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About Kolmogorov Type Inequalities for Functions Defined on a Half Line

Vladislav Babenko and Yulia Babenko

Let G be the real line $\mathbb{R}$, or the negative half-line $\mathbb{R}_-$. Let $L_p = L_p(G)$, $1 \le p \le \infty$, be the space of functions $x : G \to \mathbb{R}$, measurable and p power integrable (essentially bounded when $p = \infty$), equipped with the usual norm

$$\|x\|_p = \begin{cases} \left(\int\limits_G |x(\tau)|^p \, d\tau \right)^{1/p}, & \text{if } \ 1 \le p < \infty \\ \operatorname{ess\,sup} \{|x(\tau)| : \tau \in G\}, & \text{if } \ p = \infty. \end{cases}$$

For any fixed $r \in \mathbb{N}$, we shall denote by $L_p^r = L_p^r(G)$ the space of functions $x : G \to \mathbb{R}$ that have locally absolutely continuous derivative $x^{(r-1)}$ and such that $x^{(r)} \in L_p(G)$. For $1 \le p, \ s \le \infty$, let us denote $L_{p,s}^r = L_{p,s}^r(G) = L_s^r(G) \cap L_p(G)$.

It is known that inequalities of the type

$$\|x^{(k)}\|_q \le K \, \|x\|_p^\alpha \, \|x^{(r)}\|_s^\beta \tag{1}$$

for the norm of an intermediate derivative of the function in terms of the norms of the function itself and its higher derivative play an important role in many questions of analysis and its applications. Especially exact inequalities, i.e., inequalities with the best possible constant are important and interesting.

The first exact results have been obtained by Landau [8] (for $x \in L^2_{\infty,\infty}(\mathbb{R}_-)$, $k = 1$). One of the first complete results in this direction is due to Kolmogorov [5], [6]: for any $k \in \mathbb{N}, \ k < r$, and any function $x \in L^r_{\infty,\infty}(\mathbb{R})$,

$$\|x^{(k)}\|_\infty \le \frac{\|\varphi_{r-k}\|_\infty}{\|\varphi_r\|_\infty^{1-k/r}} \, \|x\|_\infty^{1-k/r} \, \|x^{(r)}\|_\infty^{k/r}, \tag{2}$$

where φ_r is the r-th periodic integral with zero mean value on the period of the function $\varphi_0(t) = \operatorname{sgn} \sin x$. The inequality (2) becomes an equality for any function of the kind $a\varphi_r(\lambda t)$, where $a, \lambda \in \mathbb{R}, \ \lambda > 0$.

So far, only a few cases are known where for some values of p, q, s the exact constants in inequalities (1) are found for all pairs $k, r \in \mathbb{N}$, $k < r$. Except the above mentioned Kolmogorov inequality, for $G = \mathbb{R}$ these cases are:

1) $q = p = s = 2$ (Hardy, Littlewood, and Polya [4]);

2) $q = p = s = 1$ (Stein [12]);

3) $q = \infty,\ p = s = 2$ (Taikov [13]).

For $G = \mathbb{R}_-$ these cases are:

1) $p = q = s = \infty$ (Landau [8], Schoenberg and Cavaretta[11]);

2) $p = q = s = 2$ (Lubich [9], Kupcov [7]);

3) $q = \infty,\ p = r = 2$ (Gabushin [3]).

There are also a great number of partial results related to the case of functions of small smoothness and other values of q, p, s. We refer the reader to [1], [2], [14] for references on this topic.

Let us note here that the case when the domain is a half-line is the least explored. Schoenberg and Cavaretta obtained in [11] an analog of Kolmogorov inequality in this case, but the constant in their result is not explicit.

In 1956 Olovyanishnikov [10] considered the class of $(r-1)$-monotone functions defined on the negative half-line, i.e., the class $L^{r,+}_{\infty,\infty}(\mathbb{R}_-)$ of functions that are non-negative on $\mathbb{R}_-$ with all their derivatives of order up to $r-1$. He managed to obtain for this class an exact inequality of Kolmogorov type with a simple and elegant constant. Here is his result:

Theorem 1. *For any $k, r \in \mathbb{N},\ k < r$, and for any function $x \in L^{r,+}_{\infty,\infty}$, the following inequality holds true*

$$\|x^{(k)}\|_\infty \le \frac{r!^{1-k/r}}{(r-k)!}\, \|x\|_\infty^{1-k/r}\, \|x^{(r)}\|_\infty^{k/r}.$$

In the present work we shall show that for functions from the class $L^{r,+}_{p,\infty}$, $1 \le p \le \infty(\mathbb{R}_-)$ and functions from the class $L^{r,+}_{p,1}(\mathbb{R}_-)$ (with any $p \in [1,\infty]$) exact inequalities (1), which have on the left side $\|x^{(k)}\|_q$ with any $q \in [1,\infty]$ can be obtained for all $k, r \in \mathbb{N},\ k < r$ (see Theorems 2 and 3 below).

We will show as well (Theorem 4) that imposing more restrictions on the functions, namely considering only absolutely monotone functions, we can obtain an exact Kolmogorov type inequality with an elegant constant for all possible values of the parameters r, k, q, p, s. Finally (Theorem 5), we will obtain two exact Kolmogorov type inequalities for functions from the class S (see the definition below).

For any positive a and l, let us set

$$\varphi_r(a,l,t) = \begin{cases} 0, & -\infty < t \le -l \\ \dfrac{a(l+t)^r}{r!}, & -l \le t \le 0. \end{cases} \tag{3}$$

Theorem 2. *For all $1 \le p, q \le \infty$ and for every function $x \in L^{r,+}_{p,\infty} = L^{r,+}_{p,\infty}(\mathbb{R}_-)$ the following inequality holds true*

$$\|x^{(k)}\|_q \le C_{rk}\, \|x\|_p^{\alpha}\, \|x^{(r)}\|_\infty^{1-\alpha}, \tag{4}$$

where

$$C_{rk} = \frac{\|\varphi_r^{(k)}(1,\cdot)\|_q}{\|\varphi_r(1,\cdot)\|_p^\alpha} = \frac{(r!)^\alpha (rq+1)^{\alpha/p}}{(r-k)!\big((r-k)q+1\big)^{1/q}}, \qquad \alpha = \frac{r-k+1/q}{r+1/p}.$$

The inequality (4) *becomes an equality for any function of type* (3).

Theorem 3. *For all* $1 \le p,\ q \le \infty$ *and for every function* $x \in L_{p,1}^{r,+}(\mathbb{R}_-)$ *the following exact inequality holds true*

$$\|x^{(k)}\|_q \le \frac{\|\varphi_{r-1}^{(k)}(1,\cdot)\|_q}{\|\varphi_{r-1}(1,\cdot)\|_p^\alpha} \, \|x\|_p^\alpha \, \|x^{(r)}\|_1^{1-\alpha},$$

where

$$\alpha = \frac{r-1-k+1/q}{r-1+1/p}.$$

Consider now the class of absolutely monotone functions $AM(\mathbb{R}_-)$ i.e., the class of functions $x : \mathbb{R}_- \to \mathbb{R}$ which are non-negative with their derivatives of any order.

If we impose a bit more restrictions on the functions we consider, namely if we take absolutely monotone functions, we will get complete solution of the Kolmogorov problem, i.e., we obtain exact inequality for any values of the parameters p, q, s, and any $k < r,\ k, r \in \mathbb{N}$.

Theorem 4. *For* $x \in AM(\mathbb{R}_-)$, *for any* p, q, s *and* $k < r$,

$$\|x^{(k)}\|_p \le C\, \|x\|_q^{1-\alpha} \, \|x^{(r)}\|_s^\alpha \tag{5}$$

where

$$C = \frac{q^{1/q}}{p^{1/p}} \left(\frac{s^{1/s}}{q^{1/q}}\right)^\alpha, \qquad \alpha = \frac{k+\frac{1}{q}-\frac{1}{p}}{r-\frac{1}{q}+\frac{1}{s}}.$$

The function $x(z)$ belongs to the class S if:

1. $x(z)$ is holomorphic in the upper half-plane;
2. $\Im x(z) \ge 0$ when $\Im z > 0$;
3. $x(z)$ is holomorphic and non-negative on $(-\infty, 0)$.

Theorem 5. *For* $x \in S$ *and for any* $k < r$,

$$\|x^{(k)}\|_q \le \frac{k!}{(r!)^{\frac{k+\frac{1}{q}}{r}} \big(q(k+1)+1\big)^{\frac{1}{q}}} \, \|x\|_\infty^{1-\frac{k+\frac{1}{q}}{r}} \, \|x^{(r)}\|_\infty^{\frac{k+\frac{1}{q}}{r}},$$

$$\|x^{(k)}\|_q \le \frac{k!(r+2)^{\frac{k+\frac{1}{q}}{r+1}}}{(r!)^{\frac{k+\frac{1}{q}}{r+1}} \big(q(k+1)+1\big)^{\frac{1}{q}}} \, \|x\|_\infty^{1-\frac{k+\frac{1}{q}}{r+1}} \, \|x^{(r)}\|_1^{\frac{k+\frac{1}{q}}{r+1}}.$$

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Vladislav Babenko
Dnepropetrovsk National University
Institute of Applied Mathematics
and Mechanics of NAS of Ukraine
UKRAINE
E-mail: babenko@mail.dnepr.net

Yuliya Babenko
Department of Mathematics
Vanderbilt University
Nashvile, TN 37240
USA
babenko@math.vanderbilt.edu