CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 251-258.

On the Disribution of the Zeros of an Entire Function of Exponential Type

D. P. Dryanov

1. Introduction

An entire function f is said to be of exponential type $\sigma > 0$ if for every $\varepsilon > 0$, there exists a constant $c(\varepsilon)$ such that

$$|f(z)| \leq c(\varepsilon)\, e^{(\sigma+\varepsilon)\,|z|} \qquad (z \in \mathbb{C})\,.$$

In Theorem 2 of the present article we generalize a theorem of R. J. Duffin and A. C. Schaeffer about the distribution of the zeros of a real entire function of exponential type. By using this generalization we extend a result of L. Hörmander on the local behavior of an entire function of exponential type. This is contained in Theorem 3. In the process of our research we obtain new interpolation formulae for exact recovery of an entire function of exponential type. The result is presented in Theorem 1.

2. Interpolation Formulae of Shannon - Kotelnikov Type

For the proof of our first theorem that extends an interpolation result due to Bernstein [1, (17), p. 568], [2, (2), p. 103] we need the following auxiliary lemma.

Lemma 1. *Let f be a sufficiently smooth function and suppose that we are given a finite set $S_1 := \{m_1\pi/\sigma, \dots, m_{r_1}\pi/\sigma\}$ of zeros of $\sin\sigma z$ and another finite point set $S_2 := \{z_1, \dots, z_{r_2}\}$, not containing any of the zeros of $\sin\sigma z$. Let $\{\lambda_1, \dots, \lambda_{r_1}\}$ and $\{\beta_1, \dots, \beta_{r_2}\}$ be two sets of positive integers such that $\sum_{\mu=1}^{r_1}\lambda_\mu + \sum_{\nu=1}^{r_2}\beta_\nu = m$. Then there exists a unique algebraic polynomial q_{m-1} of degree $\leq m-1$ which solves the following interpolation problem:*

$$\big(\sin\sigma z\; q_{m-1}(z)\big)^{(j)}_{z=m_\mu\pi/\sigma} = \left(f(z) - \sum_{\kappa=1}^{r_1} f\left(\frac{m_\kappa\pi}{\sigma}\right)\frac{\sin(\sigma z - m_\kappa\pi)}{\sigma z - m_\kappa\pi}\right)^{(j)}_{z=\frac{m_\mu\pi}{\sigma}}$$

for $j = 1, \dots, \lambda_\mu \quad (\mu = 1, \dots, r_1)$ *and*

$$\big(\sin\sigma z\; q_{m-1}(z)\big)^{(j)}_{z=z_\nu} = \left(f(z) - \sum_{\kappa=1}^{r_1} f\left(\frac{m_\kappa \pi}{\sigma}\right) \frac{\sin(\sigma z - m_\kappa \pi)}{\sigma z - m_\kappa \pi}\right)^{(j)}_{z=z_\nu}$$

for $j = 0, \dots, \beta_\nu - 1 \quad (\nu = 1, \dots, r_2)$.

Remark 1. The coefficients of q_{m-1} are uniquely determined by the interpolation conditions $f^{(j)}(m_\mu \pi/\sigma)$ for $j = 0, 1, \dots, \lambda_\mu \quad (\mu = 1, \dots, r_1)$, and $f^{(j)}(z_\nu)$ for $j = 0, 1, \dots, \beta_\nu - 1 \quad (\nu = 1, \dots, r_2)$.

We denote by E_σ $(\sigma > 0)$ the complex vector space of all entire functions of exponential type σ.

Theorem 1. *Let* $f \in E_\sigma$ *and let*

$$f(x) = o(x^m), \;\; |x| \to \infty \qquad (x \in \mathbb{R}),$$

where $m \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$. *Let there exist* $p_0 > 0$ *such that*

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|f(k\pi/\sigma)|^{p_0}}{|k|^{p_0\, m}} < \infty\,. \tag{1}$$

Let q_{m-1} *be the unique interpolation solution from Lemma 1 and let us define, by using the notations of Lemma 1,*

$$\Omega_m(z) := \prod_{\mu=1}^{r_1} (z - m_\mu \pi/\sigma)^{\lambda_\mu} \prod_{\nu=1}^{r_2} (z - z_\nu)^{\beta_\nu}\,,$$

and let $\omega_1 := \{m_1, \dots, m_{r_1}\}$. *Then, the following Shannon - Kotelnikov type interpolation formula holds*

$$f(z) = \sum_{\mu=1}^{r_1} f\left(\frac{m_\mu \pi}{\sigma}\right) \frac{\sin(\sigma z - m_\mu\, \pi)}{\sigma\, z - m_\mu\, \pi} + \Omega_m(z) \sin\sigma z \sum_{k \in \mathbb{Z}\setminus \omega_1} \frac{(-1)^k\, f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)\,(\sigma z - k\pi)} + \sin\sigma\, z\, q_{m-1}(z)\,.$$

Example 1. The case $\omega_1 = \{0\} \quad (r_1 = 1,\ m_1 = 0,\ \lambda_1 = m)$ and $S_2 = \emptyset$ has been considered by Bernstein [2, (2), p. 103] under the following, a bit more restrictive than (1) condition

$$f\,(k\pi/\sigma) = O\,(|k|^\alpha)\,, \;\; |k| \to \infty \qquad (\,k \in \mathbb{Z},\ \alpha < m\,)\,. \tag{2}$$

In this particular case of Theorem 1, $\Omega_m(z) = z^m$ and the corresponding interpolation formula reads as follows

$$f(z) = f(0)\frac{\sin\sigma z}{\sigma z} + (\sigma z)^m \sin\sigma z \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^k f(k\pi/\sigma)}{(k\pi)^m(\sigma z - k\pi)} + \sin\,\sigma z\, q_{m-1}(z)\,.$$

Concerning the condition (2), we take as an example the sampling sequence $f\big(2^{s^2}\pi/\sigma\big) := 2^{m\,s^2-s}\quad (s \in \mathbb{N})$, and $f(k\pi/\sigma) := 0\quad \big(k \neq 2^{s^2},\ k \in \mathbb{Z}\big)$ which does not satisfy the condition (2) but does satisfy the condition (1).

Remark 2. In the particular case $m = 1$ of Example 1, the corresponding interpolation formula has been known since 1931 [6], [3, Theorem 4, p. 47 and the footnotes on p. 47] but under more stringent conditions.

Example 2. Let $z_1, z_2, \dots, z_m$ be interpolation nodes such that $\sin\sigma z_l \neq 0$, $l = 1,\dots,m$. In this particular case of Theorem 1, we have $S_1 = \emptyset$, $\omega_1 = \emptyset$, and $S_2 = \{z_1,\dots,z_m\}$ $(r_2 = m,\ \beta_1 = \cdots = \beta_m = 1)$. Polynomial Lagrange's interpolation $q_{m-1}(z_l) = f(z_l)/\sin\sigma z_l$ $(l = 1,\dots,m)$ gives a unique polynomial q_{m-1} (see Lemma 1). Thus, $\Omega_m(z) = \prod_{\nu=1}^{m}(z - z_\nu)$ and the corresponding interpolation representation (see Theorem 1) has the form

$$f(z) = \Omega_m(z)\,\sin\sigma z \sum_{k\in\mathbb{Z}} \frac{(-1)^k f(k\pi/\sigma)}{\Omega_m(k\pi/\sigma)\,(\sigma z - k\pi)} + \sin\sigma z\, q_{m-1}(z)\,.$$

The sharpness of our result

Estimating the result of Theorem 1 we may ask the following question: Can we recover exactly a function from E_σ, which is $O(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$, and satisfying the condition (1) on the bases of samples at $\{k\pi/\sigma\}$ $(k \in \mathbb{Z})$ and an additional m-bits Hermite type of information as it is in Theorem 1? The simple example $\Omega_m(z)\,\sin\sigma\,z$ shows that this is not possible. In other words, the asymptotic $o(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$ is *the best possible* in E_σ for an exact recovery based on the interpolation values at $\{k\pi/\sigma\}$ $(k \in \mathbb{Z})$ and an additional m-bits Hermite type information, including functional values and derivatives.

3. Generalization of a Theorem of R. J. Duffin and A. C. Schaeffer

Let $E_{\sigma,\mathbb{R}}$ $(\sigma > 0)$ denote the real vector space of all entire functions from E_σ which are real on the real line. Duffin and Schaeffer [4, Theorem 1, p. 236] established the following result on the distribution of the zeros of a real entire function of exponential type.

Theorem $\mathcal{A}$. *Let $f \in E_{\sigma,\mathbb{R}}$ such that $|f(x)| \le 1$ $(x \in \mathbb{R})$, and let us denote $c_\sigma(z) := \cos \sigma z$. Then $c_\sigma(z) - f(z)$ has only real zeros, or vanishes identically. Moreover, all the zeros are simple, except perhaps at points on the real axis, where $f(x) = \pm 1$.*

Definition 1. *For a given vector*

$$\mathbf{y} := \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\} \qquad (y_k \ge 0,\ k \in \mathbb{Z})$$

of non-negative components and for a fixed $m \in \mathbb{N}_0$, we define the following convex functional set

$$\mathbf{DS}_{\sigma,m,\mathbf{y}} := \{f(z) : f \in E_{\sigma,\mathbb{R}}, f(x) = o(x^m),\ |x| \to \infty\ (x \in \mathbb{R}), \\ \text{and}\ \ |f(k\pi/\sigma)| \le y_k\ (k \in \mathbb{Z})\}.$$

Now we introduce the important for our next considerations notion for *a Chebyshev (barrier) function in the class* $\mathbf{DS}_{\sigma,m,\mathbf{y}}$.

Definition 2. *A Chebyshev function in the class $\mathbf{DS}_{\sigma,m,\mathbf{y}}$ is each function $C_\sigma(z) \in \mathbf{DS}_{\sigma,m,\mathbf{y}}$ which satisfies the following alternating interpolation conditions*

$$C_\sigma(k\pi/\sigma) = \epsilon\,(-1)^k\, y_k \qquad (k \in \mathbb{Z}),$$

where $\epsilon = \pm 1$ and $\mathbf{y} := \{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$ $(y_k \ge 0,\ k \in \mathbb{Z})$ is a vector of non-negative components.

Remark 3. Let, for some $p > 0$, we have $\sum_{k\in\mathbb{Z}\setminus\{0\}} (y_k/|k|^m)^p < \infty$ or more restrictively, $y_k = O(|k|^\alpha)$ $(k \in \mathbb{Z},\ \alpha < m)$. Then *Chebyshev functions* $C_\sigma(z)$ can be constructed by using the interpolation formula, given in Theorem 1 (see also Examples 1 and 2). So, explicit constructions of *Chebyshev barrier functions* are possible on the basis of different (m bits) Hermite type of interpolation information. For a given $\mathbf{DS}_{\sigma,m,\mathbf{y}}$, the class of all *Chebyshev functions* is uniquely determined within m Hermite type interpolation conditions.

Theorem 2. *Let $f \in \mathbf{DS}_{\sigma,m,\mathbf{y}}$ and let $C_\sigma(z) \in \mathbf{DS}_{\sigma,m,\mathbf{y}}$ be a Chebyshev function in $\mathbf{DS}_{\sigma,m,\mathbf{y}}$, i.e., $C_\sigma(k\pi/\sigma) = (-1)^k y_k$ $(k \in \mathbb{Z})$. Then, the function $C_\sigma(z) - f(z)$ vanishes identically or else:*

A. *Let m be odd. Then, besides at most $m-1$ (i.e., even number, counting multiplicities) real or complex conjugate zeros, the function $C_\sigma(z) - f(z)$ has only real zeros with the following distribution properties:*

a) *The function $C_\sigma(z) - f(z)$ has at least one and at most three zero in each interval $[k\pi/\sigma, (k+1)\pi/\sigma]$, where $k \in \mathbb{Z}$. In case of three zeros in $[k\pi/\sigma, (k+1)\pi/\sigma]$, the points $k\pi/\sigma$ and $(k+1)\pi/\sigma$ are zeros of $C_\sigma(z) - f(z)$.*

b) *There can be at most one zero of $C_\sigma(z) - f(z)$ in $(k\pi/\sigma, (k+1)\pi/\sigma)$, $k \in \mathbb{Z}$, and it can be only a zero with a multiplicity 1 (a simple zero).*

c) *A zero at any point* $k\pi/\sigma$, $k \in \mathbb{Z}$, *may be simple or double but not of higher multiplicity. If* $C_\sigma(z) - f(z)$ *has a double zero at some point* $k\pi/\sigma$, $k \in \mathbb{Z}$, *then it cannot have any zeros in*

$$((k-1)\pi/\sigma\,,\, k\pi/\sigma) \cup (k\pi/\sigma, (k+1)\pi/\sigma)\ .$$

B. *Let m be even. Then, besides at most m, even number counting multiplicities, real or complex conjugate zeros, the distribution of the zeros of* $C_\sigma(z) - f(z)$ *is as in* A.

Corollary 1. *Taking into account that the function* $f(z) \equiv 0$ *belongs to* $\mathbf{DS}_{\sigma,m,\mathbf{y}}$ *for an arbitrarily chosen parameters* $\sigma > 0$, $m \in \mathbb{N}_0$, *and* $\mathbf{y} = \{\dots, y_{-1}, y_0, y_1, \dots\}$ $(y_k \geq 0,\ k \in \mathbb{Z})$, *each Chebyshev function* C_σ *from* $\mathbf{DS}_{\sigma,m,\mathbf{y}}$ *possesses the same as* $C_\sigma - f$, $f \in \mathbf{DS}_{\sigma,m,\mathbf{y}}$ *zero distribution which is described by Theorem 2.*

Theorem $\mathcal{A}$ can be derived from the next corollary of Theorem 2 .

Corollary 2. *Let* $\mathbf{y}$ *be as in Theorem 2 and let, in the special case* $m = 1$, *the class* $\mathbf{DS}_{\sigma,1,\mathbf{y}}$ *be defined as in Definition 1. Let* $C_\sigma(z) \in \mathbf{DS}_{\sigma,1,\mathbf{y}}$ *be a Chebyshev function, i.e.,* $C_\sigma(k\pi/\sigma) = (-1)^k y_k \quad (k \in \mathbb{Z})$. *Then, for each* $f \in \mathbf{DS}_{\sigma,1,\mathbf{y}}$, *the function* $C_\sigma(z) - f(z)$ *vanishes identically or has only real zeros with a zero distribution, described by Theorem 2.*

Remark 4. Theorem $\mathcal{A}$ is the particular case $m := 1$, $y_k := 1$ $(k \in \mathbb{Z})$ and $C_\sigma(z) := \cos\sigma z$ of Corollary 2.

Remark 5. Let m be even and let $f \in E_{\sigma,\mathbb{R}}$. Then, assuming more generally that $f(x) = O(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$, or assuming more generally that, $f(x) = o(x^{m+1})$, $|x| \to \infty$ $(x \in \mathbb{R})$, by Theorem 2 A, we may have at most m additional zeros.

Example 3. Let m be odd. Then, the Chebyshev function $C_{s,\sigma}(z) := z^{2s}\cos\sigma z$ $(s = 0, 1, \dots, (m-1)/2)$ has an even number, possibly from 0 to $m-1$, additional zeros, counting multiplicities. *The Chebyshev barrier function* $C_{0,\sigma}$ has been used in Theorem $\mathcal{A}$.

Example 4. Let m be odd. Then,

$$C_\sigma(z) := (\sigma z)^m \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^k}{(k\pi)^m} \cdot \frac{\sin(\sigma z - k\pi)}{\sigma z - k\pi}$$

is another example of *a Chebyshev function* which is $o(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$. The Chebyshev function C_σ has a zero at the point 0 with multiplicity $m+1$, and from here, no zeros in $(-\pi/\sigma\,,\,0) \cup (0\,,\,\pi/\sigma)$. In addition, for $\varepsilon > 0$, $C_{\varepsilon,\sigma}(z) := \varepsilon \sin\sigma z/\sigma z + C_{m,\sigma}(z)$ is an example of *a Chebyshev function* that is $o(x^m)$ $|x| \to \infty$ $(x \in \mathbb{R})$, takes values $(-1)^k$ at $k\pi/\sigma$ $(k \in \mathbb{Z} \setminus \{0\})$, and a value ε for $z = 0$. $C_{\varepsilon,\sigma}$ has simple zeros in $(-\pi/\sigma, 0)$ and $(0, \pi/\sigma)$ and $m-1$ additional zeros, *close* to the point $z = 0$ for ε sufficiently small.

Example 5. Let m be even. Let $C_\sigma(z) := z^{2k}\, h(z)$ $(k = 0, \dots, m/2)$, where $h(z) \in E_{\sigma,\mathbb{R}}$ is uniquely determined by the interpolation conditions $h(0) = 1$, and $h(k\pi/\sigma) = (-1)^k/|k|$ $(k \in \mathbb{Z} \setminus \{0\})$ as a function which is $o(1), |x| \to \infty$ $(x \in \mathbb{R})$ (see Theorem 1). Then, the Chebyshev function $C_\sigma(x)$ is $o(x^{2k})$, $|x| \to \infty$ $(x \in \mathbb{R})$ and it has a zero with multiplicity $2k$ at the point zero $(k \in \{0, 1, \dots, m/2\})$. C_σ has one simple zero in $(-\pi/\sigma, 0)$, one simple zero in $(0, \pi/\sigma)$ and no other zeros in $(-\pi/\sigma, 0) \cup (0, \pi/\sigma)$.

The sharpness of Theorem 2

Estimating the sharpness of Theorem 2 we may ask the following question. Can we claim the same zero distribution in the case of $O(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$ functions (m odd)? We shall answer this question by two more examples.

Example 6. Let m be odd. Then, the real entire function

$$C_\sigma(z) := (z - \pi/2\sigma)^m \, \sin \sigma z$$

satisfies the condition of Theorem 2 with $y_k := 0$, $k \in \mathbb{Z}$, except that $C_\sigma(x) = O(x^m)$ but not $o(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$. Evidently, the function C_σ has $m > m-1$ additional zeros.

Example 7. Let $m \geq 1$ be odd. Let $C_\sigma(z) := z^{m-1} \, \cos z$ and let

$$f_\kappa(z) := z^{m-1} \left(\frac{\sin \sigma z}{\sigma z} - \kappa\, z \, \sin \sigma z \right) .$$

Then, the function C_σ is a Chebyshev function in $\mathbf{DS}_{\sigma,m,\mathbf{y}}$, where $y_k := (-1)^k \, (k\pi/\sigma)^{m-1}$ $(k \in \mathbb{Z}\setminus\{0\})$ and $y_0 = 0$ $(m \geq 3)$, $y_0 = 1$ $(m = 1)$. The function f_κ satisfies the conditions of Theorem 2, except that $f_\kappa(x) = O(x^m)$ but not $o(x^m)$, $|x| \to \infty$ $(x \in \mathbb{R})$. The real entire function $C_\sigma(z) - f_\kappa(z)$ $(\kappa > \sigma/3)$ has a zero with multiplicity $m+1$ at $z = 0$ and two additional simple zeros in each of the intervals $(-\pi/\sigma, 0)$ and $(0, \pi/\sigma)$, so the conclusions of Theorem 2 are violated.

4. Extension of a Result of L. Hörmander

The following result is due to Hörmander [5, Corollary, p. 26].

Theorem $\mathcal{B}$ (L. Hörmander). *Let $f \in E_{\sigma,\mathbb{R}}$ satisfy $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Furthermore, let $f(0) = M$ (and implicitly $f'(0) = 0$). Then*

$$f(x) \geq M \, \cos \sigma x \quad (-\pi/\sigma \leq x \leq \pi/\sigma) .$$

The estimate is best possible as the example $f(z) := M \, \cos \sigma z$ shows.

We prove the following theorem.

Theorem 3. *Let m be odd. Let $f \in \mathbf{DS}_{\sigma,m,\mathbf{y}}$ and let $C_\sigma(z)$ be a Chebyshev function from $\mathbf{DS}_{\sigma,m,\mathbf{y}}$. Then we have the following.*

A) *Let $f^{(j)}(0) = C_\sigma^{(j)}(0),\ j = 0, 1, \dots, m$. Then*

$$f(x) \geq C_\sigma(x) \qquad (-\pi/\sigma \leq x \leq \pi/\sigma)$$

with an equality case for some $x \in \{(-\pi/\sigma\,,\,0) \cup (0\,,\,\pi/\sigma)\}$ if and only if $f(z) \equiv C_\sigma(z)$.

B) *Let $f(0) = C_\sigma(0),\ f'(0) = C'_\sigma(0)$, and $f^{(j)}(x_s) = C_\sigma^{(j)}(x_s)$ $(j = 0, 1)$ at the real points $x_s,\ s = 1, \dots , (m-1)/2$ $(x_s \neq k\pi/\sigma\,,\ \ k \in \mathbb{Z})$. Then*

$$f(x) \geq C_\sigma(x) \qquad (-\pi/\sigma \leq x \leq \pi/\sigma)$$

with an equality case for some

$$x \in \{\{(-\pi/\sigma\,,\,0) \cup (0\,,\,\pi/\sigma)\} \setminus \{x_s,\ s = 1, \dots , (m-1)/2\}\}$$

if and only if $f(z) \equiv C_\sigma(z)$.

C) *Let $f(0) = C_\sigma(0)$ and $f'(0) = C'_\sigma(0)$. Let $f(z_s) = C_\sigma(z_s)$ at the complex points z_s $(\Im\{z_s\} \neq 0),\ s = 1, \dots , (m-1)/2$. Then*

$$f(x) \geq C_\sigma(x) \qquad (-\pi/\sigma\ \leq\ x\ \leq\ \pi/\sigma)$$

with an equality case for some $x \in \{(-\pi/\sigma\,,\,0) \cup (0\,,\,\pi/\sigma)\}$ if and only if $f(z) \equiv C_\sigma(z)$.

Remark 6. The particular case $m = 1$ and $C_\sigma(z) := \cos \sigma z$ of Theorem 3 A) implies the result, given in Theorem $\mathcal{B}$.

Remark 7. The conclusion of Theorem 3 will hold if we have $m + 1$ Hermite interpolation conditions of type A), B), and C), simultaneously. For example, let $f^{(j)}(0) = C_\sigma^{(j)}(0),\ j = 0, 1, \dots, 2l_1 + 1$ ($l_1 \geq 0$, integer) and suppose that at l_2 real points $x_s,\ s = 1, \dots , l_2,\ x_s \neq k\pi/\sigma,\ k \in \mathbb{Z}$ ($l_2 \geq 0$, integer) we have $f^{(j)}(x_s) = C_\sigma^{(j)}(x_s),\ j = 0, 1$. (We can also use Hermite interpolation conditions with *different, even number multiplicities* at each *real* interpolation node.) Let also at l_3 ($l_3 \geq 0$, integer) complex points z_s $(\Im\{z_s\} \neq 0),\ s = 1, \dots , l_3,\ f(z_s) = C_\sigma(z_s)$. (Here, we can also use Hermite interpolation conditions with *different multiplicities* at each *complex* interpolation node.) If $l_1 + l_2 + l_3 = (m-1)/2$, then the conclusion of Theorem 3 holds also.

This paper is based on some joint work done in collaboration with Dr. M. A. Qazi and Professor Q. I. Rahman.

References

[1] S. Bernstein, Sur la meilleure approximation des fonctions sur l'axe réel par des fonctions entières de degré fini. III, *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **52** (1946) 563–566.

[2] S. Bernstein, Sur la meilleure approximation des fonctions sur l'axe réel par des fonctions entières de degré fini. IV, *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **54** (1946) 103–108.

[3] M. Cartwright, On certain integral functions of order one, *Quart. J. Math. (Oxford)* **7** (1936) 46–65.

[4] R. J. Duffin and A. C. Schaeffer, Some properties of functions of exponential type, *Bull. Amer. Math. Soc.* **44** (1938) 236–240.

[5] L. Hörmander, Some inequalities for functions of exponential type, *Math. Scand.* **3** (1955) 21–27.

[6] L. Tschakaloff, Zweite Lösung der Aufgabe 105, *Jahresber. Deutsch. Math.-Verein.* **40** (1931), 11–13.

D. P. Dryanov

Départment de Mathématiques et de Statistique
Université de Montréal
Montréal H3C 3J7
CANADA
E-mail: dryanovd@dms.umontreal.ca