Discrepancy Estimates and Rational Chebyshev Approximation

HANS-PETER BLATT, RENÉ GROTHMANN AND RALITZA KOVACHEVA

Given a function \( f \), real-valued and continuous on \([-1, 1]\), let \( R_{n,m} \) be the rational function of best uniform approximation of \( f \) on \([-1, 1]\) of order \((n, m)\). Let \( m_n \leq n \), \( m_n \leq m_n + 1 \leq m_n + 1 \), \( n \to \infty \). In the present paper, results dealing with the distribution of alternation points of \( f - R_{n,m} \) are provided.

1. Introduction

Let \( I := [-1, 1] \) and let the function \( f \in C(I) \) be real-valued on \( I \). Throughout this paper, we assume that \( f \) is not rational.

Set \( \mathcal{P}_n \) for the class of polynomials with real coefficients of degree not exceeding \( n \) and \( \mathcal{R}_{n,m} \) for the collection of all rational functions \( r = p/q, \ p \in \mathcal{P}_n, \ q \in \mathcal{P}_m, \ q \not\equiv 0 \).

Given a pair \((n, m)\) of nonnegative integers \((n, m) \in \mathbb{N}\), let \( R_{n,m} (= R_{n,m}(f)) \) be the rational function of best Chebyshev approximation of \( f \) on \( I \) in the class \( \mathcal{R}_{n,m} \). Write \( R_{n,m} := \frac{P_{n,m}}{Q_{n,m}} \), where both polynomials \( P_{n,m} \) and \( Q_{n,m} \) do not have common divisors and \( Q_{n,m}(\zeta_{n,m,i}) = 0, \ i = 1, \ldots, k_{n,m} \leq m \). Fix in an arbitrary way a positive number \( R, \ R > 2 \), and normalize \( Q_{n,m} \) in the way:

\[
Q_{n,m}(z) = \prod_{|\zeta_{n,m,i}| \leq R} (z - \zeta_{n,m,i}) \prod_{|\zeta_{n,m,i}| > R} (1 - z/\zeta_{n,m,i}).
\]

Apparently, for every compact set \( K \) in \( \mathbb{C} \) there is a positive constant \( C(K) \) such that the inequality

\[
\|Q_{n,m}\|_K \leq C(K)^m
\]

holds.

Set \( d_{n,m} := \min\{n - \deg P_{n,m}, m - \deg Q_{n,m}\} \). By Chebyshev’s alternation theorem [1], \( R_{n,m} \) is unique and is characterized by the existence of
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\[ n + m + 2 - d_{n,m} \] points of alternation \( y_i^{(n,m)}; -1 \leq y_i^{(n,m)} < y_{i+1}^{(n,m)} < \cdots < y_{n+m+1-d_{n,m}}^{(n,m)} \leq 1 \) and

\[ (f-R_{n,m})(y_i^{(n,m)}) = \delta_{n,m}(-1)^i E_{n,m}(f), \ i = 0, \ldots, n+m+1-d_{n,m}, \ \delta_{n,m} = \pm 1. \]

In what follows, we shall consider sequences \( \{R_{n,m}\} \) with \( m_n \leq n, m_n \leq m_{n+1} \leq m_n + 1 \). For simplicity, we omit writing \( m_n \) (i.e., \( E_{n,m} := E_n, Q_{n,m} := Q_n, \) etc.)

Recall the well-known fact [7] that there exists an infinite sequence \( \Lambda \subset \mathbb{N} \) such that

\[ \frac{E_n + E_{n+1}}{E_n - E_{n+1}} \leq C n^2 \quad \text{as} \quad n \to \infty, \ n \in \Lambda \]

(2)

\( (C \) is a positive constant).

Write \( P_n,m(z) = A_n^{n,m} + \ldots \) and set

\[ R_{n+1} - R_n = \frac{\tilde{A}_{n+1} W_n}{Q_{n+1} Q_n}, \]

with \( W_n \) being a monic polynomial. It is easy to check that \( \deg W_n = n + m + 1 - d_{n,m} \), as well as that all its zeros are simple and interlace the alternation points \( y_{n,i}, \ i = 0, \ldots, n + m + 1 - d_{n,m} \). For \( \tilde{A}_{n+1} \) we have

\[ \tilde{A}_{n+1} = \begin{cases} A_{n+1} \prod_{|\zeta_n| > R} (\frac{1}{\zeta_n} - A_n) \prod_{|\zeta_{n+1}| > R} (\frac{1}{\zeta_{n+1}}), & \text{if} \ m_n + 1 = m_{n+1} \\ A_{n+1} \prod_{|\zeta_n| > R} (\frac{1}{\zeta_n}), & \text{if} \ m_n = m_{n+1}. \end{cases} \]

(3)

We first estimate \( \tilde{A}_{n+1} \). After keeping track of (1), we obtain

\[ |\tilde{A}_{n+1}| \leq C^{mn}_2 (E_n + E_{n+1}) 2^n, \quad n \geq n_1. \]

(4)

Apparently,

\[ (R_{n+1} - R_n)(y_{n,i})(R_{n+1} - R_n)(y_{n,i+1}) < 0, \]

\[ i = 0, \ldots, n + m_n - d_{n,m} \]

(5)

and

\[ |(R_{n+1} - R_n)(y_{n,i})| \geq E_n - E_{n+1}, \quad i = 0, \ldots, n + m_n + 1 - d_{n,m} \]

(6)

Fix now a number \( n \geq n_1 \) and introduce into considerations the quantity

\[ \rho_n := \inf_{p \in \mathcal{P}_{n+m_n-d_{n,m}}} \left\| \frac{\tilde{A}_{n+1} z^{n+m_n+1-d_{n,m}}}{Q_n(z) Q_{n+1}(z)} - \frac{p(z)}{Q_n(z) Q_{n+1}(z)} \right\|_I. \]
The set \( \{ p_{Qn, Qn+1} : p \in \mathcal{P}_{n+m_n - d_n, m_n}, Q_n \text{ and } Q_{n+1} \text{ fixed polynomials} \} \) forms a set of dimension \( n + m_n + 1 - d_n, m_n \) that satisfies the Haar conditions. Hence (2)

\[
\rho_n = \sum_{i=0}^{n+m_n+1-d_n, m_n} |\alpha_{n,i}| \cdot \left( \sum_{i=0}^{n+m_n+1-d_n, m_n} |\alpha_{n,i}| \right)^{-1} \cdot \left| \tilde{A}_{n+1} - \tilde{p}(y_{n,i}) \right|
\]

(7)

where \( \tilde{p} \in \mathcal{P}_{n+m_n - d_n, m_n} \) is arbitrary and

\[
\alpha_{n,i} = \frac{(Q_{n+1}Q_n)(y_{n,i})}{\prod_{j \neq i} (y_{n,i} - y_{n,j})}, \quad i = 0, \ldots, n + m_n + 1 - d_n, m_n.
\]

(8)

On the other hand, by (5) and (6),

\[
\rho_n \geq E_n - E_{n+1}.
\]

(9)

Let now \( \tilde{P} \) be the polynomial of degree \( n + m_n - d_n, m_n \) that interpolates \( \tilde{A}_{n+1}z^{n+m_n+1-d_n, m_n} \) at \( y_{n,i}, \quad i = 1, \ldots, n + m_n + 1 - d_n, m_n. \) Replacing in (7) \( \tilde{p} \) by \( \tilde{P} \), we obtain

\[
\rho_n = \frac{1}{\sum_{i=0}^{n+m_n+1-d_n, m_n} |\alpha_{n,i}|} \cdot |\tilde{A}_{n+1}|.
\]

Taking now into account (9) and applying (4), we arrive at

\[
\frac{1}{\sum_{i=0}^{n+m_n+1-d_n, m_n} |\alpha_{n,i}|} \geq C_3 m_n E_n - E_{n+1} 2^n (E_n + E_{n+1}).
\]

Setting \( \omega_n(z) := \prod_{i=0}^{n+m_n+1-d_n, m_n} (z - y_{n,i}) \) and combining (8) and (2), we get for \( n \geq n_2, \quad n \in \Lambda, \)

\[
\sum_{i=0}^{n+m_n+1-d_n, m_n} |\alpha_{n,i}| = \sum_{i=0}^{n+m_n+1-d_n, m_n} \frac{|(Q_nQ_{n+1})(y_{n,i})|}{|w'_n(y_{n,i})|} \leq C_4 m_n 2^n n^2.
\]

(10)

Using the Lagrange interpolation formula, we obtain from (8) that

\[
\frac{(Q_nQ_{n+1})(z)}{\omega_n(z)} = \sum_{i=0}^{n+m_n+1-d_n, m_n} \frac{\alpha_{n,i}}{z - y_{n,i}}.
\]

(11)

This formula is valid for every \( z \) with \( \omega_n(z) \neq 0. \)

From here, by means of (10), we finally arrive at

\[
\left| \frac{(Q_nQ_{n+1})(z)}{\omega_n(z)} \right| \leq C_5 m_n \frac{2^n n^2}{\text{dist}(z, I)}, \quad n \geq n_2, \quad n \in \Lambda.
\]

(12)
2. Discrepancy Results

Let $\nu_{n,m}$ be the probability measure that associates the mass $1/(n+m+2-d_{n,m})$ with each of the points $y_{n,i}$, $i = 0, \ldots, n+m+1-d_{n,m}$, and $\mu$ - the equilibrium measure on $[-1,1]$, that is: $d\mu = \frac{dx}{\pi\sqrt{1-x^2}}$, $x \in I$.

The discrepancy $D[\nu_{n,m} - \mu]$ between $\nu_{n,m}$ and $\mu$ is given by

$$D[\nu_{n,m} - \mu] := \sup_{1 \leq a < b \leq 1} |\nu_{n,m} - \mu|(a,b).$$

The first discrepancy result is the classical theorem of Kadec [7] about polynomial approximation. M. Kadec found that for every $\varepsilon > 0$ there is a positive constant $c$ such that

$$D[\nu_{n,0} - \mu] \leq c \frac{1}{n^{1/2} - \varepsilon}, \quad n \in \Lambda.$$ 

Later, H.-P. Blatt [3] sharpened Kadec’s result, showing that

$$D[\nu_{n,0} - \mu] \leq c \left(\log n\right)^2 n, \quad n \in \Lambda. \quad (13)$$

In the present paper, we will exploit formulas (11) - (12) to provide discrepancy results dealing with the distribution of alternation points in rational approximation.

Given a number $\rho > 1$, we set $E_\rho$ for the interior of the ellipse with foci at $\pm 1$ and axes $1/(2(\rho \pm 1/\rho))$; $\Gamma_\rho := \partial E_\rho$.

**Theorem 1.** Let $m_n \leq n$, $m_n \leq m_{n+1} \leq m_n + 1$ for $n = 1, 2, \ldots$. Assume there is an annulus $A_{r,R} := \{z, z \in E_R - E_r, 1 < r < R\}$ such that $Q_n(z) \neq 0$ for every $n$ starting with some number $n_0$ and $z \not\in A_{r,R}$. Then there is a positive constant $C$ such that

$$D[\nu_{n,m} - \mu] \leq C \sqrt{\frac{\ln n}{n} + \frac{m_n}{n}}, \quad n \in \Lambda.$$ 

In the special case, when $m_n = m$ for every $n \in \mathbb{N}$, we have

$$D[\nu_{n,m} - \mu] \leq \frac{(\log n)^2}{n} \quad n \in \Lambda.$$ 

We note that the assumptions of Theorem 1 are satisfied if $f$ is analytic on $I$ and admits a continuation into some ellipse as a meromorphic function with exactly $m$ poles inside the ellipse in question (multiplicities included). In this case we get the same estimate as in (13) (cf. [6]).

**Theorem 2.** Under the same conditions on $f$ and $\{m_n\}$, assume $m_n = o(n)$. Then there is a positive constant $C$ such that

$$D[\nu_{n,m} - \mu] \leq C \sqrt{\frac{m_n \ln n}{n}}, \quad n \in \Lambda.$$
3. Proofs

Lemma 1 ([5, p. 105]). Let \( p_n \in \mathcal{P}_n \) be a monic polynomial with simple zeros \( \zeta_i, i = 1, \ldots, n \) on \( I \). Assume
\[
\|p_n\|_I \leq \frac{a_n}{2^n} \quad \text{and} \quad |p'_n(\zeta_i)| \geq \frac{1}{b_n 2^n},
\]
where \( a_n > 0 \) and \( b_n > 1 \). Set \( c_n := \max\{n, a_n, b_n\} \). If \( c_n \leq e^{n/e} \), then for the measure \( \nu_{p_n} \) associated with \( \zeta_i, i = 1, \ldots, n \) we have
\[
D[\nu_{p_n} - \mu] \leq \log c_n \frac{n}{\log c_n}.
\]

Lemma 2 ([5]). Let \( p_n \in \mathcal{P}_n \) be a monic polynomial with simple zeros \( \zeta_i, i = 1, \ldots, n \) on \( I \) such that
\[
|p'_n(\zeta_i)| \geq \frac{1}{b_n 2^n}, \quad i = 1, \ldots, n
\]
with \( b_n > 1 \). Then
\[
D[\nu_{p_n} - \mu] \leq C \sqrt{\frac{\log(nb_n)}{n}}.
\]

Lemma 3 ([5]). Let \( n > 2 \), \( \Xi_n \) be a monic polynomial of degree \( n \) and the points \( \xi_i, i = 1, \ldots, n \), be such that
\[-1 \leq \xi_1 < \xi_2 < \cdots < \xi_{n-1} < \xi_n \leq 1 \]
and
\[
\Xi_n(\xi_i) \Xi_n(\xi_{i+1}) < 0, \quad i = 1, \ldots, n-1.
\]
Set \( \pi_n(z) := \prod_{i=1}^{n} (z - \xi_i) \). Then
\[
\|\pi_n\|_I \leq C n \|\Xi_n\|_I.
\]

Lemma 4 ([5]). Assume, for \( n \in \Lambda \) and \( z \in \Gamma_{1+\delta} \), \( \delta > 0 \), \( \sup \|U_{\nu_{m-n} - \mu} \| \leq \alpha(\delta) \). Then there is a positive constant \( c \) such that
\[
D[\nu_{m-n} - \mu] \leq c \alpha(\delta)^{1/2}, \quad n \in \Lambda.
\]

Proof of Theorem 1. The proof of the first part is a consequence of (10) and Lemma 2, applied to the polynomials \( w_n \). The case \( m = n \), \( \nu \rightarrow \mu \), results from (11) and Lemmas 1 and 3, after taking into account the inequality
\[
|A_{n,m}| \geq C_{m2^n}(E_n - E_{n+1}), \quad n \in \Lambda.
\]

Proof of Theorem 2. First note that the sequence \( \nu_{m,n} \) converges weakly to \( \mu \), as \( n \in \Lambda [4] \). We fix a number \( \rho > 1 \) and normalize the numerators \( Q_n \) in the way
\[
Q_n(z) = \prod_{\zeta_{n,i} \in \Xi_{\rho}} (z - \zeta_{n,i}) \prod_{\zeta_{n,i} \notin \Xi_{\rho}} (1 - z/\zeta_{n,i}) := q_n(z) Q^*_n(z).
\]
Cover each zero \( \zeta_{n,i} \) of \( q_n \) by a circle \( \Omega_{n,i} \) of radius \( 1/32m_n n^2 \) and set \( \Omega_n := \bigcup_i \Omega_{n,i} \). It is easy to see that for every \( n \) there is a number \( \kappa_n, 1 < \kappa_n < 2 \), such that \( (\Omega_n \cup \Omega_{n+1}) \cap \Gamma_{1+1/n^{\kappa_n}} = \emptyset \). By means of (12), there is a constant \( c_1 \) such that

\[
\frac{1}{|\omega_n(z)|} \leq c_1 m_n \frac{2^n n^2}{\text{dist}(z, I)} (32m_n n^2)^{2m_n}, \quad n \in \Lambda, \quad z \in E' \setminus I \setminus \Omega_n \cup \Omega_{n+1}.
\]

From here, we easily get

\[
U^\nu_n(z) - U^\mu(z) \leq c_2 m_n \frac{\log n}{n}, \quad z \in \Gamma_{1+1/n^{\kappa_n}}, \quad n \geq n_1, \quad n \in \Lambda.
\]

Lemma 4 leads now to the statement of the theorem.

References


HANS-PETER BLATT, RENÉ GROTHMANN
Katholische Universität Eichstätt
Mathematisch - Geographische Fakultät
85072 Eichstätt
GERMANY
E-mail: mga009@ku-eichstaett.de, Grothm@ku-eichstaett.de

RALITZA K. KOVACHEVA
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia
BULGARIA
E-mail: rkovach@math.bas.bg