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Discrepancy Estimates and Rational Chebyshev Approximation

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Given a function f, real-valued and continuous on $[-1,1]$, let R_{n,m_n} be the rational function of best uniform approximation of f on $[-1,1]$ of order (n,m_n). Let $m_n \le n$, $m_n \le m_{n+1} \le m_n+1$, $n\to\infty$. In the present paper, results dealing with the distribution of alternation points of $f-R_{n,m_n}$ are provided.

1. Introduction

Let $I := [-1,1]$ and let the function $f \in C(I)$ be real-valued on I. Throughout this paper, we assume that f is not rational.

Set $\mathcal{P}_n$ for the class of polynomials with real coefficients of degree not exceeding n and $\mathcal{R}_{n,m}$ for the collection of all rational functions $r = p/q,\ p \in \mathcal{P}_n$, $q \in \mathcal{P}_m,\ q \not\equiv 0$.

Given a pair (n,m) of nonnegative integers $(n,m \in \mathbb{N})$, let $R_{n,m}$ $(= R_{n,m}(f))$ be the rational function of best Chebyshev approximation of f on I in the class $\mathcal{R}_{n,m}$. Write $R_{n,m} := \frac{P_{n,m}}{Q_{n,m}}$, where both polynomials $P_{n,m}$ and $Q_{n,m}$ do not have common divisors and $Q_{n,m}(\zeta_{n,m,i}) = 0,\ i = 1,\dots,k_{n,m} \le m$. Fix in an arbitrary way a positive number $R,\ R > 2$, and normalize $Q_{n,m}$ in the way:

$$Q_{n,m}(z) = \prod_{|\zeta_{n,m,i}|\le R} (z-\zeta_{n,m,i}) \prod_{|\zeta_{n,m,i}|>R} (1 - z/\zeta_{n,m,i}).$$

Apparently, for every compact set K in $\mathbf{C}$ there is a positive constant $C(K)$ such that the inequality

$$\|Q_{n,m}\|_K \le C(K)^m \tag{1}$$

holds.

Set $d_{n,m} := \min\{n - \deg P_{n,m}, m - \deg Q_{n,m}\}$. By Chebyshev's alternation theorem [1], $R_{n,m}$ is unique and is characterized by the existence of

$n+m+2-d_{n,m}$ points of alternation $y_i^{(n,m)}$; $-1 \le y_0^{(n,m)} < y_1^{(n,m)} < \cdots < y_{n+m+1-d_{n,m}}^{(n,m)} \le 1$ and

$$(f-R_{n,m})(y_i^{(n,m)}) = \delta_{n,m}(-1)^i\, E_{n,m}(f),\;\; i=0,\ldots,n+m+1-d_{n,m},\; \delta_{n,m}=\pm 1.$$

In what follows, we shall consider sequences $\{R_{n,m_n}\}$ with $m_n \le n,\; m_n \le m_{n+1} \le m_n+1$. For simplicity, we omit writing m_n (i.e., $E_{n,m_n} := E_n$, $Q_{n,m_n} := Q_n$, etc.)

Recall the well-known fact [7] that there exists an infinite sequence $\Lambda \subset \mathbb{N}$ such that

$$\frac{E_n + E_{n+1}}{E_n - E_{n+1}} \le Cn^2 \qquad \text{as } n \to \infty,\; n \in \Lambda \tag{2}$$

(C is a positive constant).

Write $P_{n,m}(z) = A_n z_{n,m}^n + \ldots$ and set

$$R_{n+1} - R_n = \frac{\tilde{A}_{n+1} W_n}{Q_{n+1} Q_n},$$

with W_n being a monic polynomial. It is easy to check that $\deg W_n = n+m_n+1-d_{n,m_n}$, as well as that all its zeros are simple and interlace the alternation points $y_{n,i},\; i=0,\ldots,n+m+1-d_{n,m_n}$. For $\tilde{A}_{n+1}$ we have

$$\tilde{A}_{n+1} = \begin{cases} A_{n+1} \prod\limits_{|\zeta_{n,i}|>R} (-\frac{1}{\zeta_{n,i}}) - A_n \prod\limits_{|\zeta_{n+1,i}|>R} (-\frac{1}{\zeta_{n+1,i}}), & \text{if } m_n+1 = m_{n+1} \\ A_{n+1} \prod\limits_{|\zeta_{n,i}|>R} (-\frac{1}{\zeta_{n,i}}), & \text{if } m_n = m_{n+1}. \end{cases} \tag{3}$$

We first estimate $\tilde{A}_{n+1}$. After keeping track of (1), we obtain

$$|\tilde{A}_{n+1}| \le C_2^{m_n}(E_n + E_{n+1})2^n, \qquad n \ge n_1. \tag{4}$$

Apparently,

$$\begin{aligned}(R_{n+1} - R_n)(y_{n,i})\,(R_{n+1} - R_n)(y_{n,i+1}) < 0,&\\ i=0,\ldots,n+m_n-d_{n,m_n}&\end{aligned} \tag{5}$$

and

$$|(R_{n+1} - R_n)(y_{n,i})| \ge E_n - E_{n+1}, \qquad i=0,\ldots,n+m_n+1-d_{n,m_n} \tag{6}$$

Fix now a number $n \ge n_1$ and introduce into considerations the quantity

$$\rho_n := \inf_{p \in \mathcal{P}_{n+m_n-d_{n,m_n}}} \left\| \frac{\tilde{A}_{n+1} z^{n+m_n+1-d_{n,m_n}}}{Q_n(z)Q_{n+1}(z)} - \frac{p(z)}{Q_n(z)Q_{n+1}(z)} \right\|_I .$$

The set $\left\{\frac{p}{Q_nQ_{n+1}} : p \in \mathcal{P}_{n+m_n-d_{n,m_n}},\ Q_n \text{ and } Q_{n+1} - \text{fixed polynomials}\right\}$ forms a set of dimension $n+m_n+1-d_{n,m_n}$ that satisfies the Haar conditions. Hence ([2])

$$\rho_n = \frac{\displaystyle\sum_{i=0}^{n+m_n+1-d_{n,m_n}} \left|\alpha_{n,i}\Big(\frac{\tilde{A}_{n+1}y_{n,i}^{n+m_n+1-d_{n,m_n}}}{Q_{n+1}(y_{n,i})Q_n(y_{n,i})} - \frac{\tilde{p}(y_{n,i})}{Q_nQ_{n+1})(y_{n,i})}\Big)\right|}{\displaystyle\sum_{i=0}^{n+m_n+1-d_{n,m_n}} |\alpha_{n,i}|} \tag{7}$$

where $\tilde{p} \in \mathcal{P}_{n+m_n-d_{n,m_n}}$ is arbitrary and

$$\alpha_{n,i} = \frac{(Q_{n+1}Q_n)(y_{n,i})}{\prod_{j\neq i}(y_{n,i}-y_{n,j})}, \qquad i = 0,\dots,n+m_n+1-d_{n,m_n}. \tag{8}$$

On the other hand, by (5) and (6),

$$\rho_n \geq E_n - E_{n+1}. \tag{9}$$

Let now $\tilde{P}$ be the polynomial of degree $n+m_n-d_{n,m_n}$ that interpolates $\tilde{A}_{n+1}z^{n+m_n+1-d_{n,m_n}}$ at $y_{n,i},\ i = 1,\dots,n+m_n+1-d_{n,m_n}$. Replacing in (7) $\tilde{p}$ by $\tilde{P}$, we obtain

$$\rho_n = \frac{1}{\sum_{i=0}^{n+m_n+1-d_{n,m_n}} |\alpha_{n,i}|}|\tilde{A}_{n+1}|.$$

Taking now into account (9) and applying (4), we arrive at

$$\frac{1}{\sum_{i=0}^{n+m_n+1-d_{n,m_n}} |\alpha_{n,i}|} \geq C_3^{m_n}\frac{E_n - E_{n+1}}{2^n(E_n+E_{n+1})}.$$

Setting $\omega_n(z) := \prod_{i=0}^{n+m_n+1-d_{n,m_n}}(z-y_{n,i})$ and combining (8) and (2), we get for $n \geq n_2,\ n \in \Lambda$,

$$\sum_{i=0}^{n+m_n+1-d_{n,m_n}} |\alpha_{n,i}| = \sum_{i=0}^{n+m_n+1-d_{n,m_n}} \frac{|(Q_nQ_{n+1})(y_{n,i})|}{|w_n'(y_{n,i})|} \leq C_4^{m_n}\,2^n n^2. \tag{10}$$

Using the Lagrange interpolation formula, we obtain from (8) that

$$\frac{(Q_nQ_{n+1})(z)}{\omega_n(z)} = \sum_{i=0}^{n+m_n+1-d_{n,m_n}} \frac{\alpha_{n,i}}{z-y_{n,i}}. \tag{11}$$

This formula is valid for every z with $\omega_n(z) \neq 0$.

From here, by means of (10), we finally arrive at

$$\left|\frac{(Q_nQ_{n+1})(z)}{\omega_n(z)}\right| \leq C_5^{m_n}\,\frac{2^n n^2}{\operatorname{dist}(z,I)}, \qquad n \geq n_2,\ n \in \Lambda. \tag{12}$$

2. Discrepancy Results

Let ν_{n,m_n} be the probability measure that associates the mass $1/(n+m_n+2-d_{n,m_n})$ with each of the points $y_{n,i}$, $i=0,\dots,n+m_n+1-d_{n,m_n}$, and μ – the equilibrium measure on $[-1,1]$, that is: $d\mu = \frac{dx}{\pi\sqrt{1-x^2}}$, $x \in I$.

The *discrepancy* $D[\nu_{n,m}-\mu]$ between $\nu_{n,m}$ and μ is given by

$$D[\nu_{n,m}-\mu] := \sup_{1\le a<b\le 1} |\nu_{n,m}-\mu|(a,b).$$

The first discrepancy result is the classical theorem of Kadec [7] about polynomial approximation. M. Kadec found that *for every* $\varepsilon > 0$ *there is a positive constant* c *such that*

$$D[\nu_{n,0}-\mu] \le c\frac{1}{n^{1/2-\varepsilon}}, \qquad n \in \Lambda.$$

Later, H.-P. Blatt [3] sharpened Kadec's result, showing that

$$D[\nu_{n,0}-\mu] \le c\frac{(\log n)^2}{n}, \qquad n \in \Lambda. \tag{13}$$

In the present paper, we will exploit formulas (11) - (12) to provide discrepancy results dealing with the distribution of alternation points in rational approximation.

Given a number $\rho > 1$, we set $\mathcal{E}_\rho$ for the interior of the ellipse with foci at ± 1 and axes $1/2(\rho \pm 1/\rho)$; $\Gamma_\rho := \partial\mathcal{E}_\rho$.

Theorem 1. *Let* $m_n \le n$, $m_n \le m_{n+1} \le m_n + 1$ *for* $n = 1, 2, \dots$. *Assume there is an annulus* $\mathcal{A}_{r,R} := \{z, z \in \mathcal{E}_R - \mathcal{E}_r,\ 1 < r < R\}$ *such that* $Q_n(z) \ne 0$ *for every* n *starting with some number* n_0 *and* $z \notin \mathcal{A}_{r,R}$. *Then there is a positive constant* C *such that*

$$D[\nu_{n,m_n}-\mu] \le C\sqrt{\frac{\ln n}{n} + \frac{m_n}{n}}, \qquad n \in \Lambda.$$

In the special case, when $m_n = m$ *for every* $n \in \mathbf{N}$, *we have*

$$D[\nu_{n,m}-\mu] \le \frac{(\log n)^2}{n} \qquad n \in \Lambda.$$

We note that the assumptions of Theorem 1 are satisfied if f is analytic on I and admits a continuation into some ellipse as a meromorphic function with exactly m poles inside the ellipse in question (multiplicities included). In this case we get the same estimate as in (13) (cf. [6]).

Theorem 2. *Under the same conditions on* f *and* $\{m_n\}$, *assume* $m_n = o(n)$. *Then there is a positive constant* C *such that*

$$D[\nu_{n,m_n}-\mu] \le C\sqrt{\frac{m_n \ln n}{n}}, \qquad n \in \Lambda.$$

3. Proofs

Lemma 1 ([5, p. 105]). *Let $p_n \in \mathcal{P}_n$ be a monic polynomial with simple zeros $\zeta_i,\ i = 1,\dots,n$ on I. Assume*

$$\|p_n\|_I \le \frac{a_n}{2^n} \qquad \text{and} \qquad |p_n'(\zeta_i)| \ge \frac{1}{b_n 2^n},$$

where $a_n > 0$ and $b_n > 1$. Set $c_n := \max\{n, a_n, b_n\}$. If $c_n \le e^{n/e}$, then for the measure ν_{p_n} associated with $\zeta_i,\ i = 1,\dots,n$ we have

$$D[\nu_{p_n} - \mu] \le \frac{\log c_n}{n} \log \frac{n}{\log c_n}.$$

Lemma 2 ([5]). *Let $p_n \in \mathcal{P}_n$ be a monic polynomial with simple zeros $\zeta_i,\ i = 1,\dots,n$ on I such that*

$$|p_n'(\zeta_i)| \ge \frac{1}{b_n 2^n}, \qquad i = 1,\dots,n$$

with $b_n > 1$. Then

$$D[\nu_{p_n} - \mu] \le C\sqrt{\frac{\log(n b_n)}{n}}.$$

Lemma 3 ([5]). *Let $n > 2,\ \Xi_n$ be a monic polynomial of degree n and the points $\xi_i,\ i = 1,\dots,n$, be such that*

$$-1 \le \xi_1 < \xi_2 < \cdots < \xi_{n-1} < \xi_n \le 1$$

and

$$\Xi_n(\xi_i)\,\Xi_n(\xi_{i+1}) < 0, \qquad i = 1,\dots,n-1.$$

Set $\pi_n(z) := \prod_{i=1}^n (z - \xi_i)$. Then

$$\|\pi_n\|_I \le \tilde{C} n^3\, \|\Xi_n\|_I.$$

Lemma 4 ([5]). *Assume, for $n \in \Lambda$ and $z \in \Gamma_{1+\delta},\ \delta > 0$, $\sup U^{\nu_{n,m_n} - \mu} \le \alpha(\delta)$. Then there is a positive constant c such that*

$$D[\nu_{n,m_n} - \mu] \le c\alpha(\delta)^{1/2}, \qquad n \in \Lambda.$$

Proof of Theorem 1. The proof of the first part is a consequence of (10) and Lemma 2, applied to the polynomials w_n. The case $m_n = m, n = 1, 2, \dots$ results from (11) and Lemmas 1 and 3, after taking into account the inequality $|A_{n,m}| \ge C^m 2^n (E_n - E_{n+1})$, $n \in \Lambda$.

Proof of Theorem 2. First note that the sequence ν_{n,m_n} converges weakly to μ, as $n \in \Lambda$ [4]. We fix a number $\rho > 1$ and normalize the numerators Q_n in the way

$$Q_n(z) = \prod_{\zeta_{n,i} \in \overline{\mathcal{E}}_\rho} (z - \zeta_{n,i}) \prod_{\zeta_{n,i} \notin \mathcal{E}_\rho} (1 - z/\zeta_{n,i}) := q_n(z)\, Q_n^*(z).$$

Cover each zero $\zeta_{n,i}$ of q_n by a circle $\Omega_{n,i}$ of radius $1/32m_nn^2$ and set $\Omega_n := \cup_i\Omega_{n,i}$. It is easy to see that for every n there is a number κ_n, $1 < \kappa_n < 2$, such that $(\Omega_n \cup \Omega_{n+1}) \cap \Gamma_{1+1/n^{\kappa_n}} = \emptyset$. By means of (12), there is a constant c_1 such that

$$\frac{1}{|\omega_n(z)|} \le c_1^{m_n}\frac{2^nn^2}{\operatorname{dist}(z,I)}(32m_nn^2)^{2m_n}, \qquad n \in \Lambda,\ z \in \mathcal{E}_\rho \setminus I \setminus \Omega_n \cup \Omega_{n+1}.$$

From here, we easily get

$$U^{\nu_n}(z) - U^{\mu}(z) \le c_2\, m_n \frac{\log n}{n}, \qquad z \in \Gamma_{1+n^{-\kappa_n}},\ n \ge n_1,\ n \in \Lambda.$$

Lemma 4 leads now to the statement of the theorem.

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