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On Martensen Splines

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Martensen [5] introduced a method of minimal Hermite spline interpolation which is a special case of the more general result presented in [1] and [6]. We will construct the fundamental Hermite splines for a uniform mesh and derive an integral remainder. Using Boolean methods (see [3], [6], [4]) we will construct tensor product and blending Martensen splines which are an alternative to the spline spaces discussed in [7].

1. Fundamental Hermite Splines

The linear space of polynomial splines of degree p with breakpoints kh, $h > 0$, $k \in \mathbb{Z}$, is denoted by S_{p+1}^h. Martensen spline interpolation is a Hermite type interpolation method (see also Nuernberger, Dahmen and all., Bojanov) which extends piecewise linear interpolation in the most natural way. We first consider the explicit construction for cardinal splines $S_{p+1}^1 = S_{p+1}$.

Let

$$G_0(x) = x_+^0$$

be the Heaviside function. The backward difference operator is defined as

$$\nabla f(x) = f(x) - f(x-1).$$

Then

$$G_1(x) = \int_{-\infty}^{x} \nabla G_0(s)\, ds = x_+^1 - (x-1)_+^1$$

is a spline from S_2 satisfying the interpolation conditions

$$G_1(0) = 0, \qquad G_1(1) = 1.$$

The construction is continued by recursion.

Theorem 1. *The function*

$$G_p(x) = \int_{-\infty}^{x} \nabla G_{p-1}(s)\, ds$$

is a spline from S_{p+1}. It satisfies the interpolation conditions

$$\begin{aligned} D^j G_p(0) &= 0, \qquad j = 0, ..., p-1, \\ D^j G_p(p) &= 0, \qquad j = 1, ..., p-1, \\ G_p(p) &= 1. \end{aligned}$$

Proof. The assertion is true for $p = 1$. By induction we have

$$G_{p-1}(x) = \begin{cases} 0, & x < 1 \\ 1, & x > p-1. \end{cases}$$

This implies

$$G_p(x) = 0, \qquad x < 1.$$

We have for $x > p$

$$G_p(x) = \int_{p-1}^{x} G_{p-1}(s)\, ds - \int_{p-1}^{x-1} G_{p-1}(s-1)\, ds = \int_{x-1}^{x} G_{p-1}(s)\, ds = 1.$$

This completes the proof.

To construct the fundamental Hermite spline functions we define first

$$G_{0,p}(x) = G_p(x) = \int_{-\infty}^{x} \nabla G_{p-1}(s)\, ds$$

and then, by triangular recursion,

$$G_{i,p}(x) = \int_{-\infty}^{x} G_{i-1,p-1}(s-1)\, ds - \int_{-\infty}^{p} G_{i-1,p-1}(s-1)\, ds \cdot G_{0,p}(x)$$

for $0 < i < p,\ p > 1$.

Theorem 2. *The functions*

$$G_{0,p}(x), G_{1,p}(x), ..., G_{p-1,p}(x)$$

are the fundamental Hermite splines from S_{p+1} which satisfy the interpolation conditions

$$D^j G_{k,p}(0) = 0, \quad D^j G_{k,p}(p) = \delta_{j,k}, \qquad k, j = 0, ..., p-1.$$

Proof. We apply induction on p and assume $i > 0,\ p > 1$. We have for $k > 0$

$$D^k G_{i,p}(x) = D^{k-1} G_{i-1,p-1}(s-1) - \int_{-\infty}^{p} G_{i-1,p-1}(s-1)\, ds \cdot D^k G_{0,p}(x)$$

which implies

$$D^k G_{i,p}(0) = 0, \qquad D^k G_{i,p}(p) = D^{k-1} G_{i-1,p-1}(p-1) = \delta_{k,i}.$$

We have for $k = 0$:

$$G_{i,p}(0) = \int_{-\infty}^{0} G_{i-1,p-1}(s-1)\,ds - \int_{-\infty}^{p} G_{i-1,p-1}(s-1)\,ds \cdot G_{0,p}(0) = 0,$$

$$G_{i,p}(p) = \int_{-\infty}^{p} G_{i-1,p-1}(s-1)\,ds - \int_{-\infty}^{p} G_{i-1,p-1}(s-1)\,ds \cdot G_{0,p}(p) = 0.$$

This completes the proof.

Consider the spline $F_i \in S_{p+1}$ defined by

$$F_i(x) = \begin{cases} (-1)^i G_{i,p}(p-x), & x \geq 0 \\ G_{i,p}(p+x), & x \leq 0. \end{cases}$$

It is uniquely defined by the interpolation conditions

$$D^k F_i(0) = \delta_{i,k}, \quad D^k F_i(-p) = 0, \quad D^k F_i(p) = 0, \qquad i,k = 0,...,p-1.$$

Then we define $M_h(f) \in S^h_{p+1}$ by

$$M_h(f)(x) = \sum_{r=-\infty}^{\infty} \sum_{i=0}^{p-1} D^i f(hrp)\big[h^i F_i(h^{-1}(x-hrp))\big].$$

The global scaled Martensen operator M_h is uniquely defined by the interpolation conditions

$$D^i M_h(f)(hrp) = D^i f(hrp), \qquad i = 0,...,p-1,\ r \in \mathbb{Z}.$$

2. Remainder Formulas

We consider first the cardinal remainder operator

$$\overline{M}(f)(x) = f(x) - M(f)(x).$$

Recall the Taylor formula

$$f(x) = \sum_{i=0}^{p} D^i f(0) \frac{x^i}{i!} + \int_0^p \frac{(x-s)_+^p}{p!} D^{p+1} f(s)\,ds, \qquad 0 \leq x \leq p,$$

with Taylor operator

$$T(f)(x) = \sum_{i=0}^{p} D^i f(0) \frac{x^i}{i!}.$$

Note that M_h reproduces splines from S_{p+1} and in particular polynomials of degree at most p. This implies

$$\overline{M}\big(f - T(f)\big)(x) = f(x) - M(f)(x).$$

Thus, we have

$$f(x) - \sum_{i=0}^{p} D^i f(0) \frac{x^i}{i!} = \int_0^p \frac{(x-s)_+^p}{p!} D^{p+1} f(s)\, ds,$$

$$\overline{M}\big(f - T(f)\big)(x) = \int_0^p \left[\frac{(x-s)_+^p}{p!} - \sum_{i=0}^{p-1} \frac{(p-s)_+^{p-i}}{(p-i)!} G_{i,p}(x) \right] D^{p+1} f(s)\, ds.$$

Hence, the Peano kernel of the remainder functional $f(x) - M(f)(x)$ is given by

$$K(x,s) = \frac{(x-s)_+^p}{p!} - \sum_{i=0}^{p-1} \frac{(p-s)_+^{p-i}}{(p-i)!} G_{i,p}(x), \qquad 0 < x < p.$$

Let us consider now the remainder of the global scaled Martensen interpolation:

$$\begin{aligned} f(x) - M_h(f)(x) &= \mu_h(f)(h^{-1}x) - M(\mu_h(f))(h^{-1}x) \\ &= \int_0^p K(h^{-1}x, s) D^{p+1} \mu_h(f)(s)\, ds = \int_0^p K(h^{-1}x, s) h^{p+1} (D^{p+1} f)(hs)\, ds \\ &= h^p \int_0^{ph} K(h^{-1}x, h^{-1}z)(D^{p+1} f)(z)\, dz. \end{aligned}$$

Thus, we arrive at the following.

Theorem 3. *Assume $f \in C^{p+1}(\mathbb{R})$ with $\left\| D^{p+1} f \right\|_\infty < \infty$. Then*

$$\|f - M_h(f)\|_\infty \le h^{p+1} \left\| D^{p+1} f \right\|_\infty C_p, \qquad C_p = \sup_{0 \le x \le p} \int_0^p |K(x,s)|\, ds.$$

3. Bivariate Martensen Splines

Next we consider tensor product Martensen splines and follow the representations in Haemmerlin-Hoffmann, Nuernberger, and Delvos-Schempp. We introduce the parametrically extended operators for functions of bounded mixed derivatives $f \in C^{p-1,p-1}(\mathbb{R}^2)$:

$$M_{h_x}(f)(x,y) := \sum_{l} \sum_{i<p} D^{(i,0)} f(lph_x, y) W_{i,h_x}(x - lph_x),$$

$$M_{h_y}(f)(x,y) := \sum_{l'} \sum_{i'<p} D^{(0,i')} f(x, l'ph_y) W_{i',h_y}(y - l'ph_y)$$

with

$$W_{i,h_x}(x) = h_x^i F_i\left(h_x^{-1}x\right).$$

The interpolation properties of the parametrically extended Martensen operator are given by:

$$\begin{aligned} D^{(i,0)}M_{h_x}(f)(lph_x,y) &= D^{(i,0)}f(lph_x,y), && i<p,\ l\in\mathbb{Z},\\ D^{(0,i')}M_{h_y}(f)(x,l'ph_y) &= D^{(0,i')}f(x,l'ph_y), && i'<p,\ l'\in\mathbb{Z}. \end{aligned}$$

Then we define the product operator

$$\begin{aligned} &M_{h_x}M_{h_y}(f)(x,y) := M_{h_y}M_{h_x}(f)(x,y)\\ &\quad = \sum_l \sum_{l'} \sum_{i<p} \sum_{i'<p} D^{(i,i')}f(lph_x,l'ph_y)W_{i,h_x}(x-lph_x)W_{i',h_y}(y-l'ph_y). \end{aligned}$$

It has the interpolation properties

$$D^{(i,i')}M_{h_x}M_{h_y}(f)(lph_x,l'ph_y) = D^{(i,i')}(lph_x,l'ph_y), \qquad i,i'<p,l,l'\in\mathbb{Z}.$$

The remainder operators are also parametric extensions:

$$\begin{aligned} \overline{M_{h_x}}(f)(x,y) &= h_x^p \int_0^{h_x p} K(h_x^{-1}x,h_x^{-1}s)D^{(p+1,0)}f(s,y)\,ds,\\ \overline{M_{h_y}}(f)(x,y) &= h_y^p \int_0^{h_y p} K(h_y^{-1}y,h_y^{-1}t)D^{(0,p+1)}f(x,t)\,dt. \end{aligned}$$

Since

$$\overline{M_{h_x}M_{h_y}} = \overline{M_{h_x}} + \overline{M_{h_y}} - \overline{M_{h_x}M_{h_y}}$$

and

$$\begin{aligned} &\overline{M_{h_x}M_{h_y}}(f)(x,y)\\ &\quad = h_x^p\, h_y^p \int_0^{h_x p}\int_0^{h_y p} K(h_y^{-1}y,h_y^{-1}t)K(h_x^{-1}x,h_x^{-1}s)D^{(p+1,p+1)}f(s,t)\,ds\,dt, \end{aligned}$$

we obtain

Theorem 4. *Assume* $f\in C^{(p+t,p+1)}(\mathbb{R}^2)$ *and*

$$\left\|D^{(p+1,0)}f\right\|_\infty<\infty, \qquad \left\|D^{(0,p+1)}f\right\|_\infty<\infty, \qquad \left\|D^{(p+1,p+1)}f\right\|_\infty<\infty.$$

Then we have

$$\begin{aligned} \left\|f-M_{h_x}M_{h_y}(f)\right\|_\infty \le{}& h_x^{p+1}C_p\left\|D^{(p+1,0)}f\right\|_\infty + h_y^{p+1}\left\|D^{(0,p+1)}f\right\|_\infty\\ &+ h_x^{p+1}\,h_y^{p+1}\,C_p^2\left\|D^{(p+1,p+1)}f\right\|_\infty. \end{aligned}$$

Blended bivariate Martensen interpolation is defined by the Boolean sum

$$M_{h_x} \oplus M_{h_y} = M_{h_x} + M_{h_y} - M_{h_x} M_{h_y}.$$

The interpolation properties of the blending Martensen operator are given by:

$$D^{(i,0)} M_{h_x} \oplus M_{h_y}(f)(lph_x, y) = D^{(i,0)} f(lph_x, y), \qquad i < p, l \in \mathbb{Z},$$
$$D^{(0,i')} M_{h_x} \oplus M_{h_y}(f)(x, l'ph_y) = D^{(0,i')} f(x, l'ph_y), \qquad i' < p, l' \in \mathbb{Z}.$$

Since

$$\overline{M_{h_x} \oplus M_{h_y}} = \overline{M_{h_x} M_{h_y}}$$

we obtain

Theorem 5. *Assume* $f \in C^{(p+t,p+1)}(\mathbb{R}^2)$ *and* $\|D^{(p+1,p+1)} f\|_\infty < \infty$. *Then we have*

$$\|f - M_{h_x} \oplus M_{h_y}(f)\|_\infty \le h_x^{p+1}\, h_y^{p+1}\, C_p^2 \|D^{(p+1,p+1)} f\|_\infty.$$

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