by

$$\omega_r(f,\delta)_p := \sup_{0<\theta<\delta} \|(I - T_\theta)^{r/2} f\|_p, \qquad f \in L^p(S^{d-1}).$$

The modulus of smoothness $\omega_2(f,\delta)_p$ was used in many papers in the literature, so did the case of r being an even integer. For the case of $r > 0$, this definition is given in [23]; the reason that $r/2$ appears in the right hand side instead of r will become clear when we consider the equivalence of the modulus of smoothness and K-functional.

These definitions have been extended to the setting that the integrals are taken with respect to the weight function h_κ^2. The definition of the spherical means was extended in [36]. It is given in an implicit way :

Definition 2. *Let $\lambda = \lambda_\kappa$. For $0 \le \theta \le \pi$, the means T_θ^κ is defined by*

$$c_\lambda \int_0^\pi T_\theta^\kappa(f,x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta = a_\kappa \int_{S^{d-1}} f(y) V_\kappa g(\langle x,y\rangle) h_\kappa^2(y)\, d\omega(y), \tag{4.1}$$

where g is any function $[-1,1] \mapsto \mathbb{R}$ such that the integral in the right hand side is finite.

It is also possible to give a direct definition of T_θ^κ. Let us define $\chi_\theta(t)$ by

$$\int_0^\pi \chi_\theta(\cos\phi) g(\phi)\, d\phi = g(\theta),$$

a distribution function. It follows that the ordinary spherical means $T_\theta f(x)$ can be written as

$$T_\theta f(x) = \frac{1}{\sigma_{d-1}(\sin\theta)^{d-1}} \int_{S^{d-1}} f(y) \chi_\theta(\langle x,y\rangle)\, d\omega(y),$$

since $\chi_\theta(\langle x,y\rangle)$ is the characteristic function of the set $\{x \in S^d : \langle x,y\rangle = \cos\theta\}$. Using the function χ_θ, an integral representation of $T_\theta^\kappa f$ is given by

$$T_\theta^\kappa f(x) = \frac{1}{c_\lambda(\sin\theta)^{2\lambda}} a_\kappa \int_{S^{d-1}} f(y) V_\kappa[\chi_\theta(\langle x,\cdot\rangle)](y) h_\kappa^2(y)\, d\omega.$$

This also shows that T_θ^κ is indeed an extension of the ordinary spherical means T_θ. Furthermore, by the definition of $f \star_\kappa g$, we can write (4.1) as

$$(f \star_\kappa g)(x) = c_\lambda \int_0^\pi T_\theta^\kappa(f,x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta. \tag{4.2}$$

Recall that $Y_n(h_\kappa^2; f)$ denotes the projection of f to $\mathcal{H}_n^d(h_\kappa^2)$. The properties of $T_\theta^\kappa f$ are given in the following proposition ([36, 37], for ordinary spherical means ($\kappa = 0$) see [4, 19]).

Proposition 3. *The means $T_\theta^\kappa f$ satisfy the following properties:*

1. *Let $f_0(x) = 1$, then $T_\theta^\kappa f_0(x) = 1$.*
2. *For $f \in L^1(h_\kappa^2)$,*
$$Y_n(h_\kappa^2; T_\theta^\kappa f) = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} Y_n(h_\kappa^2; f);$$
in particular, $\Delta_{h,0} T_\theta^\kappa f = T_\theta^\kappa(\Delta_{h,0} f)$ if $\Delta_{h,0} f \in L^1(h_\kappa^2)$ and
$$T_\theta^\kappa f \sim \sum_{n=0}^\infty \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} Y_n(h_\kappa^2; f).$$
3. *For $0 \le \theta \le \pi$,*
$$T_\theta^\kappa f - f = \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s T_t^\kappa(\Delta_{h,0} f)(\sin t)^{2\lambda} dt.$$
4. *For $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^{d-1})$,*
$$\|T_\theta^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p} \qquad \text{and} \qquad \lim_{\theta \to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0.$$

Let us mention that the property (2) and (4.2) together give the following Funk-Hecke formula for h-harmonics: For $Y_n^h \in \mathcal{H}_n^d(h_\kappa^2)$, $g : [-1,1] \mapsto \mathbb{R}$,

$$\int_{S^{d-1}} V_\kappa f(\langle x,\cdot\rangle)(y) Y_n^h(y) h_\kappa^2(y)\, d\omega = \mu_n(f) Y_n^h(x), \qquad x \in S^{d-1},$$

where, with $\lambda = \lambda_\kappa$, $\mu_n(f)$ is defined by

$$\mu_n(f) = c_\lambda \frac{1}{C_n^\lambda(1)} \int_{-1}^1 f(t) C_n^\lambda(t)(1-t^2)^{\lambda - \frac{1}{2}} dt.$$

This property was proved first in [33] and used in the proof of property (2) in [36]. Also using the property (2) of the above proposition, we define, for $r > 0$,

$$(I - T_\theta^\kappa)^{r/2} f \sim \sum_{n=0}^{\infty} (1 - R_n^\lambda(\cos\theta))^{r/2} Y_n(h_\kappa^2; f), \qquad R_k^\lambda(t) := C_k^\lambda(t)/C_k^\lambda(1).$$

The property (4) of the Proposition 3 shows that the following definition of weighted modulus of smoothness is meaningful.

Definition 3. *Let $r > 0$. For $f \in L^p(h_\kappa^2)$, $1 < p < \infty$, or $f \in C(S^{d-1})$, define*

$$\omega_r(f,t)_{\kappa,p} := \sup_{0 \le \theta \le t} \|(I - T_\theta^\kappa)^{r/2}\|_{\kappa,p}.$$

For the Lebesgue measure ($\kappa = 0$) and r being an even integer, this definition appeared in several references; the case $r > 0$ for the Lebesgue measure appeared in [23] (see the discussion in [23] for the historical account). Some properties of $\omega_r(f,t)_{\kappa,p}$ are collected below.

Proposition 4. *The modulus of smoothness $\omega_r(f,t)_{\kappa,p}$ satisfies:*

1. $\omega_r(f,t)_{\kappa,p} \to 0$ *if* $t \to 0$;
2. $\omega_r(f,t)_{\kappa,p}$ *is monotone nondecreasing on* $(0,\pi)$;
3. $\omega_r(f+g,t)_{\kappa,p} \le \omega_r(f,t)_{\kappa,p} + \omega_r(g,t)_{\kappa,p}$;
4. *For* $0 < s < r$,
$$\omega_r(f,t)_{\kappa,p} \le 2^{[(r-s+1)/2]} \omega_s(f,t)_{\kappa,p};$$
5. *If* $(-\Delta_{h,0})^k \in L^p(h_\kappa^2)$, $k \in \mathbb{N}$, *then for* $r > 2k$
$$\omega_r(f,t)_{\kappa,p} \le c\, t^{2k} \omega_{r-2k}((-\Delta_{h,0})^k f, t)_{\kappa,p}.$$

There are several definitions of modulus of smoothness on S^{d-1} in the case of the Lebesgue measure, most of them are equivalent, as shown in [22]. In the weighted case, however, Definition 3 is the only one defined so far.

5. *K*-functional, Modulus of Smoothness and Best Approximation

Recall the definition of spherical Laplacian $\Delta_{h,0}$ defined in (2.5). Since h-harmonics are eigenfunctions of $\Delta_{h,0}$ with eigenvalues $-n(n+2\lambda_\kappa)$, we can define the fractional power of $\Delta_{h,0}$ by its harmonic expansion: for $r > 0$ define

$$(-\Delta_{h,0})^{r/2} g \sim \sum_{n=1}^{\infty} (n(n+2\lambda))^{r/2} Y_n(h_\kappa^2; g).$$

Definition 4. *Let $r > 0$. The function space $\mathcal{W}_r^p(h_\kappa^2)$ will be defined by*

$$\{f \in L^p(h_\kappa^2) : (k(k+2\lambda))^{\frac{r}{2}} Y_k(h_\kappa^2; f) = Y_k(h_\kappa^2; g) \text{ for some } g \in L^p(h_\kappa^2)\}.$$

The K-functional between $L^p(h_\kappa^2)$ and $\mathcal{W}_r^p(h_\kappa^2)$ is defined by

$$K_r(f;t)_{\kappa,p} := \inf\left\{\|f-g\|_{\kappa,p} + t^r \|(-\Delta_{h,0})^{r/2} g\|_{\kappa,p},\ g \in \mathcal{W}_r^p(h_\kappa^2)\right\}.$$

If $(-\Delta_{h,0})^{r/2} f \in L^p(h_\kappa^2)$, then clearly the function f belongs to $\mathcal{W}_r^p(h_\kappa^2)$ since we can take $g = (-\Delta_{h,0})^{r/2} f$ in the definition. If $f \in \mathcal{W}_r^p(h_\kappa^2)$, then g and $(-\Delta_{h,0})^{r/2} f$ have the same coefficients in their h-harmonic expansions, so that $g = (-\Delta_{h,0})^{r/2} f$ in the $L^p(h_\kappa^2)$ norm, which shows that $(-\Delta_{h,0})^{r/2} f \in L^p(h_\kappa^2)$. Thus, the K-functional is well-defined.

One of the main result in [37] is to show that $K_r(f;t)_{\kappa,p}$ is equivalent to the modulus of smoothness $\omega_r(f;t)_{\kappa,p}$.

Theorem 6. *For $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$,*

$$c_1, \omega_r(f;t)_{\kappa,p} \le K_r(f;t)_{\kappa,p} \le c_2\, \omega_r(f;t)_{\kappa,p}.$$

Another main result of [37] is to use the modulus of smoothness or the K-functional to characterize the weighted best approximation by polynomials on the sphere. Define

$$E_n(f)_{\kappa,p} := \inf\left\{\|f-P\|_{\kappa,p} : P \in \Pi_n^d\right\}.$$

The following theorem contains both the direct and the inverse estimates.

Theorem 7. *For $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$,*

$$E_n(f)_{\kappa,p} \le c\, \omega_r(f;n^{-1})_{\kappa,p}. \tag{5.1}$$

On the other hand,

$$\omega_r(f,n^{-1})_{\kappa,p} \le c\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{\kappa,p}. \tag{5.2}$$

Here and in the following we shall use c to denote a generic constant, which depends only on d, p, r and κ and whose value may be different from line to line.

The proof of these two theorems are easy for $r = 2$ but rather difficult in the general case of $r > 0$. In the case of the Lebesgue measure ($\kappa = 0$), the general case was stated and proved in [23], preceded by various special cases settled by several other authors. However, one of the key lemma in [23] (Lemma 3.9) was put into question in [16] and a corrected proof was given in [16] which was rather involved. The proof of these two theorems in the weighted case has been given in [37], which follows the outline of the proof in [23] but does not depend on Lemma 3.9 there.

The inverse estimate (5.2) follows as a consequence of a Bernstein type inequality, as usual. The main effort is to prove the Jackson type estimate (5.1) and the equivalence in Theorem 5.1. The proof follows the following order: one side of the equivalence, $\omega_r(f;t)_{\kappa,p} \le cK_r(f;t)_{\kappa,p}$, is established first and used to establish a Jackson type estimate,

$$E_n(f)_{\kappa,p} \le c\,\omega_r(f;\pi/(2(n+\lambda)))_{\kappa,p}, \tag{5.3}$$

which is used to established the other direction of the equivalence $K_r(f;t)_{\kappa,p} \le c\omega_r(f;t)_{\kappa,p}$. From the equivalence follows the inequality

$$\omega_r(f;\delta t)_{\kappa,p} \le c\,\max\{1,\delta^r\}\omega_r(f;t)_{\kappa,p}, \tag{5.4}$$

which is used to complete the proof of the Jackson type estimate in Theorem 6. Note that the inequality (5.4) is not an obvious consequence of the definition of $\omega_r(f;t)_{\kappa,p}$.

The first part of the equivalence, $\omega_r(f;t)_{\kappa,p} \le cK_r(f;t)_{\kappa,p}$, is essentially a consequence of the following proposition.

Proposition 5. *For* $f \in \mathcal{W}_p^r(h_\kappa^2)$,

$$E_n(f)_{\kappa,p} \le c\,n^{-r}\|(-\Delta_{h,0})^{-r/2}f\|_{\kappa,p}.$$

For each $r > 0$, by a result of [2, Theorem 1 and Theorem 3], there is a function $\phi_r(x)$ such that ϕ_r is continuous on $[-1,1)$, $\phi_r \in L^1(w_\lambda,[-1,1])$, and

$$\phi_r(t) \sim \sum_{n=1}^{\infty}(n(n+2\lambda))^{-r/2}\frac{n+\lambda}{\lambda}\,C_n^\lambda(t).$$

Using this fact, one can prove that, if $f \in \mathcal{W}_p^r(h_\kappa^2)$, $f = (-\Delta_{h,0})^{r/2}f \star_\kappa \phi_r$ in $L^p(h_\kappa^2)$. Let σ be a positive integer, $\sigma > 2\lambda+1$, so that $p_n^\sigma(w_\lambda;x,t)$ is nonnegative. Using summation by parts repeatedly on the expansion of $\phi_r(t)$, we can write

$$\phi_r(t) = \sum_{k=0}^{\infty}\Delta^{\sigma+1}\mu(k)\binom{k+\sigma}{k}p_k^\sigma(w_\lambda;t,1), \qquad \mu(k) = (k(k+2\lambda))^{-r/2}, \quad k \ge 1$$

and $\mu(0) = 0$, where $\Delta^m\mu(k)$ denotes the m-th order finite difference, defined by $\Delta\mu(t) = \mu(t) - \mu(t+1)$ and $\Delta^{m+1} = \Delta(\Delta^m)$. The polynomial used to prove Proposition 5 is constructed as $f \star_\kappa q_n$, where q_n is the n-th partial sum of the above expansion.

The idea of using summation by parts and the Cesàro means of sufficiently higher order seems to appear first in [14]; it plays an important role in the development of [23] which we followed closely in [37].

The proof of (5.3) uses the following construction: Let $\eta \in C^\infty[0,+\infty)$ be a function defined by $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ if $x \ge 2$. A sequence of operators η_n for $n \in \mathbb{N}$ is defined by

$$\eta_n f := \sum_{k=0}^{\infty} \eta\Big(\frac{k}{n}\Big) Y_k(h_\kappa^2; f) = f \star_\kappa \eta_n(w_\lambda), \qquad \eta_n(w_\lambda, t) = \sum_{n=0}^{\infty} \eta\Big(\frac{k}{n}\Big) P_k(w_\lambda; t).$$

Since $\eta(k/n) = 0$ if $k \ge 2n$, the infinite series terminates at $k = 2n-1$ so that η_n is a spherical polynomial of degree at most $2n-1$. Furthermore, if P is a spherical polynomial of degree at most n, then $Y_k(h_\kappa^2; P) = P$ for $k \ge n$ and the definition of η shows that $\eta_n P = P$. The main properties of η_n are given in the following proposition.

Proposition 6. *Let $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$. For $n \in \mathbb{N}$,*

1. *$\eta_n f \in \Pi_n^d$ and $\eta_n P = P$ for $P \in \Pi_n^d$;*
2. *$\|\eta_n f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$;*
3. *$\|f - \eta_n f\|_{\kappa,p} \le c E_n(f)_{\kappa,p}$.*

This operator is an analog of de la Vallée Poussin's delay operator for the Fourier series. There is another operator having a similar property, which is defined by

$$L_{n,m}(h_\kappa^2; f) = \frac{1}{n^m} \sum_{j=0}^{m} (2^j n)_m \prod_{i=0, i\ne j}^{m} \frac{1}{2^j - 2^i}\; S_{2^j n-1}^m(h_\kappa^2; f),$$

where $(a)_m = a(a-1)\dots(a-m+1)$. When $m=1$, $L_{n,1} = 2S_{2n-1} - S_{n-1}$ is the de la Vallée Poussin operator in the classical Fourier analysis. It is easy to see that $L_{n,m}$ also satisfies the properties in Proposition 6. However, the operators $\eta_n f$ are more suitable for establishing the following proposition.

Proposition 7. *Suppose $0 < t < \pi/(2(n+\lambda))$. For $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$,*

$$\|(-\Delta_{h,0})^{r/2} \eta_n f\|_{\kappa,p} \le c\, t^{-r} \|(I - T_\theta^\kappa)^{r/2} f\|_{\kappa,p}. \tag{5.5}$$

Furthermore,

$$\|(I - T_\theta^\kappa)^{r/2} \eta_n f\|_{\kappa,p} \le c\, t^{r} \|(-\Delta_{h,0})^{r/2} f\|_{\kappa,p}.$$

The proof of this proposition is very technical, as in [23]. Using the idea of the proof of Proposition 5, the essential part is to establish the estimates

$$\sum_{k=1}^{2n-1} \Big|\Delta^{\sigma+1}\Big[\eta\Big(\frac{k}{n}\Big)\alpha_\theta(k)\Big]\Big| k^\sigma \le c \quad \text{and} \quad \sum_{k=1}^{2n-1} \Big|\Delta^{\sigma+1}\Big[\eta\Big(\frac{k}{n}\Big)\beta_\theta(k)\Big]\Big| k^\sigma \le c,$$

where $\alpha_\theta(k)$ and $\beta_\theta(k)$ are defined by $\alpha_\theta(k) = (k(k+2\lambda))^{r/2}/(1 - R_k^\lambda(\cos\theta))^{r/2}$ and $\beta_\theta(k) = (1 - R_k^\lambda(\cos\theta))^{r/2}/(k(k+2\lambda))^{r/2}$.

Together Proposition 6 and (5.5) lead to the following result.

Proposition 8. *Suppose* $0 < t < \pi/(2(n+\lambda))$. *For any polynomial* $P_n \in \Pi_n^d$,

$$\|(-\Delta_{h,0})^{r/2} P_n\|_{\kappa,p} \le c\, t^{-r} \|(I - T_\theta^\kappa)^{r/2} P_n\|_{\kappa,p}.$$

This is called an inequality of Riesz-Bernstein-Nikolskii-Stechkin type in [23] in the case of the Lebesgue measure. It implies, in particular, the Bernstein type inequality

$$\|(-\Delta_{h,0})^{r/2} P_n\|_{\kappa,p} \le c\, n^r \|P_n\|_{\kappa,p}, \tag{5.6}$$

which is the main tool for proving the inverse estimate (5.2).

Another important consequence of the inequality (3.1) is the estimate:

$$\|(-\Delta_{h,0})^{r/2} \eta_n f\|_{\kappa,p} \le c\, n^r \omega_r(f, \pi/(2(n+\lambda)))_{\kappa,p} \tag{5.7}$$

for $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$. This is one of the main ingredients for the proof of (5.3). The proof starts with defining a sequence n_j, $j = 0, 1, 2, \ldots$, as follows:

$$n_0 = 1, \qquad n_{j+1} = \inf\{n : \omega_r(f; \pi/(2(2n+\lambda))) \le \omega_r(f; \pi/(2(2n_j+\lambda)))_{\kappa,p}/2\}$$

for $j \ge 0$. The fact that $\omega_r(f;t)_{\kappa,p}$ is monotone nondecreasing on $(0,\pi)$ shows that $n_j \to \infty$ as $j \to \infty$. Writing

$$f = \sum_{j=1}^{\infty} (\eta_{n_j} f - \eta_{n_{j-1}} f) + \eta_1 f$$

leads to

$$E_{2n_j}(f)_{\kappa,p} \le \sum_{k=j+1}^{\infty} \|\eta_{n_k} f - \eta_{n_{k-1}} f\|_{\kappa,p}.$$

Since η_n preserves polynomials of degree n and $\eta_n \eta_m f = \eta_m \eta_n f$ by definition, we can use Proposition 5 and (5.6) to obtain

$$\begin{aligned} \|\eta_{n_k} f - \eta_{n_{k-1}} f\|_{\kappa,p} &\le c\, E_{n_{k-1}}(\eta_{2n_k} f)_{\kappa,p} \\ &\le c\, n_{k-1}^{-r} \|(-\Delta_{h,0})^{r/2} \eta_{2n_k} f\|_{\kappa,p} \\ &\le c\, (n_k/n_{k-1})^r \omega_r(f; \pi/(2(2n_k+\lambda)))_{\kappa,p}, \end{aligned}$$

where the third inequality uses the fact that $n_k \le c n_{k-1}$. A limiting argument based on (5.7) and the inequality $\omega_r(f;t)_{\kappa,p} \le c K_r(f;t)_{\kappa,p}$ shows that the sequence n_k/n_{k-1} is bounded by a constant c. The inequality

$$\omega_r(f; \pi/(2(2n_k+\lambda)))_{\kappa,p} \le 2^{j-k} \omega_r(f; \pi/(2(2n_j+\lambda)))_{\kappa,p} \quad \text{for } k \ge j$$

and the fact that $\omega_r(f;t)_{\kappa,p}$ is monotonically nondecreasing completes the proof of (5.3).

Once (5.3) is proved, the inequality $K_r(f;t)_{\kappa,p} \le c\omega_r(f;t)_{\kappa,p}$ can be established by choosing $g = \eta_n f \in \mathcal{W}_p^r(h_\kappa^2)$. We note that the proof shows that $\eta_n f$ satisfies

$$\|\eta_n f - f\|_{\kappa,p} \le c\omega_r(f;n^{-1})_{\kappa,p} \text{ and } \|(-\Delta_{h,0})^{r/2}\eta_n f\|_{\kappa,p} \le cn^r\omega_r(f;n^{-1})_{\kappa,p}.$$

As a final remark, let us mention that the weighted modulus of smoothness $\omega(f,t)_{\kappa,p}$ defined in Definition 3 is not entirely satisfactory, even though Theorem 5.2 indeed justifies the definition. The main problem is the implicit nature of the definition; also, there is no natural extension to the unit ball and to the simplex (the results on these domains are stated in terms of K-functional in [37]). It would be nice if one can find some other alternative definitions, perhaps a modulus of smoothness defined using the divided differences on the sphere.

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