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Exact Jackson Inequality in L_2

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We consider Jackson inequalities between the best approximation of periodic functions by trigonometric polynomials, as well as the best approximation of functions in the space $L^2(-\infty, \infty)$ by entire functions, and the generalized modulus of continuity (in particular, the classical modulus of continuity of order $r \geq 1$) of the functions. We discuss properties of the best constant in the Jackson inequality as a function of the argument of the modulus of continuity.

1. The Jackson Inequality for Approximation of Periodic Functions by Trigonometric Polynomials in $L^2_{2\pi}$

Let $L^2_{2\pi}$ be the classical space of complex-valued, 2π-periodic, measurable, squared summable over the period functions, equipped with the norm

$$\|f\|_{L^2_{2\pi}} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx \right)^{1/2}.$$

We denote by $\mathcal{T}_n,\ n \geq 0$, the set of trigonometric polynomials

$$g(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

of order (at most) n with complex coefficients. For a function $f \in L^2_{2\pi}$, the quantity

$$E_n(f) = \inf\{\ \|f - g\|_{L^2_{2\pi}} :\ g \in \mathcal{T}_n\ \}$$

is the best approximation of the function by the set $\mathcal{T}_n$.

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The first difference with step $h \in (-\infty, \infty)$ is the operator $\Delta_h = \Delta_h^1$ defined by the formula $(\Delta_h f)(x) = f(x+h) - f(x)$. For an integer $r \geq 1$, the r-th power Δ_h^r of the operator Δ_h is called the r-th difference. It holds

$$\Delta_h^r f(x) = \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x+\nu h).$$

For a function $f \in L^2_{2\pi}$, the function

$$\omega_r(\delta, f) = \max\{\|\Delta_h^r f\|_{L^2_{2\pi}} : |h| \leq \delta\}, \qquad \delta \in [0, \infty),$$

is the modulus of continuity of f of order r. Inequalities of the form

$$E_{n-1}(f) \leq K_{n,r}(\delta)\ \omega_r(\delta, f), \qquad f \in L^2_{2\pi}, \tag{1}$$

with constants which do not depend on the function $f \in L^2_{2\pi}$ are called Jackson inequalities. In what follows we shall use the notation $K_{n,r}(\delta)$ always for the best (the least) constants in these inequalities.

Chernykh [4], [5] proved that for any $n \geq 1$ and $\delta \geq \frac{\pi}{n}$ the exact inequality

$$E_{n-1}(f) < \frac{1}{\sqrt{2}}\ \omega_1(\delta, f), \qquad f \in L^2_{2\pi},\ f \not\equiv \text{const}, \tag{2}$$

takes place. He proved also [2] that inequality (2) with the constant $1/\sqrt{2}$ does not hold for $0 < \delta < \frac{\pi}{n}$. Thus,

$$K_{n,1}(\delta) = \frac{1}{\sqrt{2}},\ \ \delta \geq \frac{\pi}{n}; \qquad\qquad K_{n,1}(\delta) > \frac{1}{\sqrt{2}},\ \ 0 < \delta < \frac{\pi}{n}. \tag{3}$$

Chernykh obtained a similar result for moduli of continuity of higher order, too. Namely, he proved [5] that the inequality

$$E_{n-1}(f) < \sqrt{1 \Big/ \binom{2r}{r}}\ \omega_r(\delta, f), \qquad f \in L^2_{2\pi},\ f \not\equiv \text{const}, \tag{4}$$

holds for any $n \geq 1$, $r \geq 2$ and $\delta \geq \frac{2\pi}{n}$. Moreover, if $n > r \geq 2$, $\frac{2\pi}{n} \leq \delta < \frac{2\pi}{r}$, then (4) is sharp.

Let us associate to a function $f \in L^2_{2\pi}$ its Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikt}, \qquad \hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt, \qquad k \in \mathbb{Z}.$$

Parseval equality $\|f\|^2_{L^2_{2\pi}} = \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2$ gives the following relation for the difference of order r with step h of a function f:

$$\|\Delta_h^r f\|^2_{L^2_{2\pi}} = \sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \varphi_r(kh);$$

here $\varphi_r(x) = 2^r(1-\cos x)^r$. Thus,

$$\omega_r(\delta, f) = \max\left\{\left(\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \varphi_r(kh)\right)^{1/2} : \ |h| \le \delta\right\}, \qquad \delta \ge 0. \tag{5}$$

Vasil'ev (see [11] and the references therein) introduced the following generalization of the modulus of continuity. Assume that a function φ is defined, bounded and non-negative on the real line. For a function $f \in L^2_{2\pi}$ define the (generalized) modulus of continuity as

$$\overline{\omega}(\delta, f) = \overline{\omega}_\varphi(\delta, f) = \max\left\{\left(\sum_{k\in\mathbb{Z}} |\hat{f}_k|^2 \varphi(kh)\right)^{1/2} : \ 0 \le |h| \le \delta\right\}, \quad \delta \ge 0. \tag{6}$$

We shall discuss the Jackson inequality with this modulus of continuity

$$E_{n-1}(f) \le K_n(\delta, \varphi)\, \overline{\omega}(\delta, f), \qquad f \in L^2_{2\pi}. \tag{7}$$

Let $K_n(\delta, \varphi)$ be the least constant in this inequality.

We shall denote by $C^+_{2\pi}$ the set of continuous, 2π-periodic, non-negative functions. Further, we denote by Φ the set of even functions $\varphi \in C^+_{2\pi}$ which satisfy the properties:

1) $\varphi(0) = 0$;
2) $\frac{1}{T}\int_0^T \varphi(t)\, dt \le \frac{1}{\pi}\int_0^\pi \varphi(t)\, dt, \ 0 < T < \pi$.

For a function $\varphi \in C^+_{2\pi}$ put $I(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, dt$. Vasil'ev [11] proved that if $\varphi \in \Phi$, then

$$K_n^2(\delta, \varphi) \le \frac{1}{I(\varphi)}, \qquad \delta \ge \frac{7\pi}{5n}. \tag{8}$$

The function φ_r belongs to the set Φ, and $I(\varphi_r) = \binom{2r}{r}$. Hence, it follows from (8), in particular, that inequality (4) holds for $\delta \ge \frac{7\pi}{5n}$. Moreover, it is proved in [11] that inequality (4) is sharp for $n > 0.7\,\pi$, $\ r \ge 2$, $\delta \ge \frac{7\pi}{5n}$. Jackson inequalities with different moduli of continuity were studied by Kozko and Rozhdestvenskii [8]. In particular, they proved that the estimate

$$K_n^2(\delta, \varphi) \ge \frac{1}{I(\varphi)}, \qquad \delta > 0, \tag{9}$$

is valid for any function $\varphi \in C^+_{2\pi}$, $\ \varphi \not\equiv 0$.

It is also of interest to study the least value

$$\mathbf{K}_n(\varphi) = \min\{K_n(\delta, \varphi) : \ \delta > 0\}$$

of the best constant in the Jackson inequality (7), and the least value $\delta_n(\varphi) = \inf\{\delta > 0 : \ K_n(\delta, \varphi) = \mathbf{K}_n(\varphi)\}$ of arguments of the modulus of continuity in inequalities with the least value $\mathbf{K}_n(\varphi)$ of the best constant. The quantity $\delta_n(\varphi)$

is called the optimal point in the Jackson inequality. It follows from the results by Vasil'ev, Kozko, and Rozhdestvenskii cited above that if $\varphi \in \Phi$, $\varphi \not\equiv 0$, then

$$\mathbf{K}_n(\varphi) = \frac{1}{I(\varphi)}, \qquad \delta_n(\varphi) \le \frac{7\pi}{5n}. \tag{10}$$

Introduce the notations $\mathbf{K}_{n,r} = \mathbf{K}_n(\varphi_r)$, $\delta_{n,r} = \delta_n(\varphi_r)$, for the corresponding quantities in the classical inequality (1). Chernykh's statement (3) means that

$$\mathbf{K}_{n,1} = \frac{1}{\sqrt{2}}, \qquad \delta_{n,1} = \frac{\pi}{n}.$$

The results by Chernykh (4) and by Vasil'ev, Kozko, and Rozhdestvenskii (10) imply that for $n \ge 1$, $r \ge 2$, we have

$$\mathbf{K}_{n,r} = \frac{1}{\sqrt{\binom{2r}{r}}}, \qquad \delta_{n,r} \le \frac{1.4\,\pi}{n}. \tag{11}$$

2. The Jackson Inequality for Approximation of Functions in the Space $L^2(-\infty, \infty)$ by Entire Functions

Results which are analogous to (1)–(11) are also known for approximation of functions in $L^2(-\infty, \infty)$ by entire functions. Let $L^2 = L^2(-\infty, \infty)$ be the space of complex-valued, measurable, square summable over the real line functions. The space L^2 is equipped with the norm

$$\|f\|_2 = \|f\|_{L_2} = \left(\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right)^{1/2}.$$

Useful means of studying problems of approximation theory in the space L^2 are the Fourier transform

$$\widehat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx$$

and Parseval equality $\|f\|_2 = \|\widehat{f}\|_2$, $f \in L^2$. Let $W(\sigma)$, $\sigma > 0$, be the set of functions $g \in L^2$ with Fourier transforms vanishing outside the interval $(-\sigma, \sigma)$. A function $g \in L^2$ belongs to the set $W(\sigma)$ if and only if it is a restriction to the real line of an entire function of exponential type σ. Let us denote by $\mathcal{K}_r(\delta, \sigma)$ the best constant in the Jackson inequality

$$E_\sigma(f) \le \mathcal{K}_r(\delta, \sigma)\, \omega_r(\delta, f), \qquad f \in L^2, \tag{12}$$

between the best approximation

$$E_\sigma(f) = \inf\{\, \|f - g\|_2 : \ g \in W(\sigma) \,\}$$

of a function $f \in L^2$ by the space $W(\sigma)$ of entire functions of exponential type σ and its modulus of continuity of order r

$$\omega_r(\delta, f) = \max\{\|\Delta_h^r f\|_{L^2(-\infty,\infty)} : \ |h| \le \delta\}, \qquad \delta \in [0, \infty). \tag{13}$$

Ibragimov and Nasibov [7] obtained the following result. For any $\sigma > 0$ and any function $f \in L^2, \ f \not\equiv 0$, the following inequalities hold true

$$E_\sigma(f) < \frac{1}{\sqrt{2}}\,\omega_1\left(\frac{\pi}{\sigma}, f\right), \qquad E_\sigma(f) < \frac{1}{2}\,\omega_2\left(\frac{\pi}{\sigma}, f\right).$$

The next result was obtained by Popov [10] independently from the result cited above. Let $\sigma > 0$. Then, for any nonzero function $f \in L^2(-\infty, \infty)$,

$$E_\sigma(f) < \frac{1}{\sqrt{2}}\,\omega_1\Big(\frac{\delta}{\sigma}, f\Big), \qquad \delta \ge \pi,$$

$$E_\sigma(f) < \sqrt{1\Big/\binom{2r}{r}}\,\omega_r\Big(\frac{\delta}{\sigma}, f\Big), \qquad \delta \ge 2\pi, \quad r \ge 2.$$

These inequalities are sharp. Thus, for the best constant $\mathcal{K}_r(\delta, \sigma)$ in (12) we have

$$\mathcal{K}_1\Big(\frac{\delta}{\sigma}, \sigma\Big) = \frac{1}{\sqrt{2}} \qquad \text{if} \quad \delta \ge \pi; \tag{14}$$

$$\mathcal{K}_r\Big(\frac{\delta}{\sigma}, \sigma\Big) = \sqrt{1\Big/\binom{2r}{r}} \qquad \text{if} \quad \delta \ge 2\pi, \quad r \ge 2. \tag{15}$$

In addition to (14) one may state that $\mathcal{K}_1\left(\frac{\delta}{\sigma}, \ \sigma\right) > \frac{1}{\sqrt{2}}$ for $0 < \delta < \pi$. This follows from Chernykh's result (3) for the periodic case; this follows also from a later result by Logan [9]. The results of Vasil'ev's paper [11] related to the periodic case yield, as a particular case, that inequality (15) holds for $\delta \ge 1.4\,\pi$.

Introduce the notations

$$\underline{\mathcal{K}}_r(\sigma) = \inf\{\mathcal{K}_r(\delta, \sigma) : \ \delta > 0\}, \quad \delta_r(\sigma) = \inf\{\delta > 0 : \ \mathcal{K}_r(\delta, \sigma) = \underline{\mathcal{K}}_r(\sigma)\}$$

for the least value of the best constant and for the optimal point in the Jackson inequality (12). It follows from results by Chernykh, Ibragimov, Nasibov, Popov, and Vasil'ev that

$$\underline{\mathcal{K}}_r(\sigma) = \sqrt{1\Big/\binom{2r}{r}}, \ r \ge 1; \quad \delta_1(\sigma) = \frac{\pi}{\sigma}; \quad \delta_r(\sigma) \le \frac{1.4\,\pi}{\sigma}, \ r > 1. \tag{16}$$

Chernykh conjectured in 1998 that $\delta_2(\sigma) = \frac{\pi}{\sigma}$. However, Arestov and Babenko showed recently that in fact

$$\delta_2(\sigma) > \frac{\pi}{\sigma}.$$

As early as in 1984, the author discussed with S. B. Stechkin the question on continuity of the constant $K_{n,1}(\delta)$ in inequality (1) with respect to $\delta \in (0, \pi]$ for $r = 1$. That time we only knew that the constant is right continuous. The fact that the constant $\mathcal{K}_r(\delta, \sigma)$ is right continuous with respect to $\delta \in (0, +\infty)$ was also mentioned in [3, p.346], [6, Remark on p.45]. Nowadays continuity of the constants $\mathcal{K}_r(\delta, \sigma)$ and $K_{n,r}(\delta)$ is known for all $r \geq 1$.

Theorem 1 (Arestov and Babenko [1]). *For any* $r \geq 1$, $\sigma > 0$, *the best constant* $\mathcal{K}_r(\delta, \sigma)$ *in inequality* (12) *is continuous with respect to* $\delta \in (0, \infty)$.

Balaganskii proved recently by a different method that the best constant $K_{n,r}(\delta)$ in inequality (1) is continuous with respect to $\delta \in (0, \pi]$ for any n and r.

3. The Jackson Inequality with the Generalized Modulus of Continuity in the Space $L^2(-\infty, \infty)$

Using a change of the variable, it is not difficult to see that the equality $\mathcal{K}_r(\delta, \sigma) = \mathcal{K}_r(\delta\sigma, 1)$ is valid for the constant $\mathcal{K}_r(\delta, \sigma)$ in (12). Therefore, it is enough to study inequality (12) for $\sigma = 1$ only. Now, let $\mathcal{K}_r(\delta)$ be the best constant in the inequality

$$E(f) \leq \mathcal{K}_r(\delta)\ \omega_r(\delta, f), \qquad f \in L^2; \tag{17}$$

here and below we use the notation $E(f) = E_1(f)$.

By virtue of Parseval equality we have (e.g. [10])

$$\|\Delta_s^r f\|_2^2 = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \varphi_r(st)\, dt, \tag{18}$$

where $\varphi_r(x) = 2^r(1 - \cos x)^r$, $x \in [0, \infty)$. Starting with formula (18), we generalize the concept of the modulus of continuity by analogy with (6). Let us denote by $C^+ = C^+[0, \infty)$ the set of continuous, bounded, non-negative functions on the half-line $[0, \infty)$. Now we define the (generalized) modulus of continuity of a function $f \in L^2$ using a function $\varphi \in C^+$, $\varphi \not\equiv 0$, as follows

$$\overline{\omega}(\delta) = \overline{\omega}(\delta, f) = \max\left\{\sqrt{F(s, f)}:\ 0 \leq s \leq \delta\right\}, \qquad \delta \in [0, \infty); \tag{19}$$

$$F(s, f) = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \varphi(s|t|)\, dt, \qquad s \in [0, \infty).$$

Under the assumptions on the function φ we made, both functions just introduced are continuous and bounded on the half-line $[0, \infty)$. By virtue of (18), the function (19) with $\varphi = \varphi_r$ coincides with the modulus of continuity (13) of the function f of order r.

Let $\mathcal{K}(\delta) = \mathcal{K}(\delta, \varphi)$ be the best constant in the (generalized) Jackson inequality

$$E(f) \le \mathcal{K}(\delta)\, \overline{\omega}(\delta, f), \qquad f \in L^2. \tag{20}$$

It may happen that inequality (20) does not hold with a finite constant; in such a case we put $\mathcal{K}(\delta) = +\infty$. Inequality (17) is a particular case of (20) for $\varphi = \varphi_r$.

Now we present effective estimates of the least value $\underline{\mathcal{K}} = \inf\{\mathcal{K}(\delta) : \delta > 0\}$ of the quantity $\mathcal{K}(\delta)$. Function (19) does not decrease with respect to $\delta \in (0, \infty)$; hence, $\mathcal{K}(\delta)$ does not increase. Therefore,

$$\underline{\mathcal{K}} = \inf\{\mathcal{K}(\delta) : \delta > 0\} = \lim_{\delta \to +\infty} \mathcal{K}(\delta). \tag{21}$$

Consider the function $\vartheta(T) = \frac{1}{T}\int_0^T \varphi(t)\, dt,\ T > 0$. Put

$$I^* = I^*(\varphi) = \overline{\lim_{T \to +\infty}}\, \vartheta(T), \qquad I_* = I_*(\varphi) = \underline{\lim_{T \to +\infty}}\, \vartheta(T).$$

Obviously, we have $0 \le I_* \le I^* \le \|\varphi\|_{C[0,\infty)}$.

Theorem 2 (Arestov). *If φ is such that $\varphi \in C^+[0, \infty)$, $\varphi \not\equiv 0$, $\varphi(0) = 0$, then the following estimates take place for the constant* (21)

$$\frac{1}{I^*(\varphi)} \le \underline{\mathcal{K}}^2 \le \frac{1}{I_*(\varphi)}. \tag{22}$$

If φ is 2π-periodic, then $I^*(\varphi) = I_*(\varphi) = I(\varphi)$, where

$$I(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, dt.$$

In particular, for $\varphi_r(x) = 2^r(1 - \cos x)^r$ we have $I(\varphi_r) = \binom{2r}{r}$, and thus (22) turns to coincide with the first statement in (16) for the classical moduli of continuity of order r.

The statement of Theorem 2 is an analogue of the first relation in (10) for the periodic case.

The following statement generalizes Theorem 1.

Theorem 3 (Arestov and Babenko [1]). *For any function $\varphi \in C^+[0, \infty)$, $\varphi \not\equiv 0$, the best constant $\mathcal{K}(\delta)$ in inequality* (20) *is continuous with respect to δ on the half-line* $(0, \infty)$.

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