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On Some Properties of Orthogonal Polynomials over an Area with a Weight Having Singularity on the Boundary Contour

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In this work the estimation of the maximum norm of orthogonal polynomials over the region with respect to the weight is analyzed. It is observed that the norm of polynomials does not change with conditions of the weight and the boundary curve.

1. Introduction

Let G be a finite, simply connected region with $0 \in G$, which is bounded by a Jordan curve $L := \partial G$. Let σ be the two-dimensional Lebesgue measure, and let $h \in L^1(G, d\sigma)$ be a positive weight function defined in G.

A system of polynomials $\{K_n(z)\}$, $\deg K_n = n$, $n = 0, 1, 2, ...$, each of them with a positive leading coefficient, is called an orthonormal system over the area of the domain G with a weight h if

$$\iint\limits_G h(z) K_n(z) \overline{K_m(z)}\, d\sigma_z = \delta_{n,m}.$$

These polynomials were first analyzed by Carleman [5] for $h(z) \equiv 1$. He studied the Faber problem concerning the generalization to simply connected region of Taylor's series. Apart from these approximation problems, the orthogonal polynomials have been a subject investigation of many mathematicians who investigated asymptotical behavior of the polynomials inside and on the closure of region G.

In the present paper we study the estimation problem for the maximum norm $\|K_n\|_{C(\overline{G})} := \max\{|K_n(z)| : z \in \overline{G}\}$ of orthogonal polynomials over the region with respect to the weight. The polynomials are defined by the pair (G, h). Therefore, variation of the norm of these polynomials depends on the properties of the region G and the weight $h(z)$. This dependence has been investigated for orthogonality along a curve in [6]–[8], and over the region in [1]–[3], [9].

2. Main Definition and Results

Throughout this paper $c, c_1, c_2, \dots$ are positive constants (in general, different in different relations), which depend on G, in general. "$a \asymp b$" is equivalent to $c_1 a \le b \le c_2 a$ with some constants c_1, c_2.

Let $w = \varphi(z)$ ($w = \Phi(z)$) be the conformal mapping of G ($\Omega := C\overline{G}$) onto the unit disk $B := \{w : |w| < 1\}$ normalized by $\varphi(0) = 0,\ \varphi'(0) > 0$ ($\Phi(\infty) = \infty,\ \Phi'(\infty) > 0$), and let $\psi := \varphi^{-1}$ ($\Psi := \Phi^{-1}$).

Definition 1. *A bounded Jordan region G is called a k-quasidisk, $0 \le k < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+k}{1-k}$, homeomorphizm of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case the curve $L := \partial G$ is called a K-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is k-quasidisk (k-quasicircle) with some $0 \le k < 1$.*

Theorem 1. *Let G be a k-quasidisk for some $0 \le k < 1$, and let the weight function $h(z)$ be uniformly bounded away from zero, i.e.,*

$$h(z) \ge c > 0. \tag{1}$$

Then, for every $n = 1, 2, \dots,$

$$\|K_n\|_{C(\overline{G})} \le c_1 n^{1+k}.$$

Definition 2. *We say that $G \in Q_\alpha$, $0 < \alpha \le 1$, if:*

a) L is a quasicircle;

b) $\Phi \in Lip\ \alpha$, $z \in \overline{\Omega}$.

Theorem 2. *Let $G \in Q_\alpha$ for some $0 < \alpha \le 1$ and assume that $h(z)$ satisfies condition* (1)*. Then, for every $n = 1, 2, \dots,$*

$$\|K_n\|_{C(\overline{G})} \le c_2 n^{1/\alpha}.$$

Now, we assume that the function $h(z)$ does not satisfy condition (1). We define $h(z)$ as follows:

$$h(z) = h_0(z) \prod_{i=1}^{m} |z - z_i|^{\gamma_i}, \tag{2}$$

where $\{z_i\}$, $i = \overline{1, m}$, is a fixed system of points on L; $\gamma_i > -2$ and $h_0(z)$ is satisfying the condition $h_0(z) \ge c > 0$.

Theorem 3. *Let G be a k-quasidisk for some $0 \le k < 1$ and let $h(z)$ satisfy condition* (2)*. Then, for every $n = 1, 2, \dots,$*

$$|K_n(z_i)| \le c_3 n^{(1+\gamma_i/2)(1+k)}, \qquad z_i \in L.$$

Theorem 4. *Suppose that $G \in Q_\alpha$ for some $\frac{1}{2} \le \alpha \le 1$ and let $h(z)$ satisfy the condition (2) with $\gamma_i > 0$. Suppose that*

$$\|K_n\|_{C(\overline{G})} \asymp |K_n(z_{i_0})|, \qquad 1 \le i_0 \le m.$$

Then, for every $n = 1, 2, \dots,$

$$\|K_n\|_{C(\overline{G})} \le c_4 n^{s_{i_0} - 1/2}, \qquad s_{i_0} := \frac{2 + \gamma_{i_0}}{2\alpha}.$$

Example. Let $G \equiv D$ and $h(z) = |z-1|^2$. Then

$$K_n(z) = \frac{2}{\sqrt{\pi(n+1)(n+2)(n+3)}} \sum_{j=0}^{n} (j+1) z^j (1 + z + \cdots + z^{n-j})$$

and

$$\|K_n\|_{C(\overline{G})} = \frac{1}{3\sqrt{\pi}} \sqrt{(n+1)(n+2)(n+3)}.$$

This shows that the exponent $s_{i_0} - \frac{1}{2}$ cannot be replaced by smaller number.

Definition 3. *We say that $\Omega \in Q(\nu)$, $0 < \nu < 1$, if:*

i) *$L := \partial\Omega = \partial G$ is quasicircle;*

ii) *For all $z \in L$, there exist $r > 0$ and $0 < \nu < 1$ such that the closed circular sector $S(z; r, \nu) := \{\zeta : \zeta = z + re^{i\theta},\ 0 \le \theta_0 < \theta < \theta_0 + \nu\}$ of radius r and opening $\nu\pi$ lies in $\overline{G}$ with vertex at z.*

This condition imposed on L gives a new geometric characterization of the curve or region. For example, if the region G^* is defined by

$$G^* := \left\{ z : z = re^{i\theta},\ 0 < r < 1,\ \frac{\pi}{2} < \theta < 2\pi \right\},$$

then the coefficient of quasiconformality k of G^* could not be obtained so easily, whereas $\Omega^* := CG^* \subset Q(\frac{1}{2})$.

Definition 4. *We say that $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$, $0 < \nu_1, \dots, \nu_m < \alpha \le 1$, if there exists a system of points $\{z_i\}$, $i = \overline{1, m}$, on L such that $\Omega \in Q(\nu_i)$ for any points $z_i \in L$ and $\Phi \in Lip\ \alpha$, $z \in \overline{\Omega} \setminus \{z_i\}$.*

Assume that the system of points $\{z_i\}$, $i = \overline{1, m}$, mentioned in (2) and Definition 4 is identically ordered on L.

Theorem 5. *Let $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ for some $0 < \nu_i < 1$ and $\alpha(2-\nu_i) \geq 1$; $h(z)$ is defined by* (2) *and, in addition,*

$$1 + \frac{\gamma_i}{2} = \frac{1}{\alpha(2-\nu_i)} \tag{3}$$

is satisfied for any points $z_i \in L$, $i = \overline{1,m}$. Then, for every $n = 1, 2, \dots,$

$$\|K_n\|_{C(\overline{G})} \leq c_5 n^{1/\alpha}. \tag{4}$$

Comparing Theorem 5 with Theorem 2 we see that when equality (3) was satisfied, the maximum norm of the polynomials $K_n(z)$ in $\overline{G}$ behaves identically, then the weight $h(z)$ and the boundary curve L both have or have not singularity. The equality given by (3) shows the interference condition of the weight and the boundary contour.

Corollary 1. *In Definition 4, if the boundary arcs of the region Ω joining the points $\{z_j\} \in L$ are arcs of the class $C(1, \alpha)$, then we can find a region with piecewise smooth boundary that contains points inside the angles $\nu_i \pi$, $0 < \nu_i < 1$. In this case relations* (3) *and* (4) *become as follows:*

$$1 + \frac{\gamma_i}{2} = \frac{1}{2-\nu_i}, \qquad i = \overline{1,m},$$
$$\|K_n\|_{C(\overline{G})} \leq c_5 n.$$

This result extends a theorem of Suetin [9, Theorem 4.6] in case $0 < \nu_i < 1$. Now, we investigate the case when (1) is not satisfied.

Theorem 6. *Let $\Omega \in Q_\alpha(\nu_1)$ for some $0 < \nu_1 < 1$ and $\alpha(2-\nu_1) \geq 1$; $h(z)$ being defined by* (2). *If*

$$1 + \frac{\gamma_1}{2} < \frac{1}{\alpha(2-\nu_1)}, \tag{5}$$

then for every $z \in \overline{G}$ and each $n = 1, 2, \dots,$

$$|K_n(z)| \leq c_4 n^{s_1} + c_5 |z - z_1|^{\sigma_1} n^{1/\alpha}, \tag{6}$$

where

$$s_1 = \frac{(2+\gamma_1)(2-\nu_1)}{2}, \qquad \sigma_1 = \frac{1}{\alpha(2-\nu_1)} - \frac{2+\gamma_1}{2}. \tag{7}$$

Since $\alpha(2-\nu_1) \geq 1$, (5) will be satisfied when $-2 < \gamma_1 < 0$. Here and from (6) we see that the order of the height of K_n at the point z_1 and at the points $z \in L$, $z \neq z_1$ for which $h(z) \to \infty$ and the curve L does not have singularity, behaves identically. The conditions (5) we will call algebraic pole conditions of order $\lambda_1 = 1 - \alpha(2-\nu_1)(1 + \frac{\gamma_1}{2})$.

This theorem can be extended to the case when L and $h(z)$ have a lot of singular points. For example, in case of two singular points we can write

$$|K_n(z)| \le c_6|z-z_1|^{\sigma_1}n^{s_2} + c_7|z-z_2|^{\sigma_2}n^{s_1} + c_8|z-z_1|^{\sigma_1}|z-z_2|^{\sigma_2}n^{1/\alpha}, \quad z \in \overline{G},$$

where s_i, σ_i, $i=1,2$, are defined as in (7), respectively.

Theorem 6 remains correct if the curve L has an algebraic pole at the point z_1 and singularities at the points $\{z_k\}$, $k \ge 2$, which are satisfying the interference conditions (3).

Theorem 7. *Let $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ for some $0<\nu_i<1$ and $\alpha(2-\nu_i) \ge 1$; $h(z)$ being defined by (2). If*

$$1 + \frac{\gamma_i}{2} > \frac{1}{\alpha(2-\nu_i)} \tag{8}$$

is satisfied for any points $z_i \in L$, $i = \overline{1,m}$, then, for every $n = 1,2,\dots$,

$$\max_{z\in\overline{G}} \prod_{i=1}^{m} |z-z_i|^{\widetilde{\mu}_i}|K_n(z)| \le c_9 n^{1/\alpha},$$
$$|K_n(z_i)| \le c_{10} n^{\widetilde{s}_i},$$

where

$$\widetilde{\mu}_i := 1 + \frac{\gamma_i}{2} - \frac{1}{\alpha(2-\nu_i)},$$
$$\widetilde{s}_i := \left(1+\frac{\gamma_i}{2}\right)(2-\nu_i), \qquad i=\overline{1,m}.$$

The conditions (8) will be satisfied anytime when $\gamma_i > 0$, $i = \overline{1,m}$. Thus, we call (8) algebraic zero conditions of order $\mu_i = \alpha(2-\nu_i)(1+\frac{\gamma_i}{2}) - 1$.

Let $A_p(h,G)$, $p>0$, denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h,G)} := \left(\iint\limits_G h(z)|f(z)|^p \, d\sigma_z \right)^{1/p} < \infty.$$

Since the polynomials $K_n(z)$ have a minimal $A_2(h,G)$-norm in the class of all polynomials $P_n(z)$, $\deg P_n \le n$, $n=1,2,\dots$, Theorems 1–7 can be generalized for this class. In this case we have relations between the norms $\|P_n\|_{C(\overline{G})}$ and $\|P_n\|_{A_p(h,G)}$. Let us give one of them.

Theorem 8. *Let $P_n(z)$, $\deg P_n \le n$, $n=1,2,\dots$, be an arbitrary polynomial and $1<p<\infty$. Then:*

a) under the conditions of Theorem 1,

$$\|P_n\|_{C(\overline{G})} \le c_{11} n^{2(1+k)/p} \|P_n\|_{A_p};$$

b) under the conditions of Theorems 1 and 5,

$$\|P_n\|_{C(\overline{G})} \leq c_{12} n^{2/(p\alpha)} \|P_n\|_{A_p}.$$

These estimations are sharp according to the exponent in the class of all polynomials P_n of degree at most n.

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