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Wavelet Decomposition and Sampling for p-adic Multiresolution Analysis

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We define a p-adic Multiresolution Analysis (MRA) on the space of p^M-periodic sequences, where $p, M \in \mathbb{Z}^+$. We present the Sampling Theorem on MRA subspaces and we discuss the existence of p-adic wavelets which provide a variety of new Discrete Transforms.

1. Introduction

It is well-known that the Discrete Fourier Transform (DFT) is one of the most widely used tools in communication, engineering and computational mathematics. Recall the following definition of the DFT of an N-periodic sequence $s(n)$:

$$\widehat{s}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n) e^{-2\pi i n k/N}, \qquad k = 0, ..., N-1.$$

The Inverse Discrete Fourier Transform is given by the formula:

$$s(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{s}(k) e^{2\pi i n k/N}, \qquad n = 0, ..., N-1.$$

The DFT Analysis will be proven very useful for defining a Multiresolution Analysis over $L^2(\mathbb{Z}/(N\mathbb{Z}))$. Recall that a *Multiresolution Analysis* of $L^2(\mathbb{R})$ (*MRA*), is a nested sequence $\{V_m \subset V_{m+1}, m \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ and a scaling function φ such that:

(i) $\overline{\bigcup_m V_m} = L_2(\mathbb{R})$ and $\bigcap_m V_m = \{\mathbf{0}\}$;

(ii) $f \in V_0 \Leftrightarrow f(2^m \cdot) \in V_m$, $m \in \mathbb{Z}$;

(iii) The set $\{\varphi(\cdot - n),\ n \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

Since in most practical applications only sampled data are available, it is natural to wander if we can build a similar construction on spaces of periodic sequences. In Section 2 we define a Multiresolution Analysis on the space of p^M-periodic sequences, so that we can be able to use two important advantages of MRA's: the existence of sampling expansions in MRA subspaces and a Wavelet Decomposition algorithm. In Section 3 we discuss the Decomposition algorithm.

2. The p-adic Multiresolution Analysis

Definition 1. *Let $p, M > 1$ be positive integers. We define $V_M = \{s(n) : n = 0, ..., p^M - 1\}$ to be the p^M-dimensional vector space of all complex p^M-periodic sequences with the usual orthonormal basis $e_k(n) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta.*

The main difficulty in defining a Multiresolution Analysis for spaces of p^M-periodic sequences is the notion of the Dilation operator for sequences. We give the following:

Definition 2. *Let $p, M > 1$ be positive integers and let φ_0 be a p^M-periodic sequence. The* Dilation operator *with respect to the sequence φ_0 is:*

$$D_{\varphi_0} : V_M \to V_M,$$

$$s \to (D_{\varphi_0}s)(n) = \frac{1}{\sqrt{p^M}} \sum_{k=0}^{p^{M-1}-1} \varphi_0(n-kp) \sum_{m=0}^{p-1} s(k+mp^{M-1}), \quad n = 0, ..., p^M-1,$$

or equivalently:

$$\widehat{(D_{\varphi_0}s)}(n) = \widehat{s}(pn)\widehat{\varphi_0}(n), \qquad n = 0, ..., p^M - 1,$$

where $\widehat{s}(n)$ is the Discrete Fourier Transform of the p^M-periodic sequence $s(n)$.

Definition 3. *We shall say that a p-*adic Multiresolution Analysis *(p-*adic MRA*) of V_M is a nested sequence $\{V_j : j = 0, ..., M-1\}$ of subspaces of V_M and a* scaling sequence *$\{\varphi_1(n)\}$ such that:*

(i) $\{c(n) : c(n) = (\alpha, ..., \alpha),\ \alpha \in \mathbf{C}\} = \mathrm{V}_0 \subset \mathrm{V}_1 \subset ... \subset \mathrm{V}_{\mathrm{M}-1} \subset \mathrm{V}_{\mathrm{M}}$*;*

(ii) V_j is the linear span of an orthonormal set $\{\varphi_{M-j}(\cdot - kp^{M-j}) : k = 0, ..., p^j - 1\}$, where for $j = M-1$ the scaling sequence $\{\varphi_1(n)\}$ is the dilation of the sequence $e_0(n) = (1, 0, ..., 0)$ and for $j = 0, ..., M-2$ the sequence $\{\varphi_{M-j}(n)\}$ is the dilation of the sequence $\{\varphi_{M-j-1}(n)\}$.

Definition 4. *Given the Dilation operator* (D_{φ_0}), *its associate sequence is given by*

$$\varphi_{M-j}(n) = \begin{cases} \frac{\varphi_0(n)}{\sqrt{p^M}} = (D_{\varphi_0} e_k)(n), & j = M-1 \\ (D_{\varphi_0}\varphi_{M-j-1})(n), & j = 0, ..., M-2. \end{cases}$$

The associate sequence is given by its DFT form in the following:

$$\widehat{\varphi_{M-j}}(k) = \begin{cases} \frac{\widehat{\varphi_0}(k)}{\sqrt{p^M}}, & j = M-1 \\ \widehat{\varphi_{M-j-1}}(pk)\widehat{\varphi_0}(k), & j = 0, ..., M-2 \end{cases}. \tag{1}$$

Proposition 1. *The Dilation operator* D_{φ_0} *satisfies the following:*

(i) $(D_{\varphi_0}T_1 s)(n) = (T_p D_{\varphi_0} s)(n)$, *where* $T_k : V_M \to V_M$, $(T_k s)(n) = s(n-k)$ *is the Translation operator for sequences;*

(ii) $\|D_{\varphi_0}\|_2 = p^{1/2}$;

(iii) Let $\hat{\varphi}_0(k_1) = ... = \hat{\varphi}_0(k_m) = 0, \ \ 0 \le m < p^N - 1$. *Then:*

$$\mathrm{Ker}\, D_{\varphi_0} = \{s(n) : \hat{s}(pn) = 0, \ n \neq k_i\};$$

(iv) Let $s(n) \in V_j$, *then* $(D_{\varphi_0} s)(n) \in V_{j-1}$.

Proof. (i) It is an immediate consequence of Definition 2.
(ii)

$$\|D_{\varphi_0} s\|_2^2 = \sum_{n=0}^{p^M-1} |\hat{s}(pn)|^2 |\hat{\varphi}_0(n)|^2 = \sum_{k=0}^{p-1} \sum_{m=0}^{p^{M-1}-1} |\hat{s}(pm)|^2 |\hat{\varphi}_0(m + kp^{M-1})|^2$$
$$= p \sum_{m=0}^{p^{M-1}-1} |\hat{s}(pm)|^2 \le p\|s(\,\cdot\,)\|_2^2.$$

Obviously:

$$\|D_{\varphi_0}\|_2 = \sup_{s \in V_M} \frac{\|D_{\varphi_0} s(\,\cdot\,)\|_2}{\|s(\,\cdot\,)\|_2} \le \sqrt{p},$$

but we can find a sequence $s_1(n)$ to obtain equality. In fact, we define $s_1(n)$ by its DFT form in the following:

$$\hat{s_1}(n) = \begin{cases} 1, & n = pk \\ 0, & n \neq pk. \end{cases}$$

(iii) It is an immediate consequence of the fact that $\widehat{(D_{\varphi_0} s)}(n) = \widehat{s}(pn)\widehat{\varphi_0}(n)$.

(iv) Let $s(n) \in V_j$. Then:

$$(D_{\varphi_0}s)(n) = \sum_{k=0}^{p^j-1} c_k (D_{\varphi_0}T_{kp^{M-j}}\varphi_{M-j})(n) = \sum_{k=0}^{p^j-1} c_k (T_{kp^{M-j+1}}D_{\varphi_0}\varphi_{M-j})(n)$$
$$= \sum_{k=0}^{p^j-1} c_k \varphi_{M-j+1}(n-kp^{M-j+1})$$
$$= \sum_{l=0}^{p^{j-1}-1}\left(\sum_{r=0}^{p-1} c_{l+rp^{j-1}}\right)\varphi_{M-j+1}(n-lp^{M-j+1}). \quad \square$$

Proposition 2. *Let φ_0 be a p^M-periodic sequence and let the collection $\{\varphi_{M-j}(n)\}_{j=0}^{M-1}$ be as in (1). If*

$$\sum_{s=0}^{p-1} |\widehat{\varphi_0}(r+sp^{M-1})|^2 = p, \qquad r = 0,...,p^{M-1}-1, \tag{2}$$

then:

(i) The collection $\{\varphi_{M-j}(\,\cdot\, - kp^{M-j}) : k = 0,...,p^j-1\}$, $(j = 0,...,M-1)$ is an orthonormal basis of V_j;

(ii) The subspaces V_j form a p-adic MRA of V_M with scaling sequence $\varphi_1(n) = \frac{\varphi_0(n)}{\sqrt{p^M}}$.

Proof. The proof was presented in [1]. We shall briefly sketch it. First we find a necessary and sufficient condition for the orthonormality of the set $\{\varphi_{M-j}(\,\cdot\, - mp^{M-j}),\ m = 0,...,p^j-1\}$:

$$\sum_{s=0}^{p^{M-j}-1} |\widehat{\varphi_{M-j}}(r+sp^j)|^2 = \frac{1}{p^j}, \qquad r = 0,...,p^j-1. \tag{3}$$

Let $j = M-1$. Using (2), (3) and Definition 4 we see that the p-translations of φ_1 form an orthonormal basis of V_{M-1}. In fact:

$$\sum_{s=0}^{p-1} |\widehat{\varphi_1}(r+sp^{M-1})|^2 = \frac{\sum_{s=0}^{p-1} |\widehat{\varphi_0}(r+sp^{M-1})|^2}{p^M} = \frac{1}{p^{M-1}}, \quad r = 0,...,p^{M-1}-1,$$

and thus Proposition 2 is valid for $j = M-1$. Now let $j = 0,...,M-2$. We suppose that the set $\{\varphi_{M-j}(\,\cdot\, - kp^{M-j})\}_{k=0}^{p^j-1}$ is an orthonormal basis of V_j and we use (3) in combination with (1) and (2) to show that the p^{M-j+1} translations of the sequence $\varphi_{M-j+1}(n)$ form an orthonormal basis of V_{j-1}. The proof follows by induction. $\square$

Proposition 3. *Let $\varphi(n)$ be a p^M-periodic sequence which satisfies the condition*

$$\sum_{m=0}^{p^{M-1}-1} \varphi(k+mp)\overline{\varphi(k+(m-r)p)} = \begin{cases} 1/p, & r=0 \\ 0, & r=1,...,p^{M-1}-1 \end{cases}$$

for $k=0,...,p-1$, then $\varphi(n)$ is the scaling sequence of a p-adic MRA.

Proof. The proof was presented in [1]. □

3. The p-adic Sampling Sequence

Definition 5. *We say that an M-dimensional subspace W of V has a* sampling basis $\{s_0,...,s_{M-1}\}$, *if there exist M positive integers $0 \le n_1 < ... < n_M < N$ such that for any sequence $a(n) \in W$ we have:*

$$a(n) = \sum_{j=0}^{M-1} a(n_j)s_j(n), \qquad 0 \le n \le N-1.$$

In particular, we say that W has a sampling sequence $s(n)$ *(which is N-periodic), if there exist M positive integers $0 \le n_1 < ... < n_M < N$ such that for any sequence $a(n) \in W$ we have:*

$$a(n) = \sum_{j=0}^{M-1} a(n_j)s(n-n_j), \qquad 0 \le n \le N-1.$$

Theorem 1. *Let $\varphi_0(n)$ be a p^M-periodic sequence which produces a p-adic MRA $\{V_j\}_{j=0}^{M-1}$. If*

$$\sum_{s=0}^{p-1} \widehat{\varphi_0}(r+sp^{M-1}) \neq 0, \qquad 0 \le r \le p^{M-1}-1, \tag{4}$$

then any sequence $f(n) \in V_j$ has the following sampling expansion:

$$f(n) = \sum_{m=0}^{p^j-1} f(mp^{M-j})s_{M-j}(n-mp^{M-j}), \qquad n=0,...,p^M-1,$$

where

$$s_{M-j}(\,\cdot\,) \leftrightarrow \frac{\sqrt{p^M}}{p^j} \cdot \frac{\widehat{\varphi_{M-j}}(\,\cdot\,)}{\sum_{r=0}^{p^{M-j}-1} \widehat{\varphi_{M-j}}(\,\cdot\,+r^j)}. \tag{5}$$

Obviously, the sequence $s_{M-j}(n)$ is the sampling sequence *of the MRA subspace V_j.*

Proof. The proof was presented in [1]. We give here a slightly alternate proof. The hypothesis (4) implies that for any $j = 0, ..., M-1$ there holds:

$$\sum_{s=0}^{p^{M-j}-1} \widehat{\varphi_{M-j}}(r + sp^j) \neq 0, \qquad 0 \leq r \leq p^j - 1. \tag{6}$$

By [2, p. 255], it suffices to prove that the set $\{K_j(\,\cdot\,, mp^{M-j}) : m = 0, ..., p^j-1\}$ is a basis for V_j, where

$$K_j(n,m) = \sum_{r=0}^{p^j-1} \varphi_{M-j}(n - rp^{M-j})\overline{\varphi_{M-j}(m - rp^{M-j})}, \qquad 0 \leq n, m \leq N-1,$$

is the reproducing kernel of V_j, or equivalrntly we try to find a positive constant $A > 0$ such that for any sequence of scalars $\{a_m\}_{m=0}^{p^j-1}$, we have

$$A \sum_{m=0}^{p^j-1} |a_m|^2 \leq \left\| \sum_{m=0}^{p^j-1} a_m K_j(\,\cdot\,, mp^{M-j}) \right\|_2^2.$$

After some calculations concerning the DFT of the kernel we find

$$\left\| \sum_{m=0}^{p^j-1} a_m \widehat{K_j}(\,\cdot\,, mp^{M-j}) \right\|_2^2 = \frac{1}{p^{M-j}} \sum_{k=0}^{p^j-1} \sum_{l=0}^{p^{M-j}-1} |\widehat{a_k}|^2 |\widehat{\varphi_{M-j}}(k + lp^j)|^2 \times \left| \sum_{s=0}^{p^{M-j}-1} \widehat{\varphi_{M-j}}(k + sp^j) \right|^2.$$

Thus $\{K_j(\,\cdot\,, mp^{M-j}) : m = 0, ..., p^j - 1\}$ is a basis of V_j, (see (6)) and it possesses a unique biorthonormal sequence $\{s_{M-j,m} : m = 0, ..., p^j - 1\}$ which is a sampling basis for V_j (see [2]). Finally,

$$\varphi_{M-j}(n) = \sum_{k=0}^{p^j-1} \varphi_{M-j}(kp^{M-j}) s_{M-j}(n - kp^{M-j})$$

and the DFT of the equality above implies (5). $\square$

4. Spectral Wavelet Decomposition

We expect that this work provides wavelet Analysis. We believe that the wavelet sequence associated with our MRA is the following:

Definition 6. *Given the Dilation operator* (D_{φ_0})*, then for* $l = 1, ..., p-1$ *its associate sequence is given by*

$$\widehat{\psi^l_{M-j}}(k) = \begin{cases} \widehat{\varphi_1}(k + lp^{M-1}), & j = M-1 \\ \widehat{\psi^l_{M-j-1}}(pk)\widehat{\varphi_0}(k), & j = 0, ..., M-2. \end{cases}$$

Assuming that $\varphi_1(n)$ is an appropriately chosen scaling sequence of a p-adic MRA, we believe that each wavelet subspace W^l_j is the linear span of $\{\psi^l_{M-j}(\,\cdot\, - kp^{M-j}) : k = 0, ..., p^j - 1\}$ and we can write

$$V_M = V_0 \bigoplus_{j=0}^{M-1} \bigoplus_{l=1}^{p-1} W^l_j,$$

which implies the following decomposition algorithm

$$s(n) = \sum_{j=0}^{M-1} \sum_{k=0}^{p^j-1} \sum_{l=0}^{p-1} \langle s(\,\cdot\,), \psi^l_{M-j}(\,\cdot - kp^{M-j})\rangle \psi^l_{M-j}(n - kp^{M-j}) + \frac{1}{p^M} \sum_{k=0}^{p^M-1} s(k).$$

References

[1] N. Atreas and C. Karanikas, Sampling formulas for spectral analysis on spaces of p^M-periodic sequences for computational applications of immune type, submitted.

[2] A. I. Zayed, "Advances in Shannon's Sampling Theory", CRC Press Inc., 1993.

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