Wavelet Decomposition and Sampling for $p$-adic Multiresolution Analysis

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We define a $p$-adic Multiresolution Analysis (MRA) on the space of $p^M\mathbb{Z}$-periodic sequences, where $p, M \in \mathbb{Z}^+$. We present the Sampling Theorem on MRA subspaces and we discuss the existence of $p$-adic wavelets which provide a variety of new Discrete Transforms.

1. Introduction

It is well-known that the Discrete Fourier Transform (DFT) is one of the most widely used tools in communication, engineering and computational mathematics. Recall the following definition of the DFT of an $N$-periodic sequence $s(n)$:

$$\hat{s}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n) e^{-2\pi i nk/N}, \quad k = 0, ..., N - 1.$$  

The Inverse Discrete Fourier Transform is given by the formula:

$$s(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{s}(k) e^{2\pi i nk/N}, \quad n = 0, ..., N - 1.$$  

The DFT Analysis will be proven very useful for defining a Multiresolution Analysis over $L^2(\mathbb{Z}/(N\mathbb{Z}))$. Recall that a Multiresolution Analysis of $L^2(\mathbb{R})$ (MRA), is a nested sequence $\{V_m \subset V_{m+1}, m \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ and a scaling function $\varphi$ such that:

(i) $\bigcup_m V_m = L^2(\mathbb{R})$ and $\bigcap_m V_m = \{0\}$;

(ii) $f \in V_0 \iff f(2^m \cdot) \in V_m, m \in \mathbb{Z}$;

(iii) The set $\{\varphi(\cdot - n), n \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$. 
Since in most practical applications only sampled data are available, it is natural to wonder if we can build a similar construction on spaces of periodic sequences. In Section 2 we define a Multiresolution Analysis on the space of $p^M$-periodic sequences, so that we can be able to use two important advantages of MRA’s: the existence of sampling expansions in MRA subspaces and a Wavelet Decomposition algorithm. In Section 3 we discuss the Decomposition algorithm.

2. The $p$-adic Multiresolution Analysis

Definition 1. Let $p, M > 1$ be positive integers. We define $V_M = \{ s(n) : n = 0,...,p^M - 1 \}$ to be the $p^M$-dimensional vector space of all complex $p^M$-periodic sequences with the usual orthonormal basis $e_k(n) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta.

The main difficulty in defining a Multiresolution Analysis for spaces of $p^M$-periodic sequences is the notion of the Dilation operator for sequences. We give the following:

Definition 2. Let $p, M > 1$ be positive integers and let $\varphi_0$ be a $p^M$-periodic sequence. The Dilation operator with respect to the sequence $\varphi_0$ is:

$$D_{\varphi_0} : V_M \to V_M,$$

$$s \to (D_{\varphi_0} s)(n) = \frac{1}{\sqrt{p^M}} \sum_{k=0}^{p^M-1} \varphi_0(n-kp) \sum_{m=0}^{p-1} s(k+mp^{M-1}), \quad n = 0,...,p^M-1,$$

or equivalently:

$$\widehat{(D_{\varphi_0} s)}(n) = \hat{s}(pn)\hat{\varphi_0}(n), \quad n = 0,...,p^M-1,$$

where $\hat{s}(n)$ is the Discrete Fourier Transform of the $p^M$-periodic sequence $s(n)$.

Definition 3. We shall say that a $p$-adic Multiresolution Analysis ($p$-adic MRA) of $V_M$ is a nested sequence $\{V_j : j = 0,...,M-1\}$ of subspaces of $V_M$ and a scaling sequence $\{\varphi_1(n)\}$ such that:

(i) $\{c(n) : c(n) = (\alpha,\ldots,\alpha), \alpha \in \mathbb{C}\} = V_0 \subset V_1 \subset \ldots \subset V_{M-1} \subset V_M$;

(ii) $V_j$ is the linear span of an orthonormal set $\{\varphi_{M-j}(\cdot - kp^{M-j}) : k = 0,...,p^{j-1}\}$, where for $j = M - 1$ the scaling sequence $\{\varphi_1(n)\}$ is the dilation of the sequence $e_0(n) = (1,0,...,0)$ and for $j = 0,...,M-2$ the sequence $\{\varphi_{M-j}(n)\}$ is the dilation of the sequence $\{\varphi_{M-j-1}(n)\}$. 
Definition 4. Given the Dilation operator \( D_{\varphi_0} \), its associate sequence is given by
\[
\varphi_{M-1}(n) = \begin{cases} 
\varphi_0(n) = (D_{\varphi_0}e_k)(n), & j = M - 1 \\
(D_{\varphi_0}\varphi_{M-1})(n), & j = 0, ..., M - 2.
\end{cases}
\]
The associate sequence is given by its DFT form in the following:
\[
\hat{\varphi}_{M-1}(k) = \begin{cases} 
\hat{\varphi}_0(k) / \sqrt{p^M}, & j = M - 1 \\
\hat{\varphi}_{M-1}(pk)\hat{\varphi}_0(k), & j = 0, ..., M - 2.
\end{cases}
\]

Proposition 1. The Dilation operator \( D_{\varphi_0} \) satisfies the following:

(i) \((D_{\varphi_0}T_1s)(n) = (T_pkD_{\varphi_0}s)(n)\), where \( T_k : V_M \to V_M \), \((T_k s)(n) = s(n - k)\) is the Translation operator for sequences;

(ii) \( \|D_{\varphi_0}\|_2 = p^{1/2} \);

(iii) Let \( \varphi_0(k_1) = ... = \varphi_0(k_m) = 0, \ 0 \leq m < p^N - 1 \). Then:
\[
\text{Ker } D_{\varphi_0} = \{ s(n) : \hat{s}(pm) = 0, \ n \neq k_i \};
\]

(iv) Let \( s(n) \in V_j \), then \((D_{\varphi_0}s)(n) \in V_{j-1}\).

Proof. (i) It is an immediate consequence of Definition 2.

(ii) 
\[
\|D_{\varphi_0}s\|_2^2 = \sum_{n=0}^{p^M-1} |s(pm)|^2 |\varphi_0(n)|^2 = \sum_{k=0}^{p-1} \sum_{m=0}^{p^{M-1}-1} |s(pm)|^2 |\varphi_0(m + kp^{M-1})|^2
\]
\[
= p \sum_{m=0}^{p^{M-1}-1} |s(pm)|^2 \leq p \|s(\cdot)\|_2^2.
\]

Obviously:
\[
\|D_{\varphi_0}\|_2 = \sup_{s \in V_M} \frac{\|D_{\varphi_0}s(\cdot)\|_2}{\|s(\cdot)\|_2} \leq \sqrt{p},
\]
but we can find a sequence \( s_1(n) \) to obtain equality. In fact, we define \( s_1(n) \) by its DFT form in the following:
\[
s_1(n) = \begin{cases} 1, & n = pk \\
0, & n \neq pk.
\end{cases}
\]

(iii) It is an immediate consequence of the fact that \((D_{\varphi_0}s)(n) = \hat{s}(pm)\hat{\varphi}_0(n)\).
\((D_{\varphi_0}s)(n) = \sum_{k=0}^{p^j-1} c_k (D_{\varphi_0} T_{kp^j} \varphi_{M-j})(n) = \sum_{k=0}^{p^j-1} c_k (T_{kp^j M-j+1} D_{\varphi_0} \varphi_{M-j})(n) = \sum_{k=0}^{p^j-1} c_k \varphi_{M-j+1}(n - kp^j)\)
\[= \sum_{l=0}^{p^j-1} \left( \sum_{r=0}^{p-1} c_{l+rp^j} \right) \varphi_{M-j+1}(n - lp^j). \]

**Proposition 2.** Let \(\varphi_0\) be a \(p^M\)-periodic sequence and let the collection \(\{\varphi_{M-j}(n)\}_{j=0}^{M-1}\) be as in (1). If
\[\sum_{s=0}^{p^j-1} |\hat{\varphi}_0(r + sp^{M-1})|^2 = p, \quad r = 0, ..., p^j - 1, \] then:

(i) The collection \(\{\varphi_{M-j}(\cdot - kp^j) : k = 0, ..., p^j - 1\}, (j = 0, ..., M - 1)\) is a p-adic MRA of \(V_M\) with scaling sequence \(\varphi_1(n) = \frac{\varphi_0(n)}{\sqrt{p^M}}\).

(ii) The subspaces \(V_j\) form a p-adic MRA of \(V_M\) with scaling sequence \(\varphi_1(n) = \frac{\varphi_0(n)}{\sqrt{p^M}}\).

Proof. The proof was presented in [1]. We shall briefly sketch it. First we find a necessary and sufficient condition for the orthonormality of the set \(\{\varphi_{M-j}(\cdot - mp^j) : m = 0, ..., p^j - 1\}\):
\[\sum_{s=0}^{p^j-1} |\hat{\varphi}_1(r + sp^j)|^2 = \frac{1}{p^j}, \quad r = 0, ..., p^j - 1. \] (3)

Let \(j = M - 1\). Using (2), (3) and Definition 4 we see that the \(p\)-translations of \(\varphi_1\) form an orthonormal basis of \(V_{M-1}\). In fact:
\[\sum_{s=0}^{p^j-1} |\hat{\varphi}_1(r + sp^{M-1})|^2 = \sum_{s=0}^{p^j-1} |\hat{\varphi}_0(r + sp^j)|^2 = \frac{1}{p^j}, \quad r = 0, ..., p^j - 1, \]
and thus Proposition 2 is valid for \(j = M - 1\). Now let \(j = 0, ..., M - 2\). We suppose that the set \(\{\varphi_{M-j}(\cdot - kp^j)\}_{k=0}^{p^j-1}\) is an orthonormal basis of \(V_j\) and we use (3) in combination with (1) and (2) to show that the \(p^j\)translations of the sequence \(\varphi_{M-j+1}(n)\) form an orthonormal basis of \(V_{j-1}\). The proof follows by induction. \(\Box\)
Proposition 3. Let \( \varphi(n) \) be a \( p^M \)-periodic sequence which satisfies the condition
\[
\sum_{m=0}^{pM-1} \varphi(k + mp)\varphi(k + (m - r)p) = \begin{cases} 
1/p, & r = 0 \\
0, & r = 1, \ldots, p^M - 1
\end{cases}
\]
for \( k = 0, \ldots, p - 1 \), then \( \varphi(n) \) is the scaling sequence of a \( p \)-adic MRA.

Proof. The proof was presented in [1]. □

3. The \( p \)-adic Sampling Sequence

Definition 5. We say that an \( M \)-dimensional subspace \( W \) of \( V \) has a sampling basis \( \{s_0, \ldots, s_{M-1}\} \), if there exist \( M \) positive integers \( 0 \leq n_1 < \ldots < n_M < N \) such that for any sequence \( a(n) \in W \) we have:
\[
a(n) = \sum_{j=0}^{M-1} a(n_j)s_j(n), \quad 0 \leq n \leq N - 1.
\]
In particular, we say that \( W \) has a sampling sequence \( s(n) \) (which is \( N \)-periodic), if there exist \( M \) positive integers \( 0 \leq n_1 < \ldots < n_M < N \) such that for any sequence \( a(n) \in W \) we have:
\[
a(n) = \sum_{j=0}^{M-1} a(n_j)s(n - n_j), \quad 0 \leq n \leq N - 1.
\]

Theorem 1. Let \( \varphi_0(n) \) be a \( p^M \)-periodic sequence which produces a \( p \)-adic MRA \( \{V_j\}_{j=0}^{M-1} \). If
\[
\sum_{s=0}^{p-1} \hat{\varphi}_0(r + sp^M) \neq 0, \quad 0 \leq r \leq p^M - 1 - 1,
\]
then any sequence \( f(n) \in V_j \) has the following sampling expansion:
\[
f(n) = \sum_{n=0}^{p^j-1} f(mp^j - j)(n - mp^j - j), \quad n = 0, \ldots, p^j - 1,
\]
where
\[
s_{M-j}(\cdot) \leftrightarrow \sqrt{p^j} \cdot \sum_{r=0}^{p^M-j-1} \frac{\varphi_{M-j}(\cdot + r)}{\varphi_{M-j}(\cdot + r^j)}.
\]
Proof. The proof was presented in [1]. We give here a slightly alternate proof. The hypothesis (4) implies that for any \( j = 0, \ldots, M - 1 \) there holds:

\[
\sum_{s=0}^{p^M-j-1} \hat{\varphi}_{M-j}(r + sp^j) \neq 0, \quad 0 \leq r \leq p^j - 1.
\] (6)

By [2, p. 255], it suffices to prove that the set \( \{ K_j(\cdot, mp^{M-j}) : m = 0, \ldots, p^j - 1 \} \) is a basis for \( V_j \), where

\[
K_j(n, m) = \sum_{r=0}^{p^j-1} \varphi_{M-j}(n - rp^{M-j}) \overline{\varphi_{M-j}(m - rp^{M-j})}, \quad 0 \leq n, m \leq N - 1,
\]

is the reproducing kernel of \( V_j \), or equivalently we try to find a positive constant \( A > 0 \) such that for any sequence of scalars \( \{a_m\}_{m=0}^{p^j-1} \), we have

\[
A \sum_{m=0}^{p^j-1} |a_m|^2 \leq \left\| \sum_{m=0}^{p^j-1} a_m K_j(\cdot, mp^{M-j}) \right\|_2^2.
\]

After some calculations concerning the DFT of the kernel we find

\[
\left\| \sum_{m=0}^{p^j-1} a_m \hat{K}_j(\cdot, mp^{M-j}) \right\|_2^2 = \frac{1}{p^{M-j}} \sum_{k=0}^{p^M-j-1} \sum_{l=0}^{p^M-j-1} |\hat{a}_k|^2 |\hat{\varphi}_{M-j}(k + lp^j)|^2 \times \left| \sum_{s=0}^{p^M-j-1} \hat{\varphi}_{M-j}(k + sp^j) \right|^2.
\]

Thus \( \{ K_j(\cdot, mp^{M-j}) : m = 0, \ldots, p^j - 1 \} \) is a basis of \( V_j \), (see (6)) and it possesses a unique biorthonormal sequence \( \{ s_{M-j,m} : m = 0, \ldots, p^j - 1 \} \) which is a sampling basis for \( V_j \) (see [2]). Finally,

\[
\varphi_{M-j}(n) = \sum_{k=0}^{p^j-1} \varphi_{M-j}(kp^{M-j}) s_{M-j}(n - kp^{M-j})
\]

and the DFT of the equality above implies (5). \( \Box \)

4. Spectral Wavelet Decomposition

We expect that this work provides wavelet Analysis. We believe that the wavelet sequence associated with our MRA is the following:
**Definition 6.** Given the Dilation operator \( (D_{\phi_0}) \), then for \( l = 1, ..., p - 1 \) its associate sequence is given by

\[
\hat{\psi}_{M-j}^l(k) = \begin{cases} 
\hat{\phi}_1(k + lp^{M-1}), & j = M - 1 \\
\hat{\psi}_{M-j-1}^l(pk)\hat{\varphi}_0(k), & j = 0, ..., M - 2.
\end{cases}
\]

Assuming that \( \varphi_1(n) \) is an appropriately chosen scaling sequence of a \( p \)-adic MRA, we believe that each wavelet subspace \( W_{l,j} \) is the linear span of \( \{\psi_{M-j}(\cdot - kp^{M-j}) : k = 0, ..., p^j - 1\} \) and we can write

\[
V_M = V_0 \bigoplus_{j=0}^{M-1} \bigoplus_{l=1}^{p-1} W_{l,j},
\]

which implies the following decomposition algorithm

\[
s(n) = \sum_{j=0}^{M-1} \sum_{k=0}^{p^j-1} \sum_{l=0}^{p-1} \langle s(\cdot), \psi_{M-j}(\cdot - kp^{M-j}) \rangle \psi_{M-j}(n - kp^{M-j}) + \frac{1}{p^{M-1}} \sum_{k=0}^{p^M-1} s(k).
\]

**References**


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