Analyticity of Moduli of Continuity

O. A. DOVGOSHEY AND L. L. POTOMKINA *

For any real-analytic function $f$ on a closed interval of the real axis, Perelman [3] showed that the modulus of continuity of $f$ is an analytic function on a neighborhood of the origin. We study conditions of analyticity for a modulus of continuity of piecewise analytic functions.

1. Introduction and notation

The modulus of continuity is one of the main structural characteristics of a function, and questions associated with the study of the modulus of continuity are considered in a great number of works (for example, see monographs [1], [4–8], and the bibliography there). In particular, it was proved in Perelman’s article [3] that for any real-analytic function $f(x)$, defined on a closed interval of the real axis, the modulus of continuity $\omega(f; \delta)$ is an analytic function at zero. The subject of this report is the analysis of conditions of analyticity for the modulus of continuity of piecewise analytic functions.

As is well-known, the function $f : [a, b] \to \mathbb{R}$ is real-analytic (r.-a.) on $[a, b]$ if it is analytic at each point $x \in [a, b]$, i.e., if it can be expanded in convergent power series in a certain neighborhood of $x$. A continuous function $f : [a, b] \to \mathbb{R}$ is piecewise analytic (p.-a.) on $[a, b]$, if there is a partition $a = x_0 < x_1 < \ldots < x_{n+1} = b$ such that the restrictions $f|_{[x_i, x_{i+1}]}$ are r.-a. functions.

Let $f(x)$ be p.-a. function with a right derivative $f'_r(x)$ and a left one $f'_l(x)$. Let us denote

$$
\Theta(x) := \begin{cases}
|f'_r(a)|, & \text{for } x = a \\
\max \{|f'_r(x)|; |f'_l(x)|\}, & \text{for } x \in (a, b) \\
|f'_l(b)|, & \text{for } x = b,
\end{cases}
$$

and

$$
m := \sup_{x \in [a, b]} \Theta(x),
$$

(1)

*The research of O.A. Dovgoshey was supported by grant No. 01.07/00241 from DFFD of Ukraine.
Analyticity of Moduli of Continuity

\[ M_r := \{ x \in [a, b] : |f'_x(x)| = m \} \quad \text{and} \quad M_l := \{ x \in (a, b) : |f'_x(x)| = m \} \]

\[ M := M_l \cup M_r. \]

Since \( f(x) \) is p.-a. function on \([a, b]\), the relation \( M \neq \emptyset \) holds. For each \( t_0 \in [a, b] \) and every sufficiently small \( \varepsilon > 0 \) the function \( f'_x(x) \) is expanded on \([t_0 - \varepsilon, t_0]\) by Taylor’s formula

\[ f'_x(t_0) = f'_x(t_0) + k_l(x - t_0)^{d_l} + o(x - t_0)^{d_l}, \]

where \( d_l = d_{lt_0} \) is the multiplicity of the zero of the function \((f'_x(x) - f'_x(t_0))\) at \( t_0 \), and

\[ k_l = k_{lt_0} := \frac{1}{(d_l)!} f^{(1+d_l)}(t_0). \]

If for each \( x \in [t_0 - \varepsilon, t_0] \) we have \( f'_x(t_0) = f'_x(t_0) \), then we set \( d_{lt_0} := \infty \), \( k_{lt_0} := 0 \). Analogously, for \( t_0 \in [a, b] \), let \( d_r = d_{rt_0}, k_r = k_{rt_0} \). Then

\[ f'_r(x) = f'_r(t_0) + k_r(x - t_0)^{d_r} + o(x - t_0)^{d_r}. \]

Let the function \( d(x) \) be defined on the set \( M \) by

\[ d(x) := \begin{cases} d_{lx}, & \text{for } x \not\in M_l \setminus M_r \\ \max\{d_{lx}; d_{rx}\}, & \text{for } x \in M_l \cap M_r \\ d_{rx}, & \text{for } x \in M_r \setminus M_l. \end{cases} \]

Let \( \vdash \) be the symbol for the binary relation, defined on \( \mathbb{N} \cup \{\infty\} \) by the rule: \((a \vdash b)\) if and only if either \( a \) and \( b \) are natural numbers and \( b \) divides \( a \) or \( a = \infty \) and \( b \) is an arbitrary element of \( \mathbb{N} \cup \{\infty\} \).

Let us introduce the following notation

\[ D_N := M_l \cap M_r \cap \{ x \in (a, b) : f'_x(x) = f'_x(x), \ (d_{lx}; d_{rx}), \ (d_{lx}; d_{lx}) \}, \]

i.e., for \( x \in D_N \) no one from the “numbers” \( d_{rx}, d_{lx} \) is a “divider” of the other one and \( f'_x(x) = f'_x(x) = \pm m. \)

Assume that

\[ d := \sup_{x \in M} d(x), \tag{2} \]

and

\[ M_{dl} := M_l \cap \{ x \in (a, b) : d_{lx} = d \}, \quad M_{dr} := M_r \cap \{ x \in [a, b) : d_{rx} = d \}, \quad M_d := M_{dl} \cup M_{dr}. \]

Let the function \( k(x) \) be defined on the set \( M_d \) by the following rule: If \( d = \infty \), then

\[ \forall x \in M_d \quad k(x) = 0, \tag{3} \]
but for $d < \infty$

$$k(x) := \begin{cases} 
|k_{lx}|, & \text{for } x \in M_d \setminus M_{dr} \\
(|k_{lx}|^{-1/d} + |k_{rx}|^{-1/d})^{-d}, & \text{for } x \in M_d \cap M_{dr} \text{ and } f'_l(x) = f'_r(x) \\
\max\{|k_{lx}|; |k_{rx}|\}, & \text{for } x \in M_d \cap M_{dr} \text{ and } f'_l(x) = -f'_r(x) \\
|k_{rx}|, & \text{for } x \in M_{dr} \setminus M_d. 
\end{cases}$$

(4)

2. Main Results

**Theorem 1.** Let $f(x)$ be a p.-a. function on $[a, b]$. If either

$$\sup_{x \in D_N} d(x) < \sup_{x \in M \setminus D_N} d(x)$$

or

$$\sup_{x \in D_N} d(x) = \sup_{x \in M \setminus D_N} d(x) \quad \text{but} \quad \inf_{x \in D_N \cap M_d} k(x) > \inf_{x \in M_d \setminus D_N} k(x),$$

then $\omega(f; \delta)$ is analytic at zero. If either

$$\sup_{x \in D_N} d(x) > \sup_{x \in M \setminus D_N} d(x)$$

or

$$\sup_{x \in D_N} d(x) = \sup_{x \in M \setminus D_N} d(x) \quad \text{but} \quad \inf_{x \in D_N \cap M_d} k(x) < \inf_{x \in M_d \setminus D_N} k(x),$$

then $\omega(f; \delta)$ is not analytic at zero.

**Remark 1.** If $D_N = \emptyset$ or $M \setminus D_N = \emptyset$, then we have $\sup_{x \in \emptyset} d(x) = -\infty$.

It is clear that $\sup_{x \in D_N} d(x) < \infty$, and the relation $d = \infty$ is equivalent with $f'(x) \equiv m$ or $f'(x) \equiv -m$ on an interval $(a_1, b_1) \subset (a, b)$. Let $I$ be a set of points in $(a, b)$ where $f(x)$ is not analytic. Clearly $D_N \subseteq I$ and $f(x)$ is r.-a. function if and only if $I = \emptyset$.

**Proposition 1.** Let $f(x)$ be a p.-a. function on $[a, b]$. If

$$\forall x \in M \cap I: (f'_l(x) = f'_r(x)) \Rightarrow \left((d_{rx} : d_{lx}) \lor (d_{lx} : d_{rx})\right),$$

then $\omega(f; \delta)$ is analytic at zero.
Remark 2. Perelman’s theorem about analyticity of the modulus of continuity of a r.-a. function is a special case of Proposition 1.

Proposition 2. Let \( f(x) \) be a p.-a. function on \([a, b]\). Suppose that \( M \) consists of a single point \( x_0 \in (a, b) \), and that \( f'(x_0) = f''(x_0) \). Then \( \omega(f; \delta) \) is analytic at zero if and only if either \( d_{r\omega} \); \( d_{l\omega} \) or \( d_{r\omega} \); \( d_{l\omega} \).

Corollary 1. Let \( f(x) \) be a r.-a. function on \([a, b]\) and

\[
n_1(x) := |f(x)|, \quad n_2(x) := f_+(x), \quad n_3(x) := f_-(x),
\]

where \( f_+(x) = \frac{1}{2}(|f(x)| + f(x)) \), \( f_-(x) = \frac{1}{2}(|f(x)| - f(x)) \). Then the functions \( \omega(f; \delta) \), \( i = 1, 2, 3 \), are analytic at zero.

Proof. If \( x \in I \), \( i = 1, 2, 3 \), then

\[
n_{i\prime}(x) n_{i\prime}(x) (n_{i\prime}(x) + n_{i\prime}(x)) = 0.
\]

This implies \( D_N = 0 \) for each \( i \).

Corollary 2. Let \( f_1(x) \) be a r.-a. on \([a, b]\) and let \( f_2(x) \) be a r.-a. on \([b, c]\) such that \( f_1(b) = f_2(b) \). Let \( f(x) := f_1(x) \) on \([a, b]\) and \( f(x) = f_2(x) \) on \([b, c]\).

If the angle between the arcs \( y = f_1(x) \) and \( y = f_2(x) \) is not equal to zero, then \( \omega(f; \delta) \) is analytic at zero.

Proof. The assertion follows from the condition \( f_{\prime\prime}(b) \neq f_{\prime\prime}(b) \).

Corollary 3. Let \( f \) be a spline of order \( m \leq 3 \) or a spline with defect \( d = 1 \) on \([a, b]\). Then \( \omega(f; \delta) \) is an analytic function at zero.

The proof follows from the definitions of order and defect of a spline function (see, for example [2, p. 7]).

Let

\[
k := \inf_{x \in M_d} k(x), \quad (5)
\]

where \( k(x) \) is the function defined by (3) and (4).

The following theorem gives some information about the modulus of continuity of an arbitrary p.-a. function.

Theorem 2. Let \( f(x) \) be a p.-a. function on \([a, b]\). Then there are a natural number \( s = s(f) \geq 1 \) and a positive number \( \varepsilon = \varepsilon(f) > 0 \) for which \( \omega(f; \delta) \) can be expanded on \([0, \varepsilon]\) in a power series of the variable \( \delta^{1/s} \). The formula

\[
\omega(f; \delta) = m\delta - \frac{k}{d+1}\delta^{d+1} + O(\delta^{d+1+1/s}),
\]

holds, where the constants \( m, d \) and \( k \) are defined by formulae (1), (2) and (5). If the function \( \omega(f; \delta) \) is not analytic at zero, then

\[
d \geq 3, \quad s \leq d - 1, \quad d + 1 + \frac{1}{s} \geq \frac{9}{2}.
\]
Corollary 4. Let \( f(x) \) be a p.-a. function on \([a, b]\). Then there is \( \varepsilon = \varepsilon(f) > 0 \) such that on \((0, \varepsilon] \) the function \( \delta^{-1} \omega(f; \delta) \) is nonincreasing and \( \omega(f; \delta) \) is concave.

Example 1. Let

\[
 f'(x) := \begin{cases} 
 1 - x^2, & \text{for } -1 \leq x \leq 0 \\ 
 1 - x^3, & \text{for } 0 \leq x \leq 1.
\end{cases}
\]

It may be proved, that on a neighborhood of zero we have

\[
 \omega(f; \delta) = \delta - \frac{Y^9(\delta)}{3} - \frac{Y^8(\delta)}{4},
\]

where

\[
 Y(\delta) = \sum_{n=1}^{\infty} k_n \delta^{n/2}, \quad k_1 = 1, \quad k_2 = -\frac{1}{2}, \quad ..., \quad k_{n+2} = \frac{9n^2 - 4}{4n^2 + 3n + 2} k_n.
\]

Hence, on a neighborhood of zero

\[
 \omega(f; \delta) = \delta - \frac{\delta^4}{4} + \frac{2}{3} \delta^{3/2} - \frac{3}{2} \delta^5 + ...
\]

Consequently, the constants in Theorem 2 and Corollary 3 are exact.

The following theorem is about analyticity of the modulus of continuity \( \omega_k(f; \delta) \) of order \( k \), for natural \( k \geq 2 \).

**Theorem 3.** Let \( f(x) \) be a r.-a. function on \([a, b]\), and let \( k \) be a natural number. Then \( \omega_k(f; \delta) \) is an analytic function at the origin.

The following lemma is used for the proof of Theorem 1.

**Lemma 1.** Let \( f(x) \) be continuous on \([a, b]\), and \( 0 < \delta < b - a \). Assume that \( x_1, x_2 \) is a pair of points, for which

\[
 \omega(f; \delta) = |f(x_1) - f(x_2)|, \quad 0 < x_1 - x_2 \leq \delta, \quad [x_1, x_2] \subset (a, b).
\]

If \( f \) is differentiable at the points \( x_1 \) and \( x_2 \), then \( f'(x_1) = f'(x_2) \), and if we have \( f'(x_1) = f'(x_2) \neq 0 \), then \( x_1 - x_2 = \delta \).

We can prove an analogous lemma for the modulus \( \omega_k(f; \delta) \) with \( k \geq 2 \).
References


O. A. Dovgoshey
IAMM of NAS of Ukraine
Roze Luxemburg Street 74
83114, Donetsk
UKRAINE
E-mail: dovgoshey@iamm.ac.donetsk.ua

L. L. Potomkina
IAMM of NAS of Ukraine
Roze Luxemburg Street 74
83114, Donetsk
UKRAINE