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Alternation Points in Rational Chebyshev Approximation

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We consider real rational Chebyshev approximations of a real-valued function on $[-1, 1]$. The paper is concerned with the relation of the normalized counting measure of alternation sets with the normalized counting measure of the poles of the best approximants and the equilibrium distribution on $[-1, 1]$.

1. Introduction

Let $f \in C[-1, 1]$ be real-valued and $\mathcal{R}_{n,m}$ denote the collection of real rational functions of the form $r = p/q$, where $\deg p \leq n$, $\deg q \leq m$, $q \neq 0$. It is known that for each pair of nonnegative integers (n, m) there exists a unique $r^*_{n,m}$ that is a best uniform approximation to f on $[-1, 1]$ in the sense that

$$\|f - r^*_{n,m}\| < \|f - r\| \quad \text{for all } r \in \mathcal{R}_{n,m},\ r \neq r^*_{n,m}$$

where $\|\cdot\|$ denotes the sup-norm on $[-1, 1]$. Writing

$$r^*_{n,m} = p^*_n/q^*_m$$

where p^*_n and q^*_m have no common factors, let

$$d_{n,m} := \min\{n - \deg p^*_n,\ m - \deg q^*_m\}.$$

Then there exist $m + n + 2 - d_{n,m}$ points $x^{(n,m)}_k$,

$$-1 \leq x^{(n,m)}_1 < \cdots < x^{(n,m)}_{n+m+2-d_{n,m}} \leq 1$$

that satisfy

$$\lambda(-1)^k (f - r^*_{n,m})(x^{(n,m)}_k) = \|f - r^*_{n,m}\| \quad \text{for } 1 \leq k \leq n + m + 2 - d_{n,m},$$

where $\lambda = +1$ or $\lambda = -1$ is fixed.

Such an *alternation set* is in general not unique. For each pair (n, m) let

$$A_{n,m} = A_{n,m}(f) = \{x_k^{(n,m)}\}_{k=0}^{n+m+2-d_{n,m}}$$

denote an arbitrary, but fixed alternation set for the best approximation $r^*_{n,m}$ to f from $\mathcal{R}_{n,m}$. Our result concerns the asymptotic distribution of the sets $A_{n,m}$ as $n \to \infty$ where $m = m(n) \leq n$.

We denote by $\nu_{n,m}$ the normalized counting measure of $A_{n,m}$. For the case of polynomial approximation, Kadec [4] has shown that there exists a subset Λ of $\mathbb{N}$ such that

$$\nu_{n,0} \overset{*}{\to} \mu \qquad \text{as } n \in \Lambda,\ n \to \infty$$

where μ is the equilibrium distribution of $[-1, 1]$.

For rational approximation, Borwein, Grothmann, Kroó, and Saff [2] have shown that denseness in $[-1, 1]$ of a subsequence of alternation sets $A_{n,m}$ holds. To be more precise, let

$$x_0^{(n,m)} := -1, \qquad x_{n+m+3-d_{n,m}}^{(n,m)} := 1,$$

and let

$$\varrho_{n,m} := \max_{0 \leq k \leq n+m+3-d_{n,m}} \left(x_{k+1}^{(n,m)} - x_k^{(n,m)} \right)$$

denote the denseness of $A_{n,m}$ in $[-1, 1]$. Then the result of [2] is the following: Let

$$m = m(n) \quad \text{and} \quad m(n) \leq m(n+1) \leq m(n) + 1 \tag{1}$$

and

$$\frac{n - m(n)}{\log n} \to \infty \qquad \text{as } n \to \infty.$$

Then

$$\liminf_{n \to \infty} \varrho_{n,m(n)} = 0.$$

Under the condition (1) and

$$\frac{m(n)}{n} \overset{n \to \infty}{\longrightarrow} c, \qquad 0 \leq c < 1,$$

Kroó and Peherstorfer [5] have proved that there exists a subset $\Lambda \subset \mathbb{N}$ with

$$\liminf_{n \in \Lambda, n \to \infty} \nu_{n,m(n)}([\alpha, \beta]) \geq \frac{1-c}{1+c}\, \mu([\alpha, \beta])$$

for any subinterval $[\alpha, \beta] \subset [-1, 1]$.

The results give no information about denseness in the important case $n = m$. Braess, Lubinsky, and Saff [3] have considered this situation and obtained asymptotic results for the alternation sets by taking into account the location of the poles of the best approximation $r^*_{n,m}$: Let $\gamma_n(\varepsilon)$ denote the number of poles of $r^*_{n,m}$ outside the ε-neighbourhood of $[-1,1]$. If $\varepsilon > 0$ and

$$\gamma_n(\varepsilon)/\log n \stackrel{n\to\infty}{\longrightarrow} \infty,$$

then, in [3], it was proved that

$$\liminf_{n\to\infty} \varrho_{n,n} = 0;$$

if $\varepsilon > 0$ and

$$\liminf_{n\to\infty} \frac{\gamma_n(\varepsilon)}{n} > 0,$$

then there exists a constant $c > 0$ and $\Lambda \subset \mathbb{N}$ such that

$$\liminf_{n\in\Lambda, n\to\infty} \nu_{n,n}([\alpha,\beta]) \geq c\mu([\alpha,\beta]).$$

The purpose of the present paper is to consider both situations $m = n$ and $m < n$ simultaneously by using the normalized counting measure of the alternation set $A_{n,m}$ together with the normalized counting measure of the poles of $r^*_{n,m}$ and the equilibrium measure μ of $[-1,1]$.

2. The Potential Theoretical Approach

In the following, let $m = m(n)$ and

$$m(n) \leq m(n+1) \leq m(n)+1, \qquad m(n) \leq n.$$

We denote the minimal error of the approximation of f with respect to $\mathcal{R}_{n,m(n)}$ by

$$E_n := \|f - r^*_{n,m(n)}\|$$

and let us abbreviate

$$r^*_n := r^*_{n,m(n)}$$

Then

$$r^*_{n+1} - r^*_n = \frac{q^*_{n+1}q^*_{m(n)} - p^*_n q^*_{m(n+1)}}{q^*_{m(n)}q^*_{m(n+1)}} =: \frac{P_n}{Q_n}$$

where

$$Q_n := q^*_{m(n)}q^*_{m(n+1)},$$

and

$$\text{grad } P_n = n + m(n) + 1 - d_{n,m(n)}, \qquad \text{grad } Q_n \leq \text{ grad } P_n.$$

At the alternation set $A_{n,m(n)}$ we have

$$\lambda(-1)^k \frac{P_n}{Q_n}\left(x_k^{(n,m(n))}\right) \geq E_n - E_{n+1},$$

i.e., the minimal error ϱ of the approximation of p_n with respect to the polynomials of degree $n + m(n) - d$ and the weight function $1/Q_n(x)$ on $A_{n,m(n)}$ satisfies, by de la Vallée-Poussin's theorem,

$$\varrho \geq E_n - E_{n+1}. \tag{2}$$

Let

$$P_n(x) = a_n x^{n+m+1-d_{n,m(n)}} + \cdots$$

and

$$Q_n(x) = b_n \prod_{i=1}^{l_n}(x - y_{i,n}) = b_n \widetilde{w}(x) \tag{3}$$

where $y_{1,n}, \ldots, y_{l_n,n}$ are all the poles of r_n^* and r_{n+1}^* according to their multiplicities. Then

$$\rho = \frac{|a_n|}{\sum_{k=1}^{n+m(n)+2-d_{n,m(n)}} |Q_n(x_k^{(n,m(n))})/\omega'(x_k^{(n,m(n))})|} \tag{4}$$

where

$$\omega(x) = \prod_{k=1}^{n+m(n)+2-d_{m,n(m)}} (x - x_k^{(n,m(n))}).$$

If $r_n^* \neq r_{n+1}^*$, then Lagrange interpolation together with (2)–(4) leads to the bound

$$\left|\frac{\widetilde{w}(x)}{w(x)}\right| \leq D(x)\left|\frac{a_n}{b_n}\right| \frac{1}{E_n - E_{n+1}}, \qquad y \notin [-1,1], \tag{5}$$

where

$$D(x) = \max_k \left|\frac{1}{x - x_k^{(n,m(n))}}\right|.$$

Moreover, it is possible to obtain an upper bound for $|a_n/b_n|$, namely

$$\begin{aligned}\log\left|\frac{a_n}{b_n}\right| \leq \sum_{i=1}^{l_n} G(y_{i,n}) + \log(E_n + E_{n+1})\\ + (n + m + 1 - d_{n,m(n)} - l)\log\frac{1}{2}\end{aligned} \tag{6}$$

where $G(z)$ is Green's function for $\overline{\mathbb{C}} \setminus [-1,1]$.

Let

$$\begin{aligned} \nu_n &:= \text{normalized counting measure of the alternation set } A_{n,m(n)}, \\ \tau_n &:= \text{normalized pole counting measure of } r_n^* \cup r_{n+1}^*, \end{aligned}$$

and

$$\alpha_n = \frac{l_n}{n+m+2-d_{n,m(n)}}.$$

Then inserting (6) into (5) we obtain

$$\begin{aligned} (U^{\nu_n} - \alpha_n U^{\tau_n} - (1-\alpha_n)U^{\mu})(z) &\leq \frac{1}{n}\log A_n + (1-\alpha_n)G(z) \\ &+ \frac{1}{n+m+2-d_{n,m(n)}}\left(\sum_{i=1}^{l_n} G(y_{i,n}) + \log D(z)\right). \end{aligned}$$

Here U^{ν_n}, U^{τ_n} and U^{μ} denote the logarithmic potentials of ν_n, τ_n and μ.

Next, we use the balayage measure $\widehat{\tau_n}$ of τ_n on $[-1,1]$; i.e., $\widehat{\tau_n}$ is the unique measure supported on $[-1,1]$ for which $\|\widehat{\tau_n}\| = \|\widehat{\tau}\|$ and

$$\begin{aligned} U^{\widehat{\tau_n}}(z) &\leq U^{\tau_n}(z) + c, \qquad z \in \mathbb{C}, \\ U^{\widehat{\tau_n}}(z) &= U^{\tau_n}(z) + c, \qquad z \in [-1,1], \\ c &= \int_{\mathbb{C}} G(t)d\tau_n(t) = \frac{1}{l_n}\sum_{i=1}^{l_n} G(y_{i,n}). \end{aligned}$$

Then the following final result holds.

Theorem. *Let* $m = m(n)$ *and*

$$m(n) \leq m(n+1) \leq m(n)+1, \qquad m(n) \leq n.$$

Then there exists a subsequence Λ *of* $\mathbb{N}$ *such that*

$$\nu_n - \alpha_n\widehat{\tau_n} - (1-\alpha_n)\mu \xrightarrow{*} 0 \qquad \text{as } n \in \Lambda \to \infty$$

in the weak**-topology.*

It is interesting to give an interpretation of this result. In the polynomial case, $m = 0$, the distribution of the alternation points is, at least for a subsequence $\Lambda \subset \mathbb{N}$, governed by the equilibrium distribution on $[-1,1]$ which is the distribution of a point mass 1 at ∞.

In the rational case, this distribution is disturbed by the point measures of the poles of the approximants which attract alternation points. Hence, if poles are approaching parts of the interval $[-1,1]$ then these parts contain more alternation points than determined by the equilibrium measure of these parts.

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