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On Kolmogorov Type Inequalities for the Norms of Intermediate Derivatives of Functions of Many Variables

Vladislav Babenko, Nikolai Korneichuck and Sergei Pichugov

Let G be the real line $\mathbb{R}$ or the unit circle $\mathbb{T}$ which is realized as interval $[-\pi, \pi]$ with identified endpoints, and let $L_p(G)$, $1 \le p \le \infty$, be the space of all measurable functions $x : G \to \mathbb{R}$ such that $\|x\|_{L_p(G)} < \infty$, where

$$\|x\|_{L_p(G)} := \left(\int_G |x(t)|^p \, dt \right)^{1/p}, \qquad 1 < p < \infty,$$

and

$$\|x\|_{L_\infty(G)} := \sup_{t \in G} \text{vrai} \, |x(t)|.$$

Given $r \in \mathbb{N}$ and $1 \le s \le \infty$, let us denote by $L_s^r(G)$ the space of all functions x such that their derivatives $x^{(r-1)}$ ($x^{(0)} := x$) are locally absolutely continuous and $x^{(r)} \in L_s(G)$. Given $1 \le p \le \infty$, we set

$$L_{p,s}^r(G) = L_p(G) \cap L_s^r(G).$$

Note that $L_s^r(\mathbb{T}) \subset L_p(\mathbb{T})$ for any p.

Inequalities for the norms of the intermediate derivatives of the functions $x \in L_{p,s}^r(G)$, $G = \mathbb{R}$ or $G = \mathbb{T}$, i.e., inequalities of the form

$$\|x^{(k)}\|_{L_q(G)} \le K \, \|x\|_{L_p(G)}^{\alpha} \, \|x^{(r)}\|_{L_s(G)}^{\beta}, \tag{1}$$

where $k, r \in \mathbb{Z}_+$, $k < r$, especially inequalities with the best possible constants K, are of great importance for many branches of mathematics.

An inequality of this type appeared for the first time in the paper of Hardy and Littlewood [6] of 1912. First exact inequalities have been obtained by Landau [10] in 1913. After that numerous papers by many mathematicians have been devoted to obtaining exact inequalities for the norms of intermediate derivatives.

One of the first complete results giving an exact inequality of the form (1) is due to Kolmogorov [8] (see also [9]), after whom the exact inequalities (1) are called *inequalities of Kolmogorov type*.

We refer the reader to [11], [1], [2] for surveys of the known results on exact inequalities of Kolmogorov type for the functions $x \in L^r_{p,s}(G)$, where $G = \mathbb{R}$ or $G = \mathbb{T}$, as well as for the corresponding references.

General conditions for the existence of inequalities of the form (1) in the case $G = \mathbb{R}$ were given in [5]: for given r, k and q, p, s inequality (1) holds true for functions $x \in L^r_{p,s}(G)$ if and only if

$$\frac{r-k}{p} + \frac{k}{s} \geq \frac{r}{q}, \tag{2}$$

and in this case the exponent α in (1) is uniquely defined by the relation

$$\alpha = \frac{r - k - s^{-1} + q^{-1}}{r - s^{-1} + p^{-1}}. \tag{3}$$

The general conditions of existence of inequalities of the form (1) with a constant K which is independent of the function x in the case $G = \mathbb{T}$, $p, q, s \geq 1$, are different from (2) and (3). It was proved in [7] that inequality (1) takes place for any function $x \in L^r_s(\mathbb{T})$, any $k, r \in \mathbb{Z}_+$, $k < r$, and any q, p, s if and only if

$$\alpha \leq \alpha_{cr} := \min\left\{1 - \frac{k}{r},\ \frac{r - k - s^{-1} + q^{-1}}{r - s^{-1} + p^{-1}}\right\}. \tag{4}$$

Note that inequalities of the type (1) for periodic functions with $\alpha = \alpha_{cr}$ are of substantial interest, since namely such inequalities have the most important applications. Besides that, in many cases it is easy to obtain exact inequalities with $\alpha < \alpha_{cr}$ and the best possible constant K from the best possible inequalities of the type (1) with $\alpha = \alpha_{cr}$.

In [3] Babenko, Kofanov, and Pichugov have shown that in the case when inequality of Kolmogorov type is possible for nonperiodic functions defined on the real line, the exact constant in such inequality is less than or equal to the exact constant in analogous inequality for periodic functions. Namely, let

$$K(G) = K_{k,r}(G; q, p, s; \alpha) := \sup_{\substack{x \in L^r_{p,s}(G) \\ x^{(r)} \neq 0}} \frac{\|x^{(k)}\|_{L_q(G)}}{\|x\|^{\alpha}_{L_p(G)}\ \|x^{(r)}\|^{1-\alpha}_{L_s(G)}}$$

be the exact constant in the inequality of Kolmogorov type. Then

$$K(\mathbb{R}) \leq K(\mathbb{T}) \qquad \text{if} \quad \frac{r - k + 1/q - 1/s}{r + 1/p - 1/s} < 1 - \frac{k}{r},$$

and

$$K(\mathbb{R}) = K(\mathbb{T}) \qquad \text{if} \quad \frac{r - k + 1/q - 1/s}{r + 1/p - 1/s} = 1 - \frac{k}{r}.$$

Let now G be the space $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$; $\|x\|_p = \|x\|_{L_p(G)}$, $1 \leq p \leq \infty$; $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$, $k, r \in \mathbb{N}$, $k < r$;

$$x^{(\gamma)} := \frac{\partial^{\gamma_1 + \ldots \gamma_d} x}{\partial t_1^{\gamma_1} \ldots \partial t_d^{\gamma_d}}$$

(we mean derivatives in Sobolev sense); M, M_1, M_2 are constants independent of f.

The problem on Kolmogorov type inequality for functions of many variables can be formulated in the following rather general form:

Given $\gamma, \gamma^1, \dots \gamma^m,\ m \le d$, estimate $\|x^{(\gamma)}\|_q$ with the help of

$$\|x\|_p,\ \|x^{(\gamma^1)}\|_{s_1}, \dots , \|x^{(\gamma^m)}\|_{s_m}.$$

We refer the reader to [4] for some known results about possibilities of such estimations.

For functions $x \in L^{r\gamma}_{p,s}(G) := \{f \in L_p(G) : f^{(r\gamma)} \in L_s(G)\}$ we consider the problem of estimating $\|x^{(k\gamma)}\|_q$ with the help of $\|x\|_p$ and $\|x^{(r\gamma)}\|_s$.

Theorem 1. *Given $d > 1$, $\gamma = (\gamma_1, \dots \gamma_d) \in \mathbb{N}^d$, $k, r \in \mathbb{N}$, $k < r$, and any $\delta > 0$,*

$$\sup\left\{\|x^{(k\gamma)}\|_\infty\ :\ x \in L^{r\gamma}_{\infty,\infty}(\mathbb{R}^d),\ \|x\|_\infty \le \delta,\ \|x^{(r\gamma)}\|_\infty = 1\right\} = \infty.$$

On the other hand, the following theorem holds.

Theorem 2. *For every $x \in L^{r\gamma}_{\infty,\infty}(\mathbb{T}^d)$,*

$$\|x^{(k\gamma)}\|_\infty \le M\,\|x\|_\infty^{1-k/r}\,\|x^{(r\gamma)}\|_\infty^{k/r}\left(1 + \ln^+ \frac{\|x^{(r\gamma)}\|_\infty}{\|x\|_\infty}\right)^{d-1}. \tag{5}$$

On the other hand, there exists $\beta \ge (d-1)\max\{1-k/r,\ k/r\}$ such that

$$\sup_{\substack{x \in L^{r\gamma}_{\infty,\infty}(\mathbb{T}^d)\\ x\ne 0}} \frac{\|x^{(k\gamma)}\|_\infty}{\|x\|_\infty^{1-k/r}\,\|x^{(r\gamma)}\|_\infty^{k/r}\left(1 + \ln^+ \frac{\|x^{(r\gamma)}\|_\infty}{\|x\|_\infty}\right)^{\beta}} \ge M_1 > 0.$$

It is interesting to compare Theorems 1 and 2 with the above presented results about comparison of $K(\mathbb{R})$ and $K(\mathbb{T})$.

It follows from (5) that if $x \in L^{r\gamma}_{\infty,\infty}(\mathbb{T}^d)$ has zero mean value on $[0, 2\pi]$ in any variable (the rest of variables are fixed), then for $x \in L^{r\gamma}_{\infty,\infty}(\mathbb{T}^d)$ and any $\epsilon > 0$,

$$\|x^{(k\gamma)}\|_\infty \le M_2\,\|x\|_\infty^{1-k/r-\epsilon}\,\|x^{(r\gamma)}\|_\infty^{k/r+\epsilon}.$$

Let now $1 < p, s < \infty$, $\gamma = (1, 1, \dots , 1) = \mathbf{1}$.

Theorem 3. *For every $x \in L^{r\mathbf{1}}_{p,s}(\mathbb{T}^d)$,*

$$\|x^{(k\mathbf{1})}\|_\infty \le M\,\|x\|_p^{\frac{r-k-1/s}{r-1/s+1/p}}\,\|x^{(r\mathbf{1})}\|_s^{\frac{k+1/p}{r-1/s+1/p}}\left(1 + \ln^+ \frac{\|x^{(r\mathbf{1})}\|_s}{\|x\|_p}\right)^{d-1}.$$

On the other hand, there exists $\beta \ge \frac{r/p+k/s-k/p}{r-1/s+1/p}$ such that

$$\sup_{\substack{x \in L^{r\mathbf{1}}_{p,s}(\mathbb{T}^d)\\ x\ne 0}} \frac{\|x^{(k\mathbf{1})}\|_\infty}{\|x\|_p^{\frac{r-k-1/s}{r-1/s+1/p}}\,\|x^{(r\mathbf{1})}\|_s^{\frac{k+1/p}{r-1/s+1/p}}\left(1 + \ln^+ \frac{\|x^{(r\mathbf{1})}\|_s}{\|x\|_p}\right)^{\beta}} \ge M_1 > 0.$$

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Vladislav Babenko, Sergei Pichugov

Dnepropetrovsk National University
Institute of Applied Mathematics
and Mechanics of NAS of Ukraine
UKRAINE
E-mail: babenko@mail.dnepr.net

Nikolai Korneichuk

Institute of Mathematics of NAS of Ukraine
Tereshchenkovskaya Street 3
252601 Kiev – 4
UKRAINE
E-mail: korneich@imath.kiev.ua