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Analyticity of Moduli of Continuity

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For any real-analytic function f on a closed interval of the real axis, Perelman [3] showed that the modulus of continuity of f is an analytic function on a neighborhood of the origin. We study conditions of analyticity for a modulus of continuity of piecewise analytic functions.

1. Introduction and notation

The modulus of continuity is one of the main structural characteristics of a function, and questions associated with the study of the modulus of continuity are considered in a great number of works (for example, see monographs [1], [4–8], and the bibliography there). In particular, it was proved in Perelman's article [3] that for any real-analytic function $f(x)$, defined on a closed interval of the real axis, the modulus of continuity $\omega(f;\delta)$ is an analytic function at zero. The subject of this report is the analysis of conditions of analyticity for the modulus of continuity of piecewise analytic functions.

As is well-known, the function $f : [a,b] \to \mathbb{R}$ is real-analytic (r.-a.) on $[a,b]$ if it is analytic at each point $x \in [a,b]$, i.e., if it can be expanded in convergent power series in a certain neighborhood of x. A continuous function $f : [a,b] \to \mathbb{R}$ is piecewise analytic (p.-a.) on $[a,b]$, if there is a partition $a = x_0 < x_1 < ... < x_{n+1} = b$ such that the restrictions $f|_{[x_i,x_{i+1}]}$ are r.-a. functions.

Let $f(x)$ be p.-a. function with a right derivative $f'_r(x)$ and a left one $f'_l(x)$. Let us denote

$$\Theta(x) := \begin{cases} |f'_r(a)|, & \text{for } x = a \\ \max\{|f'_r(x)|; |f'_l(x)|\}, & \text{for } x \in (a,b) \\ |f'_l(b)|, & \text{for } x = b, \end{cases}$$

and

$$m := \sup_{x\in[a,b]} \Theta(x), \tag{1}$$

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$$M_r := \{x \in [a,b) : |f_r'(x)| = m\}, \qquad M_l := \{x \in (a,b] : |f_l'(x)| = m\},$$
$$M := M_l \cup M_r.$$

Since $f(x)$ is p.-a. function on $[a,b]$, the relation $M \neq \emptyset$ holds. For each $t_0 \in (a,b]$ and every sufficiently small $\varepsilon > 0$ the function $f_l'(x)$ is expanded on $[t_0 - \varepsilon, t_0]$ by Taylor's formula

$$f_l'(x) = f_l'(t_0) + k_l(x - t_0)^{d_l} + o(x - t_0)^{d_l},$$

where $d_l = d_{lt_0}$ is the multiplicity of the zero of the function $(f_l'(x) - f_l'(t_0))$ at t_0, and

$$k_l = k_{lt_0} := \frac{1}{(d_l)!} f_l^{(1+d_l)}(t_0).$$

If for each $x \in [t_0 - \varepsilon, t_0]$ we have $f_l'(x) = f_l'(t_0)$, then we set $d_{lt_0} := \infty$, $k_{lt_0} := 0$. Analogously, for $t_0 \in [a,b)$, let $d_r = d_{rt_0}$, $k_r = k_{rt_0}$. Then

$$f_r'(x) = f_r'(t_0) + k_r(x - t_0)^{d_r} + o(x - t_0)^{d_r}.$$

Let the function $d(x)$ be defined on the set M by

$$d(x) := \begin{cases} d_{lx}, & \text{for } x \in M_l \setminus M_r \\ \max\{d_{lx}; d_{rx}\}, & \text{for } x \in M_l \cap M_r \\ d_{rx}, & \text{for } x \in M_r \setminus M_l. \end{cases}$$

Let $\vdots$ be the symbol for the binary relation, defined on $\mathbb{N} \cup \{\infty\}$ by the rule: $\big(a \vdots b\big)$ if and only if either a and b are natural numbers and b divides a or $a = \infty$ and b is an arbitrary element of $\mathbb{N} \cup \{\infty\}$.

Let us introduce the following notation

$$D_N := M_l \cap M_r \cap \big\{x \in (a,b) : f_r'(x) = f_l'(x),\ \rceil(d_{lx} \vdots d_{rx}),\ \rceil(d_{rx} \vdots d_{lx})\big\},$$

i.e., for $x \in D_N$ no one from the "numbers" d_{rx}, d_{lx} is a "divider" of the other one and $f_l'(x) = f_r'(x) = \pm m$.

Assume that

$$d := \sup_{x \in M} d(x), \tag{2}$$

and

$$M_{dl} := M_l \cap \{x \in (a,b] : d_{lx} = d\}, \qquad M_{dr} := M_r \cap \{x \in [a,b) : d_{rx} = d\},$$
$$M_d := M_{dl} \cup M_{dr}.$$

Let the function $k(x)$ be defined on the set M_d by the following rule: If $d = \infty$, then

$$\forall x \in M_d \qquad k(x) = 0, \tag{3}$$

but for $d < \infty$

$$k(x) := \begin{cases} |k_{lx}|, & \text{for } x \in M_{dl} \setminus M_{dr} \\ \left(|k_{lx}|^{-1/d} + |k_{rx}|^{-1/d}\right)^{-d}, & \text{for } x \in M_{dl} \cap M_{dr} \text{ and } f'_l(x) = f'_r(x) \\ \max\{|k_{lx}|; |k_{rx}|\}, & \text{for } x \in M_{dl} \cap M_{dr} \text{ and } f'_l(x) = -f'_r(x) \\ |k_{rx}|, & \text{for } x \in M_{dr} \setminus M_{dl}. \end{cases} \tag{4}$$

2. Main Results

Theorem 1. *Let $f(x)$ be a p.-a. function on $[a, b]$. If either*

$$\sup_{x \in D_N} d(x) < \sup_{x \in M \setminus D_N} d(x)$$

or

$$\sup_{x \in D_N} d(x) = \sup_{x \in M \setminus D_N} d(x) \qquad \text{but} \qquad \inf_{x \in D_N \cap M_d} k(x) > \inf_{x \in M_d \setminus D_N} k(x),$$

then $\omega(f; \delta)$ is analytic at zero. If either

$$\sup_{x \in D_N} d(x) > \sup_{x \in M \setminus D_N} d(x)$$

or

$$\sup_{x \in D_N} d(x) = \sup_{x \in M \setminus D_N} d(x) \qquad \text{but} \qquad \inf_{x \in D_N \cap M_d} k(x) < \inf_{x \in M_d \setminus D_N} k(x),$$

then $\omega(f; \delta)$ is not analytic at zero.

Remark 1. If $D_N = \emptyset$ or $M \setminus D_N = \emptyset$, then we have $\sup_{x \in \emptyset} d(x) = -\infty$.

It is clear that $\sup_{x \in D_N} d(x) < \infty$, and the relation $d = \infty$ is equivalent with $f'(x) \equiv m$ or $f'(x) \equiv -m$ on an interval $(a_1, b_1) \subset (a, b)$. Let I be a set of points in (a, b) where $f(x)$ is not analytic. Clearly $D_N \subseteq I$ and $f(x)$ is r.-a. function if and only if $I = \emptyset$.

Proposition 1. *Let $f(x)$ be a p.-a. function on $[a, b]$. If*

$$\forall x \in M \cap I : (f'_r(x) = f'_l(x)) \quad \Rightarrow \quad \Big(\big(d_{rx} \vdots d_{lx}\big) \vee \big(d_{lx} \vdots d_{rx}\big) \Big),$$

then $\omega(f; \delta)$ is analytic at zero.

Remark 2. Perelman's theorem about analyticity of the modulus of continuity of a r.-a. function is a special case of Proposition 1.

Proposition 2. *Let $f(x)$ be a p.-a. function on $[a,b]$. Suppose that M consists of a single point $x_0 \in (a,b)$, and that $f'_r(x_0) = f'_l(x_0)$. Then $\omega(f;\delta)$ is analytic at zero if and only if either $d_{rx_0} \vdots\, d_{lx_0}$ or $d_{lx_0} \vdots\, d_{rx_0}$.*

Corollary 1. *Let $f(x)$ be a r.-a. function on $[a,b]$ and*

$$f_1(x) := |f(x)|, \qquad f_2(x) := f_+(x), \qquad f_3(x) := f_-(x),$$

where $f_+(x) = \frac{1}{2}\big(|f(x)| + f(x)\big)$, $f_-(x) = \frac{1}{2}\big(|f(x)| - f(x)\big)$. Then the functions $\omega(f_i;\delta)$, $i = 1,2,3$, are analytic at zero.

Proof. If $x \in I$, $i = 1,2,3$, then

$$f'_{il}(x)\, f'_{ir}(x)\, \big(f'_{ir}(x) + f'_{il}(x)\big) = 0.$$

This implies $D_N = \emptyset$ for each i.

Corollary 2. *Let $f_1(x)$ be a r.-a. on $[a,b]$ and let $f_2(x)$ be a r.-a. on $[b,c]$ such that $f_1(b) = f_2(b)$. Let $f(x) := f_1(x)$ on $[a,b]$ and $f(x) = f_2(x)$ on $[b,c]$. If the angle between the arcs $y = f_1(x)$ and $y = f_2(x)$ is not equal to zero, then $\omega(f;\delta)$ is analytic at zero.*

Proof. The assertion follows from the condition $f'_l(b) \neq f'_r(b)$.

Corollary 3. *Let f be a spline of order $m \le 3$ or a spline with defect $d = 1$ on $[a,b]$. Then $\omega(f;\delta)$ is an analytic function at zero.*

The proof follows from the definitions of order and defect of a spline function (see, for example [2, p. 7]).

Let

$$k := \inf_{x \in M_d} k(x), \tag{5}$$

where $k(x)$ is the function defined by (3) and (4).

The following theorem gives some information about the modulus of continuity of an arbitrary p.-a. function.

Theorem 2. *Let $f(x)$ be a p.-a. function on $[a,b]$. Then there are a natural number $s = s(f) \ge 1$ and a positive number $\varepsilon = \varepsilon(f) > 0$ for which $\omega(f;\delta)$ can be expanded on $[0,\varepsilon]$ in a power series of the variable $\big(\delta^{1/s}\big)$. The formula*

$$\omega(f;\delta) = m\delta - \frac{k}{d+1}\delta^{d+1} + O\big(\delta^{d+1+1/s}\big),$$

holds, where the constants m, d and k are defined by formulae (1), (2) and (5). If the function $\omega(f;\delta)$ is not analytic at zero, then

$$d \ge 3, \qquad s \le d-1, \qquad d + 1 + \frac{1}{s} \ge \frac{9}{2}.$$

Corollary 4. *Let $f(x)$ be a p.-a. function on $[a, b]$. Then there is $\varepsilon = \varepsilon(f) > 0$ such that on $(0, \varepsilon]$ the function $\delta^{-1}\omega(f;\delta)$ is nonincreasing and $\omega(f;\delta)$ is concave.*

Example 1. Let

$$f'(x) := \begin{cases} 1 - x^2, & \text{for } \; -1 \le x \le 0 \\ 1 - x^3, & \text{for } \; 0 \le x \le 1. \end{cases}$$

It may be proved, that on a neighborhood of zero we have

$$\omega(f;\delta) = \delta - \frac{Y^9(\delta)}{3} - \frac{Y^8(\delta)}{4},$$

where

$$Y(\delta) = \sum_{n=1}^{\infty} k_n \delta^{n/2}, \quad k_1 = 1, \quad k_2 = -\frac{1}{2}, \quad ..., \quad k_{n+2} = \frac{3}{4} \cdot \frac{9n^2 - 4}{4n^2 + 3n + 2} k_n.$$

Hence, on a neighborhood of zero

$$\omega(f;\delta) = \delta - \frac{\delta^4}{4} + \frac{2}{3}\delta^{9/2} - \frac{3}{2}\delta^5 + ...$$

Consequently, the constants in Theorem 2 and Corollary 3 are exact.

The following theorem is about analyticity of the modulus of continuity $\omega_k(f;\delta)$ of order k, for natural $k \ge 2$.

Theorem 3. *Let $f(x)$ be a r.-a. function on $[a, b]$, and let k be a natural number. Then $\omega_k(f;\delta)$ is an analytic function at the origin.*

The following lemma is used for the proof of Theorem 1.

Lemma 1. *Let $f(x)$ be continuous on $[a, b]$, and $0 < \delta < b - a$. Assume that x_1, x_2 is a pair of points, for which*

$$\omega(f;\delta) = |f(x_1) - f(x_2)|, \qquad 0 < x_1 - x_2 \le \delta, \;\; [x_1, x_2] \subset (a, b).$$

If f is differentiable at the points x_1 and x_2, then $f'(x_1) = f'(x_2)$, and if we have $f'(x_1) = f'(x_2) \ne 0$, then $x_1 - x_2 = \delta$.

We can prove an analogous lemma for the modulus $\omega_k(f;\delta)$ with $k \ge 2$.

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