

New Nonexistence Results for Spherical Designs

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We obtain bounds for the smallest and largest inner products of distinct points of spherical τ -designs of relatively small cardinalities and odd strengths τ . In many cases, the restrictions obtained imply new nonexistence results. Our method works well in small dimensions as well as when the dimension tends to infinity. For $\tau = 3$ and $\tau = 5$, we obtain new asymptotic bounds on the minimum possible odd size of τ -designs.

1. Introduction

The spherical designs have been introduced in 1977 by Delsarte-Goethals-Seidel [7] on the analogy of the classical combinatorial designs. A spherical τ -design $C \subset \mathbf{S}^{n-1}$ is a finite subset of the unit sphere \mathbf{S}^{n-1} in \mathbb{R}^n such that

$$\frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, \dots, x_n)$ of degree at most τ (i.e., the average of f over the set is equal to the average of f over \mathbf{S}^{n-1}). The number τ is called *the strength* of C .

Let us denote by $B(n, \tau)$ (resp., by $B_{\text{odd}}(n, \tau)$) the minimum possible cardinality (resp. odd cardinality) of a τ -design on \mathbf{S}^{n-1} . The following Fisher-type lower bound on $B(n, \tau)$ was obtained by Delsarte-Goethals-Seidel [7, Theorem 5.11 and Theorem 5.12]

$$B(n, \tau) \geq R(n, \tau) = \begin{cases} \binom{n+e-1}{e} + \binom{n+e-2}{e-1}, & \text{if } \tau = 2e \\ 2\binom{n+e-2}{e-1}, & \text{if } \tau = 2e - 1. \end{cases} \quad (1)$$

In this paper we consider designs of odd strength $\tau = 2e - 1$, $e \geq 2$ is integer, and odd cardinality $|C|$. This case turned out to be more difficult for realization.

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For $\tau = 2e - 1$ we derive from (1) the estimate

$$B_{\text{odd}}(n, 2e - 1) \geq 2 \binom{n + e - 2}{e - 1} + 1.$$

First nonexistence results for $(2e - 1)$ -designs of odd size were proved in [5] (see also [4]). In this paper we continue this investigation by refining the approach from [5].

The following definition for spherical designs is crucial for our approach. A code $C \subset \mathbf{S}^{n-1}$ is a spherical τ -design if and only if for any point $y \in \mathbf{S}^{n-1}$ and any real polynomial $f(t)$ of degree at most τ , the equality

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C| \tag{2}$$

holds, where

$$f_0 = c_n \int_{-1}^1 f(t)(1 - t^2)^{(n-3)/2} dt, \quad c_n = \frac{\Gamma(n-1)}{2^{n-2}(\Gamma(\frac{n-1}{2}))^2}$$

(f_0 is the first coefficient in the expansion of $f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22]). As usual, $\langle x, y \rangle$ denotes the standard scalar product in \mathbb{R}^n .

We use (2) when y belongs to the design. Then (2) becomes

$$\sum_{x \in C \setminus \{y\}} f(\langle x, y \rangle) = f_0 |C| - f(1). \tag{3}$$

In this paper, we propose a method for investigation of $(2e - 1)$ -designs of odd size. As results, we derive some restrictions on the structure of such designs which are expressed as bounds on inner products of their points. Sometimes this implies nonexistence results. Interestingly, nonexistence follows already in first open cases (i.e., in small dimensions) and when the dimension n tends to infinity but the strength τ is fixed.

In Section 2 we describe our approach. We start with the assumption of the existence of certain $(2e - 1)$ -design $C \subset \mathbf{S}^{n-1}$ with prescribed odd cardinality. Then we prove that some special triple of points of C do always exist. The remaining analysis is restricted to these three points with using suitable polynomials in (3).

Applications for $\tau = 3$ and $\tau = 5$ are shown in Sections 3 and 4. It becomes clear how this investigation could be continued for higher strengths.

2. Method of Investigation

Let $C \subset \mathbf{S}^{n-1}$ be a $(2e - 1)$ -design. For arbitrary point $x \in C$, we denote

$$I(x) = \{\langle x, y \rangle : y \in C \setminus \{x\}\} = \{t_1(x), t_2(x), \dots, t_{|C|-1}(x)\},$$

where $-1 \leq t_1(x) \leq t_2(x) \leq \dots \leq t_{|C|-1}(x) < 1$.

It follows from [9, Section 4] (see also [10, 4]) that for every fixed “size” $|C| > R(n, 2e - 1)$ there exist uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \dots < \alpha_{e-1} < 1$ and $\rho_0, \rho_1, \dots, \rho_{e-1}, \rho_i > 0$ for $i = 0, 1, \dots, e - 1$, such that the equality

$$f_0 = \frac{f(1)}{|C|} + \sum_{i=0}^{e-1} \rho_i f(\alpha_i)$$

is true for every real polynomial $f(t)$ of degree at most $2e - 1$. The numbers $\alpha_i, i = 0, 1, \dots, e - 1$, are all roots of the equation $P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0$, where $P_i(t) = P_i^{(n-1)/2, (n-3)/2}(t)$ is the Jacobi polynomial [1, 12]. The weights ρ_i can be found by

$$\rho_i = - \left(\prod_{0 \leq j \leq e-1, j \neq i} (1 - \alpha_j^2) \right) / \left(\alpha_i |C| \prod_{0 \leq j \leq e-1, j \neq i} (\alpha_i^2 - \alpha_j^2) \right).$$

These facts were used in [4] for obtaining restrictions on the structure of designs. For example, it was proved in [4] that $t_1(x) \leq \alpha_0$ (see Lemma 2 below) and $t_{|C|-1}(x) \geq \alpha_{e-1}$ for every point $x \in C$.

In what follows, we associate every feasible size $|C| \geq R(n, 2e - 1) + 1$ to the corresponding numbers $\alpha_0, \alpha_1, \dots, \alpha_{e-1}$ and $\rho_0, \rho_1, \dots, \rho_{e-1}$.

Our approach is the following. First, we show for odd cardinalities $|C|$ that some special triples (x, y, z) of points of C appear. Then we use suitable polynomials in (3) in order to derive bounds on some inner products in the sets $I(x), I(y)$ and $I(z)$. At the third step, we organize an iterative process by using the new bounds and (other) suitable polynomials in (3). The results are again bounds on inner products in $I(x), I(y)$ and $I(z)$. In many cases this implies nonexistence of the design under target.

Step 1. Our first step is based on the following simple observation.

Lemma 1. *Let $C \subset \mathbf{S}^{n-1}$ be a τ -design of odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$.*

Proof. Let Γ be the directed graph with vertices the points of C and edges $x \rightarrow y$ if and only if $t_1(x) = \langle x, y \rangle$. It is easy to see that cycles in Γ are possible only of length two. Since $|\Gamma| = |C|$ is odd, we must have $y \leftrightarrow x \leftarrow z$ which completes the proof. \square

Lemma 2 ([4]). *If $C \subset \mathbf{S}^{n-1}$ is a $(2e - 1)$ -design, then $t_1(x) \leq \alpha_0$ for every point $x \in C$.*

It follows by Lemmas 1 and 2 that there exists a point $x \in C$ such that $t_1(x) \leq t_2(x) \leq \alpha_0$. This observation was used in [4] to prove that $\rho_0 |C| \geq 2$

is a necessary condition for existence of C . We upgrade this result by more detailed investigation of the triple (x, y, z) .

Step 2. Inequalities of the type $t_1(x) = \langle x, y \rangle \leq t_2(x) = \langle x, z \rangle \leq a$ may mean that the points y and z are close each other. Indeed, it is easy to see that $\langle y, z \rangle \geq 2a^2 - 1$. If $2a^2 - 1 > \alpha_{e-1}$, we actually have obtained new bounds on $t_{|C|-1}(y)$ and $t_{|C|-1}(z)$. In turn, these new bounds give better estimations $t_1(y) \leq a' < a$ and $t_1(z) \leq a' < a$. This leads to an improvement $t_1(x) \leq t_2(x) \leq a' < a$.

Step 3. If $2\alpha_0^2 - 1 > \alpha_{e-1}$, we can start and further organize the following iterative process, applying in fact Step 2 as many times as necessary. Set $\delta_0 = \alpha_0$ and let us have $\delta_1 = a'$ by applying Step 2 for $a = \alpha_0$. Now $2\delta_1^2 - 1 > 2\delta_0^2 - 1$ is a better lower bound for $t_{|C|-1}(y)$ and $t_{|C|-1}(z)$ and implies by a second application of Step 2 better upper bounds $t_2(x) \leq \delta_2$. We can continue this process, checking (at each iteration) the existence of C .

Theorem 1. *If there exists a real nonnegative polynomial $f(t)$ of degree at most $2e - 1$ which decreases in the interval $[-1, \alpha_0)$ and*

$$2f(\delta_i) > f_0|C| - f(1) \quad (4)$$

for some $i \geq 0$, then C does not exist.

Proof. Assume that such a polynomial f exists and consider (3) for this polynomial and $x \in C$. The assertion then follows since the left-hand side of (3) is at least $2f(t_2(x)) \geq 2f(\delta_i)$ for all $i \geq 0$. This means that C could not exist if (4) is satisfied. \square

The logic of Theorem 1 is the following. If $\lim_{i \rightarrow +\infty} \delta_i = -\infty$ (we actually need much weaker results – see Example 1), then C does not exist. Otherwise, we have some new bounds of $\ell(C) = \min\{\langle x, y \rangle : x, y \in C\}$ and $s(C) = \max\{\langle x, y \rangle : x, y \in C, x \neq y\}$.

Theorem 2. *If $\lim_{i \rightarrow +\infty} \delta_i = \delta > -\infty$, then $-1 \leq \ell(C) \leq \delta$ and $1 > s(C) \geq 2\delta^2 - 1 > \alpha_{e-1}$.*

Proof. We have $\ell(C) \leq t_1(z) \leq \delta$ and $s(C) \geq t_{|C|-1}(z) \geq 2\delta^2 - 1$. \square

3. Some Results for 3-designs

3.1. Small Cases

Let $\tau = 3$ and $C \subset \mathbf{S}^{n-1}$ be a 3-design of cardinality $|C| = R(n, 3) + k = 2n + k$. Now α_0 and α_1 are the roots of the equation $n(n + k - 1)X^2 + n(n - 1)X - k = 0$ (see [5, Eq. (8)]).

Bajnok [2, 3] shows that 3-designs of any even size exist. He also constructs 3-designs of any odd cardinality greater than or equal to $5n/2$ for $n \geq 7$, 11 for $n = 3, 4$ and 15 for $n = 5, 6$. On the other hand, it was shown in [5] that $k \geq 3$ in all dimensions, $k \geq 5$ for $n \geq 11$, $k \geq 7$ for $n \geq 19$, etc.

Therefore, we may assume that k is odd, $k = 3$ for $n = 3, 5, 7, 8, 9, 10$, $k = 5$ for $11 \leq n \leq 18$, $k = 7$ for $15 \leq n \leq 2x$, etc. Let x, y, z be points in C such that $t_1(x) = t_1(y) \leq t_2(x) = t_1(z) \leq \alpha_0$. For Step 2, we assume that $\mu_0 = 2\alpha_0^2 - 1 > \alpha_1$.

Lemma 3. *For any real $a \in (\alpha_0, \alpha_1)$, we have $t_1(z) \leq F(a)$, where*

$$F(a) = -2 \frac{n\alpha_0^2 a^2 + [2n(2\alpha_0^2 - 2\alpha_0^4 - 1) + |C|] a + n\alpha_0^2(4\alpha_0^4 - 6\alpha_0^2 + 3)}{(|C| - 2)a^2 + 4n\alpha_0^2 a + 2n(2\alpha_0^2 - 2\alpha_0^4 - 1) + |C|}.$$

Proof. The equality follows from (3) for the function $f(t) = (t - t_1(z))(t - a)^2$ and $z \in C$. \square

In each concrete case, the optimal value of $F(a)$ can be found numerically by *Maple*. According to Step 2, we denote $\delta_1 = \min\{F(a) : a \in (\alpha_0, \alpha_1)\}$. Then $t_1(y) \leq t_1(z) \leq \delta_1$ whence $t_{|C|-1}(z) \geq \langle y, z \rangle \geq 2\delta_1^2 - 1 = \mu_1$. For the iterative process of Step 3, we use the analog of Lemma 3 by using μ_1 instead of μ_0 to obtain $t_1(y) \leq t_1(z) \leq \delta_2$ and so on.

To check the existence of C , we use Theorem 1 with $f(t) = t^2$. Hence, we have to check if $2\delta_i^2 > |C|/n - 1$. The whole iteration process was realized by a simple *Maple* program which is available upon request.

Example 1. Let us consider the cases $n = 9$ and $n = 10$, $k = 3$ in both dimensions. If $C \subset \mathbf{S}^9$ is a 23-point 3-design, then $\alpha_0 \approx -0.78197$, $\alpha_1 \approx 0.03197$. Thus $2\alpha_0^2 - 1 > \alpha_1$ and even at the first iteration we obtain $\delta_1 = -0.81202$, whence $2\delta_1^2 = 1.31875 > 1.3 = \frac{23}{10} - 1$. Therefore C does not exist. Analogously, for a putative 21-point 3-design on $C \subset \mathbf{S}^8$, we obtain $2\delta_4^2 = 1.35909 > \frac{4}{3} = \frac{21}{9} - 1$ (i.e., four iterations are needed). Therefore such designs do not exist.

There were 144 open cases in dimensions $3 \leq n \leq 50$. We rule out 50 of them. The first nonexistence results (Example 1) show that there are no 3-designs of 21 points in nine dimensions ($k = 3$), 23 points in ten dimensions ($k = 3$), 35 points in fifteen dimensions ($k = 5$), etc. The problem of finding all possible cardinalities of 3-designs is therefore solved in dimensions $n = 4, 6, 9$ and 10, only one open case remains in dimensions $n = 3, 5, 7, 8, 21, 22$ and for $11 \leq n \leq 18$, etc.

3.2. Asymptotic Results

Boyvalenkov-Danev-Nikova prove in [5] that

$$B_{\text{odd}}(n, 3) \gtrsim (1 + 2^{1/3})n \approx 2.2599n \tag{5}$$

as n tends to infinity. Bajnok's construction [2, 3] shows that $B_{\text{odd}}(n, 3) \leq 2.5n$.

Therefore we have to consider $k/n = \gamma \in [2^{1/3} - 1, 0.5)$ as n tends to infinity. Already the first applications of our iterative process gives better asymptotic results than (5). Asymptotically, we have $\alpha_0 \sim -1/(1 + \gamma)$ and $\alpha_1 \sim 0$. Now Lemma 3 with $a = 0$ gives

$$t_2(x) \leq \delta_1 \sim -\frac{2(\gamma^5 + 8\gamma^4 + 19\gamma^3 + 13\gamma^2 - 2\gamma + 1)}{\gamma(\gamma^2 + 4\gamma + 5)^2(\gamma + 1)^4}.$$

Solving by Maple the equation $2\delta_1^2 = 1 + \gamma$ we obtain that $B_{\text{odd}}(n, 3) \gtrsim 2.2949n$ as n tends to infinity. We were able to implement four iterations to obtain the following assertion.

Theorem 3. *We have $B_{\text{odd}}(n, 3) \gtrsim 2.3227n$.*

Therefore, $2.3227n \leq B_{\text{odd}}(n, 3) \leq 2.5n$ asymptotically. Our conjecture is that the upper bound gives the exact value of $B_{\text{odd}}(n, 3)$ either in small dimensions and as n tends to infinity.

4. Some Results for 5-designs

4.1. Small Cases

Let $C \subset \mathbf{S}^{n-1}$ be a 5-design of size $|C| = R(n, 5) + k = n^2 + n + k$.

Constructions of 5-designs were described by Reznick [11] and Hardin-Sloane [8] (see also Sloane's home page www.research.att.com/~njas/).

Let k be odd and x, y, z be the points from Lemma 1. We assume that $\mu_0 = 2\alpha_0^2 - 1 > \alpha_2$ as Step 2 requires. Then the new bound on $t_2(x) = t_1(z)$ is given by the following lemma which can be proved as Lemma 3.

Lemma 4. *For any real a and b , we have $t_1(z) \leq F(a, b)$, where*

$$F(a, b) = \frac{2a|C|[(n+2)b+3] - n(n+2)[(1+a+b)^2 + (2\alpha_0^2 - 1)K]}{|C|[n(n+2)b^2 + (n+2)(a^2 + 2b) + 3] - n(n+2)[(1+a+b)^2 + K]},$$

and $K = [(2\alpha_0^2 - 1)^2 + a(2\alpha_0^2 - 1) + b]^2$ provided the denominator in the last fraction is positive.

The iterative process can be continued as in the case $\tau = 3$ by using $f(t) = (t - \alpha_1)^2(t - \alpha_2)^2$ in Theorem 1. The first nonexistence result shows that 33-point 5-designs on \mathbf{S}^4 do not exist. We tested the first open cases (with $\rho_0|C| \geq 2$) in each dimension $3 \leq n \leq 20$ until nonexistence proof is still possible. In this way, 53 designs were proved not to exist.

4.2. Asymptotic Results

Asymptotic consequences from [5, 4] say that $B_{\text{odd}}(n, 5) \gtrsim \frac{1+2^{1/5}}{2} n^2 \approx 1.0743 n^2$ as n tends to infinity. Using the same argument as in Section 3.2, we are able to improve this.

Theorem 4. *We have $B_{\text{odd}}(n, 5) \gtrsim 1.09309 n^2$.*

Proof. The assertion follows by Theorem 1 with $f(t) = (t - \alpha_1)^2(t - \alpha_2)^2$ and the asymptotic behaviour of δ_1 . \square

5. General Asymptotic Results

Asymptotically, we have

$$B_{\text{odd}}(n, 2e - 1) \gtrsim \frac{2}{(e - 1)!} n^{e-1}$$

by the Delsarte-Goethals-Seidel bound. This was improved to

$$B_{\text{odd}}(n, 2e - 1) \gtrsim \frac{1 + 2^{1/(2e-1)}}{(e - 1)!} n^{e-1}$$

in [5] and the condition $\rho_0|C| \geq 2$ implies the same asymptotic result. We improved this for $e = 2$ and $e = 3$ in the previous two sections. Our attempts to find improvement in general led us to the following assertion.

Let x_0 be a root of the equation

$$2 \left(x^{4e-2} + (2 - x^2)^{2e-1} \right)^{2e-2} = x^{4e-3} \left(x^{4e-3} - (2 - x^2)^{2e-2} \right)^{2e-2}.$$

Then $B_{\text{odd}}(n, 2e - 1) \gtrsim \left(\frac{2}{(e-1)!} + \gamma_e \right) n^{e-1}$, where $\gamma_e = (x_0 - 1)/(e - 1)!$. The first values of γ_e are $\gamma_2 = 0.29244$, $\gamma_3 = 0.09309$ (as in Section 4.2) and $\gamma_4 = 0.02314$.

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