CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 184-189.

The Complete Asymptotic Expansion for the Gamma Operators and Their Left Quasi-Interpolants

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Our purpose is to study the local rate of convergence of the Gamma operators. The talk presents the complete asymptotic expansion for the Gamma operators. We investigate their asymptotic behavior also concerning simultaneous approximation. All expansion coefficients are explicitly calculated. It turns out that Stirling numbers play an important role. Moreover, we deal with linear combinations of Gamma operators having a better degree of approximation than the operators themselves. Using divided differences we define general classes of linear combinations, special cases of which were recently introduced and investigated by other authors. Finally, we study the left quasi-interpolants of the Gamma operators in the sense of Sablonnière.

1. Introduction

Müller's Gamma operators are given by

$$G_n f(x) := \frac{x^{n+1}}{n!} \int_0^\infty t^n e^{-x\,t} f\left(\frac{n}{t}\right) dt \qquad (n = 1, 2, \dots) \tag{1}$$

for all functions $f : (0, \infty) \to \mathbb{R}$ for which the integral on the right-hand side of (1) exists for all $x \in (0, \infty)$. These operators have been introduced in [7] and investigated in subsequent papers [6], [8] and [10]. Newer results on Gamma operators can be found in [5] and [3].

The purpose of this paper is the study of the local rate of convergence of the operators (1). We investigate their asymptotic behavior also for simultaneous approximation. As main result we derive the complete asymptotic expansion

$$(G_n f)^{(r)}(x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty} \frac{c_k^{(r)}(f; x)}{n^k} \qquad (n \to \infty),$$

provided f possesses derivatives of sufficiently high order at x and satisfies certain growth conditions. Furthermore, we study general classes of linear combinations of Gamma operators having a better degree of approximation than Gamma operators themselves.

Recently, Müller [9] introduced left quasi-interpolants of the Gamma operators in the sense of Sablonniere. The asymptotic behaviour of these quasi-interpolants will be presented in the last section.

2. Asymptotic Simultaneous Approximation by Gamma Operators

Let $W(0,\infty)$ be the space of all locally bounded and integrable functions $f:(0,\infty)\to\mathbb{R}$ satisfying the growth conditions $|f(t)|\le Me^{a/t}$ $(0<t<R_1)$ and $|f(t)|\le Mt^b$ $(t>R_2)$ for some positive constants a, b, M, R_1, R_2. It is obvious that, for each $f\in W(0,\infty)$ and a fixed $x\in(0,\infty)$, the Gamma operators $G_nf(x)$ are well-defined for all $n\ge\max\{\lfloor a/x\rfloor+1,\lfloor b\rfloor\}$ (cf. [6, Lemmas 2.5 and 2.9]).

For $q\in\mathbb{N}$ and $x\in(0,\infty)$, let $A[q;x]$ be the class of all functions $f\in W(0,\infty)$ which are q times differentiable at x. The following theorem presents the complete asymptotic expansion for the image functions of the Gamma operators and also of their derivatives.

Theorem 1. *Let $q\in\mathbb{N}$ and $x\in(0,\infty)$. For each function $f\in A[2q;x]$ the sequence $\{G_nf(x)\}_{n\ge1}$ possesses the asymptotic expansion*

$$G_nf(x)=f(x)+\sum_{k=1}^{q}\frac{c_k(f;x)}{n^k}+o(n^{-q})\qquad(n\to\infty),$$

where the coefficients $c_k(f;x)$ are given by

$$c_k(f;x)=\sum_{s=2}^{2k}\frac{x^sf^{(s)}(x)}{s!}T(s,k)\qquad(k=1,2,\dots),$$

with

$$T(s,k)=\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}\sigma_{r-1+k}^{r-1}. \tag{2}$$

Moreover, for any fixed $r\in\mathbb{N}_0$, if $f\in A[2(q+r);x]$, the asymptotic expansion of the r-th derivatives $(G_nf)^{(r)}$ can be obtained by term-by-term differentiation, i.e.,

$$(G_nf)^{(r)}(x)=f^{(r)}(x)+\sum_{k=1}^{q}\frac{c_k^{(r)}(f;x)}{n^k}+o\left(n^{-q}\right)\qquad(n\to\infty).$$

The quantities σ_n^m in (2) denote the Stirling numbers of second kind. For the case $r=0$ in (2), we interpret σ_{-1+k}^{-1} as $\delta_{k,0}$.

An immediate consequence of Theorem 1 is the following Voronovskaya type result for the simultaneous approximation by the Gamma operators.

Corollary 1. *Let $r \in \mathbb{N}_0$ and $x \in (0,\infty)$. For $f \in A[2\,(r+1)\,;x]$, the r-th derivative $(G_n f)^{(r)}(x)$ satisfies*

$$\lim_{n\to\infty} n\left((G_n f)^{(r)}(x) - f^{(r)}(x)\right) = \frac{x^2}{2}\, f^{(r+2)}(x) + rxf^{(r+1)}(x) + r(r-1)f^{(r)}(x).$$

Remark. The special case $r=0$ can be found in [6, Theorem 5.3].

3. Linear Combinations of Gamma Operators

For a given $m \in \mathbb{N}$, we fix certain integers $1 \le \beta_0 < \cdots < \beta_m$ and consider special linear combinations of the type

$$G_{n,m} := \sum_{j=0}^{m} \alpha_j\, G_{\beta_j n}, \tag{3}$$

where in some cases the coefficients α_j may depend on n and m. The α_j are determined by certain conditions on $G_{n,m}$. Relation (3) can be written in terms of divided differences in the form $G_{n,m} = [n\,\beta_0,\dots,n\,\beta_m;\ g(z)\,G_z]$, for a certain function g, such that $\alpha_j = g(n\beta_j)/\omega_{n,m}'(n\beta_j)$ $(j=0,\dots,m)$, where $\omega_{n,m}(z) = (z-n\,\beta_0)\dots(z-n\,\beta_m)$.

As a consequence of Theorem 1, for $x>0$, $r \in \mathbb{N}_0$, and for each $f \in A[2\,(q+r)\,;x]$, we have the asymptotic expansion

$$(G_{n,m}f)^{(r)}(x) = f^{(r)}(x) + \sum_{k=1}^{q} c_k^{(r)}(f;x)\,[n\,\beta_0,\dots,n\,\beta_m;\ z^{-k}g(z)] + o(n^{-q}) \tag{4}$$

as $n\to\infty$, provided $[n\,\beta_0,\dots,n\,\beta_m;\ g(z)] = \sum_{j=0}^{m}\alpha_j = 1$ and $\alpha_j = O(1)$ as $n\to\infty$ $(j=0,\dots,m)$.

3.1. Optimal Order of Convergence

We are looking for constants $\alpha_j = \alpha_j(m)$ independent of n such that $(G_{n,m}f)(x) = f(x) + O\left(n^{-m-1}\right)$ as $n\to\infty$, for all functions $f \in W(0,\infty)$ sufficiently smooth at x. To this end we define $G_{n,m} := [n\,\beta_0,\dots,n\,\beta_m;\ z^m\ G_z]$.

Obviously the coefficients $\alpha_j(m) = \frac{(n\,\beta_j)^m}{\omega_{n,m}'(n\,\beta_j)} = \prod_{\substack{\nu=0\\ \nu\ne j}}^{m}(1-\beta_\nu/\beta_j)^{-1}$ $(j=0,\dots,m)$ are independent of n and $[n\,\beta_0,\dots,n\,\beta_m;\ z^m] = 1$. Let e_r

denote the functions given by $e_r(x) = x^r$ $(r \in \mathbb{R},\ x > 0)$. By (4), for $n \to \infty$, we obtain

$$(G_{n,m}f)^{(r)}(x) = f^{(r)}(x) + \sum_{k=m+1}^{q} \frac{c_k^{(r)}(f;x)}{n^k} [\beta_0, \ldots, \beta_m; e_{m-k}] + o\left(n^{-q}\right).$$

In particular, if $q = m+1$, we have the Voronovskaya type formula

$$\lim_{n\to\infty} n^{m+1} \left((G_{n,m}f)^{(r)}(x) - f^{(r)}(x)\right) = (-1)^m \left(\prod_{\nu=0}^{m} \beta_\nu^{-1}\right) c_{m+1}^{(r)}(f;x).$$

3.2. Preservation of Polynomials up to a Certain Degree

Define the operators $G_{n,m}$ by $G_{n,m} := [n\,\beta_0, \ldots, n\,\beta_m;\ (z-1)^{\underline{m}}\, G_z]$, where $x^{\underline{m}}$ denotes the falling factorial, i.e., $x^{\underline{m}} := x(x-1)\ldots(x-m+1)$, $x^{\underline{0}} := 1$. The coefficients $\alpha_j(n,m)$ are given by $\alpha_j(n,m) = (n\beta_j - 1)^{\underline{m}} \prod_{\substack{\nu=0\\ \nu\neq j}}^{m} (n\beta_j - n\beta_\nu)^{-1}$ $(j = 0, \ldots, m)$ (see also [2]).

Furthermore, for $n > m$, we have $[n\,\beta_0, \ldots, n\,\beta_m;\quad (z-1)^{\underline{m}}] = 1$ and $\sum_{j=0}^{m} |\alpha_j(n,m)| < C(m)$.

Theorem 2. *For $x > 0$, $r, q \in \mathbb{N}_0$, and for $f \in A[2(q+m+1+r);x]$, we have the asymptotic expansion*

$$\begin{aligned}(G_{n,m}f)^{(r)}(x) \quad &= \quad f^{(r)}(x) + \sum_{k=0}^{q} \frac{[\beta_0, \ldots, \beta_m; e_{-k-1}]}{n^{m+1+k}} \sum_{i=0}^{m+1} S_{m+1}^{i} c_{k+i}^{(r)}(f;x) \\ &\quad + o\left(n^{-q-m-1}\right)\end{aligned}$$

as $n \to \infty$. In particular, if $q = 0$, we obtain the Voronovskaya type formula

$$\lim_{n\to\infty} n^{m+1} \left((G_{n,m}f)^{(r)}(x) - f^{(r)}(x)\right) = (-1)^m \left(\prod_{\nu=0}^{m} \beta_\nu^{-1}\right) \sum_{i=0}^{m+1} S_{m+1}^{i} c_i^{(r)}(f;x).$$

The special cases $\beta_j = 2^j$ $(j = 0, \ldots, m)$ for odd m and $\beta_j = j+1$ $(j = 0, \ldots, m)$ were previously considered in [3] and [4, Section 5], respectively.

4. Left Gamma Quasi-Interpolants

In his recent paper [9] Müller introduced and studied left quasi-interpolants of his Gamma operators (1). For each integer $k \geq 0$, the left Gamma quasi-interpolants (LGQI) are given by

$$G_n^{[k]} f = \sum_{j=0}^{k} \alpha_j^n \cdot (G_n f)^{(j)}$$

with the polynomials $\alpha_j^n(x) = \left(\frac{x}{n}\right)^j L_j^{(n-j)}(n)$, where $L_j^{(a)}$ denote the Laguerre polynomials. The operators $G_n^{[k]}$ $(k = 0, 1, 2, \ldots)$ are well-defined for $n \geq \max\{2, k\}$ and turn out to be bounded operators on the space $L_p(0, \infty)$, for $1 \leq p \leq \infty$ ([9, Theorem 2]). In the special case $k = 0$ the LGQI reduce to the Gamma operators $G_n^{[0]} \equiv G_n$.

The following theorem presents the complete asymptotic expansion for the LGQI.

Theorem 3. *Let $k \in \mathbb{N}_0$ and $x > 0$. For each function $f \in A[2(q+k); x]$, the LGQI possess the asymptotic expansion*

$$G_n^{[k]}(f; x) = f(x) + \sum_{\nu=\lfloor k/2\rfloor+1}^{q} \frac{a_\nu^{[k]}(f; x)}{n^\nu} + o\left(n^{-q}\right) \qquad (n \to \infty),$$

where the coefficients $a_\nu^{[k]}(f; x)$ are given by

$$a_\nu^{[k]}(f; x) = \sum_{\ell=k+1}^{2\nu} b_{\nu,\ell}^{[k]} x^\ell f^{(\ell)}(x) \qquad (\nu = 1, 2, \ldots)$$

with

$$b_{\nu,\ell}^{[k]} = \frac{1}{\ell!} \sum_{m=0}^{\nu} \sum_{j=m}^{\min\{2m,k\}} \binom{\ell}{j} C_{m,2m-j} \sum_{s=0}^{\ell} \binom{j}{s} T(\ell - s, \nu - m)$$

and

$$T(s, \nu) = \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \sigma_{r-1+\nu}^{r-1}. \tag{5}$$

The quantities σ_n^m and $C_{n,m}$ denote the Stirling numbers of second kind and the Jordan coefficients of first kind, respectively. For the case $r = 0$ in (5), we interpret $\sigma_{-1+\nu}^{-1}$ as $\delta_{\nu,0}$.

As a consequence of Theorem 3, we get the following Voronovskaya type result on the limit $\lim_{n\to\infty} n^{\lfloor k/2\rfloor+1}\left(G_n^{[k]}(f; x) - f(x)\right)$. In order to simplify the notation we distinguish between even and odd values of k.

Corollary 2. *Let $k \in \mathbb{N}_0$ and $x > 0$. For each $f \in A[2(q+k); x]$, the LGQI satisfy*

$$\begin{aligned}
&\lim_{n\to\infty} n^{k+1}\left(G_n^{[2k]}(f; x) - f(x)\right) \\
&\qquad = \frac{(-1)^k}{2^{k+1}(k+1)!}\left(\frac{4}{3}k(k+1)x^{2k+1}f^{(2k+1)}(x) + x^{2k+2}f^{(2k+2)}(x)\right), \\
&\lim_{n\to\infty} n^{k+1}\left(G_n^{[2k+1]}(f; x) - f(x)\right) = \frac{(-1)^k}{2^{k+1}(k+1)!}x^{2k+2}f^{(2k+2)}(x).
\end{aligned}$$

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