CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 259-266.

Walsh-Fourier Series with Respect to Weights

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It is known that the trigonometric Fourier series converges in L^p-norm if $1 < p < \infty$. This claim is false for $p = 1$ and $p = \infty$. Similar statements are true for the Walsh-system. The domain of parameter p for the L^p-norm convergence in the case of orthogonal polynomials depends on the weight function. For example, for Legendre polynomials, the L^p-norm convergence holds only for $4/3 < p < 4$ (see [5]).

Bl. Sendov introduced in 1999 a generalization of the Walsh-system, the so-called "Walsh-similar" functions (see [10]). These are special cases of the Walsh system with respect the weight ρ ($W^\rho = \Psi^\rho = (\psi_n^\rho, n \in \mathbb{N})$), which was introduced by Schipp (see [7], [8]).

We show that if the weight function ρ belongs to the class $Lip\,(\alpha, W)$ $(0 < \alpha \leq 1)$, and $\rho \geq \rho_0 > 0$, then the W^ρ-Walsh-Fourier-series is convergent in L^p_ρ-norm if $1 < p < \infty$. We study the behavior of such W^ρ-systems too, whose weight function has not positive lower bound. For example, for $\rho(x) = x^\alpha$ $(\alpha > -1)$ the W^ρ-system is not uniformly bounded. We give an exact estimation for the norm of the functions ψ_n^ρ, and show for $p_0^\alpha = \frac{2(\alpha+1)\ln 2}{\ln(2^{\alpha+1}-1)}$ that if $1 \leq p \leq p_0$ or $p_0' \leq p \leq \infty$, then there exists a function $f \in L^p_\rho$ with divergent W^ρ-Fourier series in L^p_ρ-norm.

1. Introduction

In this paper we fix a weight function $\rho \in L^1([0,1))$, $\rho \geq 0$, with $\int_0^1 \rho(x)dx = 1$ and investigate dyadic martingales, with respect to the probability measure spaces $(\mathbb{I}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the collection of Lebesgue-measurable sets in $\mathbb{I} := [0,1)$ and $\mu(A) = \int_A \rho(x)\,dx$ $(A \in \mathcal{A})$. Let us denote by

$$\mathcal{I}_n := \{(k2^{-n}, (k+1)2^{-n}) : k = 0, 1, \dots, 2^n - 1\} \qquad (n \in \mathbb{N})$$

the set of dyadic intervals with length 2^{-n}, and let $\mathcal{A}_n$ be the σ-algebra, generated by $\mathcal{I}_n$. The set of real, $\mathcal{A}_n$-measurable functions, defined on $\mathbb{I}$ is denoted by $L(\mathcal{A}_n)$. Obviously

$$L(\mathcal{A}_n) = \operatorname{span}\{\chi_I : I \in \mathcal{I}n\} \qquad (n \in \mathbb{N}),$$

*Supported by the Grant OTKA/T032719.

where χ_I is the characteristic function of the set I. The conditional expectation E_n^ρ with respect to $\mathcal{A}_n$ is of the form

$$(E_n^\rho f)(x) = \frac{\int_I f(s)\rho(s)\,ds}{\int_I \rho(s)\,ds} \qquad (x \in I \in \mathcal{A}_n,\ f \in L_\rho^1(\mathbb{I})).$$

In the case $\rho = 1$ we shall use the notation $E_n := E_n^1$ $(n \in \mathbb{N})$. Obviously

$$E_n^\rho f = \frac{E_n(\rho f)}{E_n \rho} \qquad (f \in L_\rho^1(\mathbb{I}),\ n \in \mathbb{N}). \tag{1}$$

The sequence $\Phi = (\phi_n, n \in \mathbb{N})$ is called a normalized dyadic martingale difference sequence in the probability space $(\mathbb{I}, \mathcal{A}, \mu)$, if

$$i)\ \phi_n \in L(\mathcal{A}_{n+1}), \quad ii)\ E_n^\rho(\phi_n) = 0, \quad iii)\ E_n^\rho(|\phi_n|^2) = 1 \qquad (n \in \mathbb{N}). \tag{2}$$

It is clear that in the case $\rho = 1$ the Rademacher system $(R = (r_n, n \in \mathbb{N}))$ satisfies (2). In the general case, taking the standardization of r_n in the space $(\mathbb{I}, \mathcal{A}, \mu)$, we get the system

$$\phi_n := \frac{r_n - E_n^\rho r_n}{\sqrt{E_n^\rho(|r_n - E_n^\rho r_n|^2)}} = \frac{r_n - b_n}{\sqrt{1 - b_n^2}}, \quad b_n := E_n^\rho r_n = \frac{E_n(\rho r_n)}{E_n(\rho)} \qquad (n \in \mathbb{N}) \tag{3}$$

satisfying (2). (See [8].)

The product system of the system $\Phi^\rho = (\phi_n, n \in \mathbb{N})$ (see [1], [6], [9]) is defined by

$$\psi_m := \prod_{k=0}^{\infty} \phi_k^{m_k} \qquad (m \in \mathbb{N}), \tag{4}$$

where the numbers $m_k \in \{0, 1\}$ are the digits in the dyadic representation $m = \sum_{k=0}^{\infty} m_k 2^k$. Especially the product system of the Rademacher system is the Walsh system in Paley's enumeration, which is orthonormed in $L^2(\mathbb{I})$ (see [9]). It is known that the product system $\Psi^\rho = (\psi_n, n \in \mathbb{N})$ is orthonormal in L_ρ^2 (see [8]). The weight function ρ can be written in the form

$$\rho = \prod_{j=0}^{\infty}(1 + b_j r_j), \qquad b_j = \frac{E_j(\rho r_j)}{E_j(\rho)} \qquad (j \in \mathbb{N}) \tag{5}$$

(see [8]). By (3), the reciprocal of the function ϕ_n is of the form

$$\phi_n^{-1} := 1/\phi_n = \frac{r_n + b_n}{\sqrt{1 - b_n^2}} \qquad (n \in \mathbb{N}).$$

The functions

$$\rho_0^- := 1, \qquad \rho_n^- := \prod_{j=0}^{n-1}(1 - b_j r_j) \qquad (n \in \mathbb{N}^*)$$

form a dyadic martingale with respect to the Lebesgue-space, and the martingale $(\rho_n^-, n \in \mathbb{N})$ is L^1-bounded. This implies that the infinite product $\prod_{j=0}^{\infty}(1 - b_j r_j)$ converges a.e. to the limit

$$\rho^- := \prod_{j=0}^{\infty}(1 - b_j r_j), \tag{6}$$

and $\rho^- \geq 0$ and $\rho^- \in L^1(\mathbb{I})$ (see [2], [3]). It is proved in [8] that if the maximal function of the martingale $(\rho_n^-, n \in \mathbb{N})$ satisfies

$$\sup_{n\in\mathbb{N}} \rho_n^- \in L^1(\mathbb{I}),$$

then Ψ^{-1} is an orthonormal system with respect to weight function ρ^-. We introduce the martingale transform operators

$$T_m^{\rho^-} f := \sum_{k=0}^{\infty} a_m^k E_k^{\rho^-}(\phi_k^{-1} f)\phi_k^{-1} \qquad (f \in L^1_{\rho^-}(\mathbb{I}),\ m \in \mathbb{N}), \tag{7}$$

where the $\mathcal{A}_k$-measurable coefficients are defined by

$$a_m^k := m_k \prod_{j=0}^{k-1} \frac{(1 - b_j r_j)^{1-m_j}}{(1 + b_j r_j)^{1-m_j}} \qquad (m, k \in \mathbb{N}). \tag{8}$$

It is proved in [8] that for any function $f \in L^1_\rho(\mathbb{I})$ we have

$$S_m^\rho f = \psi_m T_m^{\rho^-}(f\rho\psi_m/\rho^-) \qquad (m \in \mathbb{N}). \tag{9}$$

2. Results

Recall that the functions $f : [0, 1) \to \mathbb{R}$ belong to the class $Lip\,(\alpha, W)$ $(0 < \alpha \leq 1)$ if there exists a constant $K > 0$ such that $|f(x+h) - f(x)| \leq K\,h^\alpha$ for all $x, h \in \mathbb{I}$.

Lemma 1. *(i) If $\rho \geq \rho_0 > 0$ on $\mathbb{I}$ and $\rho \in Lip\,(\alpha, W)$ $(0 < \alpha \leq 1)$, then $\|b_n\|_\infty < 1$ $(n \in \mathbb{N})$ and $\sum_{n=0}^{\infty} \|b_n\|_\infty < \infty$.*

(ii) If $\sum_{n=0}^{\infty} \|b_n\|_\infty < \infty$ and $\|b_n\|_\infty < 1$ $(n \in \mathbb{N})$, then the function sequences a_m^k and Ψ_m are uniformly bounded, and the functions ρ, ρ^- and $\frac{\rho}{\rho^-}$ are bounded.

From Lemma 1 and (9) we get

Theorem 1. *If $\rho \geq \rho_0 > 0$ on $\mathbb{I}$ and $\rho \in Lip\,(\alpha, W)$ $(0 < \alpha \leq 1)$, then*

$$\|S_n^\rho f\|_{L_\rho^p} \leq \|f\|_{L_\rho^p}$$

for $0 < p < \infty$.

Lemma 2. *If $\rho(x) = x^\alpha$ $(\alpha > -1)$, then*

$$b_n(x) = \frac{2\left(k+\frac{1}{2}\right)^{\alpha+1} - k^{\alpha+1} - (k+1)^{\alpha+1}}{(k+1)^{\alpha+1} - k^{\alpha+1}} \tag{10}$$

if $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ $(k = 0, \dots, 2^n - 1)$.

Lemma 3. *Let $x \in [2^{-\ell}, 2^{-\ell+1})$ $(\ell = 1, 2, \dots)$ and $\rho(x) = x^\alpha$.*

(i) *If $n < \ell$ and $\alpha > -1$, then $b_n(x) = \left(\frac{1}{2}\right)^\alpha - 1$.*

(ii) *If $n \geq \ell$ and $\alpha > -1$, $\alpha \neq 1$, then there exists a constant B_α depending only on α, such that $|b_n(x)| \leq B_\alpha 2^{-n+\ell}$.*

(iii) *If $n \geq \ell$ and $\alpha = 1$, then $2^{-n+\ell-3} < |b_n(x)| \leq 2^{-n+\ell-2}$.*

Lemma 4. *Let α_0 be a positive real number with $2^{\alpha_0 - 2}\,\alpha_0 = 1$. If $\rho(x) = x^\alpha$ $(\alpha_0 \geq \alpha > -1)$ and $x \in [2^{-\ell}, 2^{-\ell+1})$ $(\ell = 1, 2, \dots)$, then*

$$\psi_m(x) = w_m(x)(2^{\alpha+1} - 1)^{\frac{1}{2}\sum_{k=0}^{\ell-1} m_k} \cdot C_{m,\ell}^\alpha(x), \tag{11}$$

where $w(x)$ is the n-th Walsh function in Paley's enumeration, $|C_{m,\ell}^\alpha(x)| \leq K_\alpha$, and K_α is a constant depending only on α.

Let $0 < N \in \mathbb{N}$, $m_N := \sum_{k=0}^{N-1} 2^k$. It follows from (11) that

$$|\psi_m(x)| \leq K_\alpha (2^{\alpha+1} - 1)^{\ell/2}, \tag{12}$$

and

$$|\psi_{m_N}(x)| \geq \frac{1}{K_\alpha}(2^{\alpha+1} - 1)^{\ell/2},$$

if $\rho(x) = x^\alpha$ $(\alpha > -1, \alpha \neq 0)$, and if $x \in [2^{-\ell}, 2^{-\ell+1})$ $(\ell = 1, 2, \dots)$. From these estimations we get the following

Theorem 2. *Let $\rho(x) = x^\alpha$ $(\alpha_0 \geq \alpha > 0)$, where α_0 is a positive real number with $2^{\alpha_0-2}\alpha_0 = 1$, and $p_0^\alpha := \frac{2(\alpha+1)\ln 2}{\ln(2^{\alpha+1}-1)}$. There exists a function $f \in L_\rho^p$ with divergent W^ρ-Fourier series in L_ρ^p-norm if $1 \leq p \leq p_0$ or $p_0' \leq p \leq \infty$.*

3. Proofs

Proof of Lemma 1. It follows immediately from (3) that if $\rho \geq \rho_0 > 0$ on $\mathbb{I}$, then $\|b_n\|_\infty < 1$ $(n \in \mathbb{N})$. We get by definition that in this case $|E_n\rho| \geq \rho_0$, and it is known that if $\rho \in Lip\,(\alpha, W)$ $(0 < \alpha \leq 1)$, then $\|E_n\rho - \rho\|_\infty \leq C_\alpha 2^{-n\alpha}$, where C_α is a constant depending only on α (see [9]). Using these estimations we get

$$|b_n| = \left|\frac{E_n(\rho r_n)}{E_n\rho}\right| = \left|\frac{r_n E_n(\rho r_n)}{E_n\rho}\right| = \left|\frac{E_{n+1}\rho - E_n\rho}{E_n\rho}\right|$$
$$\leq \frac{|E_{n+1}\rho - \rho| + |E_n\rho - \rho|}{|E_n\rho|} \leq \frac{C_\alpha 2^{-(n+1)\alpha} + C_\alpha 2^{-n\alpha}}{\rho_0} = A_\alpha 2^{-n\alpha},$$

where $A_\alpha = C_\alpha(2^{-\alpha} + 1)/\rho_0$ and taking the supremum we get $\|b_n\|_\infty \leq A_\alpha 2^{-n\alpha}$. Since the series $\sum_{n=0}^\infty 2^{-n\alpha}$ is convergent if $\alpha > 0$, it follows that the series $\sum_{n=0}^\infty \|b_n\|_\infty$ is convergent too, and (i) is proved. Since the series $\sum_{n=0}^\infty \|b_n\|_\infty$ is convergent, it follows that there exists a constant $\tilde{K}(< 1)$, and $N \in \mathbb{N}$ with $\|b_n\|_\infty \leq \tilde{K}$ for $n > N$. Since $|b_j| < 1$ $(j = 0, 1, \dots, N)$ and b_j is continuous on the group, and the group is compact, it follows that $\|b_j\|_\infty < 1$ $(j = 0, \dots, N)$, and we get that for all $n \in \mathbb{N}$, $\|b_n\|_\infty \leq K := \max\{\tilde{K}, \|b_0\|_\infty, \dots, \|b_n\|_\infty\} < 1$.

Since $|b_n| < 1$ $(n \in \mathbb{N})$, we get by (8) that

$$|a_m^k| = m_k \prod_{j=0}^{k-1} \left|\frac{1 - b_j r_j}{1 + b_j r_j}\right|^{1-m_j} \leq m_k \prod_{j=0}^{k-1} \left(\frac{1 + |b_j|}{1 - |b_j|}\right)^{1-m_j} \leq \prod_{j=0}^{k-1} \frac{1 + |b_j|}{1 - |b_j|}. \tag{13}$$

Since $1 + u \leq e^u$ $(u \in \mathbb{R})$, and if $|u| \leq K < 1$, then $\frac{1}{1-u} \leq e^{su}$, where $s = \frac{1}{1-K}$, we get by (13) that

$$|a_m^k| \leq \prod_{j=0}^{k-1} e^{|b_j|} e^{s|b_j|} \leq e^{(s+1)\sum\limits_{j=0}^{k-1} \|b_j\|_\infty}, \tag{14}$$

if $|b_j| \leq K < 1$ $(j \in \mathbb{N})$, and $s = \frac{1}{1-K}$.

Since $|b_n| < 1$, and $r_n^2 = 1$ we can write in view of (3) the functions ϕ_n in the form

$$\phi_n = \frac{r_n(1 - r_n b_n)}{\sqrt{1 - r_n^2 b_n^2}} = r_n \sqrt{\frac{1 - r_n b_n}{1 + r_n b_n}}. \tag{15}$$

From (15) and (4) we get similarly as in (13) and (14) that

$$|\psi_m| \leq e^{\frac{s+1}{2}\sum\limits_{k=0}^{\infty} \|b_k\|_\infty},$$

where $|b_k| \leq K < 1$ $(k \in \mathbb{N})$, and $s = \frac{1}{1-K}$. Since the series $\sum_{n=0}^\infty \|b_n\|_\infty$ is convergent, the function series a_m^k and ψ_m are uniformly bounded. We get similarly by (5) and (6) that the function $\frac{\rho}{\rho^-}$ is bounded. $\square$

Proof of Theorem 1. It follows from (7) and (9) that the partial sums can be expressed by martingale transforms operators. It is known (see [9]) that if in the martingale transforms operators the function sequences a_m^k are uniformly bounded, then the martingale transforms operator is bounded in L^p-norm if $1 < p < \infty$. Since the function sequences ψ_m are uniformly bounded too and the function $\frac{\rho}{\rho^-}$ is bounded (see Lemma 1), it follows by (9) that the partial sums $S_m^\rho f$ are bounded in L^p-norm $(1 < p < \infty)$. $\square$

Proof of Lemma 2. Suppose that $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ $(k = 0, \ldots, 2^n - 1)$. We get with easy enumeration that

$$(E_n\rho)(x) = 2^n \int_{k2^{-n}}^{(k+1)2^{-n}} t^\alpha \, dt = \frac{(k+1)^{\alpha+1} - k^{\alpha+1}}{2^{n\alpha}(\alpha+1)}, \tag{16}$$

and

$$\begin{aligned}(E_n\rho r_n)(x) &= 2^n \int_{k2^{-n}}^{(k+1)2^{-n}} t^\alpha r_n(t) \, dt \\ &= \frac{2(k+1/2)^{\alpha+1} - (k+1)^{\alpha+1} - k^{\alpha+1}}{2^{n\alpha}(\alpha+1)}.\end{aligned} \tag{17}$$

Using (16) and (17), we get our statement by (1). $\square$

Proof of Lemma 3. If $n < \ell$, then for all $n \in \mathbb{N}$ $x \in [0, 2^{-n})$, and so $k = 0$ in (10), and $b_n(x) = (1/2)^\alpha - 1$. If $n \geq \ell$, then $2^{n-\ell} \leq k < k+1 \leq 2^{n-\ell+1}$ in (10). Taking the function $f(u) = u^{\alpha+1}$ and using the Lagrange theorem in (10) we get that there exist real numbers $\xi_1 \in (k, k+1/2)$, $\xi_2 \in (k+1/2, k+1)$ and $\xi_3 \in (k, k+1)$ such that

$$b_n(x) = \frac{1}{2} \cdot \frac{f'(\xi_1) - f'(\xi_2)}{f'(\xi_3)} = \frac{\xi_1^\alpha - \xi_2^\alpha}{2\xi_3^\alpha}. \tag{18}$$

Applying the Lagrange theorem again for the function $g(u) = u^\alpha$ and for the interval (ξ_1, ξ_2), we get by (18) that there exists a real number $\eta \in (\xi_1, \xi_2)$ for which

$$b_n(x) = \frac{(\xi_1 - \xi_2)g'(\eta)}{2\xi_3^\alpha} = \frac{(\xi_1 - \xi_2)\alpha\eta^{\alpha-1}}{2\xi_3^\alpha}. \tag{19}$$

Since $\xi_1 < \xi_2$, we conclude that $b_n(x)$ has negative sign if $\alpha > 0$ and has positive sign if $-1 < \alpha < 0$. If $\alpha > 1$, then we get by (19) that

$$|b_n(x)| \leq \frac{\alpha\xi_2^{\alpha-1}}{2k^\alpha} < \frac{\alpha(k+1)^{\alpha-1}}{2k^\alpha} \leq 2^{\alpha-2}\alpha 2^{-n+\ell}. \tag{20}$$

If $0 < \alpha < 1$, then we get similarly by (19) that

$$|b_n(x)| \leq \frac{\alpha\xi_1^{\alpha-1}}{2k^\alpha} < \frac{\alpha(k+1/2)^{\alpha-1}}{2k^\alpha} \leq \frac{\alpha}{2} \, 2^{-n+\ell}. \tag{21}$$

If $-1<\alpha<0$, then we get similarly by (19) that

$$|b_n(x)| \leq \frac{|\alpha|\xi_1^{\alpha-1}}{2(k+1)^\alpha} < \frac{|\alpha|k^{\alpha-1}}{2(k+1)^\alpha} \leq \frac{|\alpha|}{2}\, 2^{-\alpha}\cdot 2^{-n+\ell}. \tag{22}$$

With (20), (21), and (22), (ii) is proved. If $\alpha=1$, then we get by (10) that $b_n(x) = -\frac{1}{4k+2}$, if $x\in\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]$ $(k=0,\dots,2^n-1)$, which is a monotone function on $(0,1)$ and (iii) follows easily. □

Proof of Lemma 4. Suppose that $x\in[2^{-\ell},2^{-\ell+1})$ $(\ell=1,2,\dots)$. Let α_0 be a positive real number with $2^{\alpha_0-2}\alpha_0=1$, and let

$$B_\alpha := \begin{cases} \alpha 2^{\alpha-2}, & \text{if } 1<\alpha\leq\alpha_0\\ \alpha/2, & \text{if } 0<\alpha\leq 1\\ \frac{|\alpha|}{2}\,2^{-\alpha}, & \text{if } -1<\alpha\leq 0.\end{cases}$$

Then, for all $n\in\mathbb{N}$, $\|b_n\|_\infty \leq K := \max\{|(1/2)^\alpha-1|, B_\alpha\}<1$.

Since $r_k(x)=1$ if $k<\ell$, we get by (4), (15) and Lemma 3 that

$$\begin{aligned}\psi_m(x) &= \prod_{k=0}^{\infty}(r_k(x))^{m_k}\prod_{k=0}^{\ell-1}\left(\sqrt{\frac{1-b_k(x)}{1+b_k(x)}}\right)^{m_k}\prod_{k=\ell}^{\infty}\left(\sqrt{\frac{1-r_k(x)b_k(x)}{1+r_k(x)b_k(x)}}\right)^{m_k}\\ &= w_m(x)\,(2^{\alpha+1}-1)^{\frac12\sum_{k=0}^{\ell-1}m_k}\cdot C_{m,\ell}^\alpha(x),\end{aligned}$$

where $C_{m,\ell}^\alpha(x)=\prod_{k=\ell}^\infty\left(\sqrt{\frac{1-r_k(x)b_k(x)}{1+r_k(x)b_k(x)}}\right)^{m_k}$. Then, by Lemma 3, we obtain

$$|C_{m,\ell}^\alpha(x)| \leq \prod_{k=\ell}^\infty\sqrt{\frac{1+|b_k(x)|}{1-|b_k(x)|}} \leq \prod_{k=\ell}^\infty\sqrt{e^{(s+1)|b_k(x)|}} \leq e^{(s+1)B_\alpha} =: K_\alpha,$$

and Lemma 4 is proved. □

Proof of Theorem 2. It was proved by Newman and Rudin in [4] that a necessary condition for the L_ρ^p-convergence of the W^ρ-Fourier series of any function $f\in L_\rho^p$ is

$$\|\psi_m\|_{L_\rho^p}\cdot\|\psi_m\|_{L_\rho^{p'}} = O(1), \tag{23}$$

where $1/p+1/p'=1$. We denote by $\chi_\ell(x)$ the characteristic function of the interval $[2^{-\ell},2^{-\ell+1})$ $(\ell=1,2,\dots)$. If $\rho(x)=x^\alpha$ $(\alpha_0\geq\alpha>-1)$, $0<N\in\mathbb{N}$, $m_N:=\sum_{k=0}^{N-1}2^k$, then we get by (12) that

$$\begin{aligned}\|\psi_{m_N}\|_{L_\rho^p}^p &= \int_0^1|\psi_{m_N}(x)|^p\rho(x)\,dx = \int_0^1\sum_{\ell=1}^\infty\chi_\ell(x)|\psi_{m_N}(x)|^p\rho(x)\,dx\\ &\geq \int_0^1\sum_{\ell=1}^N\chi_\ell(x)\frac{(2^{\alpha+1}-1)^{\ell p/2}}{K_\alpha^p}\,\rho(x)\,dx\\ &= \frac{2^{\alpha+1}-1}{(\alpha+1)K_\alpha^p}\sum_{\ell=1}^N\left(\frac{(2^{\alpha+1}-1)^{p/2}}{2^{\alpha+1}}\right)^\ell =: C_{\alpha,p}\sum_{\ell=1}^N q_{\alpha,p}^\ell.\end{aligned}\tag{24}$$

The last series in (24) is divergent if $q_{\alpha,p} \geq 1$. This inequality is true, if $p \geq \frac{2(\alpha+1)\ln 2}{\ln(2^{\alpha+1}-1)} = p_0^\alpha$, and $0 < \alpha \leq \alpha_0$. It follows by (23) that the necessary condition for the L_ρ^p-convergence of the W^ρ-Fourier series of any function $f \in L_\rho^p$ is that $p_0' < p < p_0$, and Theorem 2 is proved. $\square$

Acknowledgment. I wish to express my gratitude to Professor Ferenc Schipp and Professor Péter Simon for their help and advice in preparing this paper.

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