CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 275-281.

Decompositions of Hilbert Spaces: Local Construction of Global Frames

MASSIMO FORNASIER *

We introduce a decomposition of a Hilbert space $\mathbb{H}$ into a quasi-orthogonal family of closed subspaces. We shall investigate conditions in order to derive bounded families of corresponding *quasi-projectors* or *resolutions of the identity operator*. Given a local family of atoms, or generalized basis, for each subspace, we show that the union of the local atoms can generate a global frame for the Hilbert space. Corresponding duals can be calculated in a flexible way by means of the systems of quasi-projectors.

Decomposition methods were introduced by Frazier and Jawerth [8] in order to construct wavelet-type bases for Besov spaces. A general presentation of this kind of methods was proposed by Gröbner and Feichtinger [4, 6, 9] showing how many different classical and widely used Banach spaces (L^p-spaces, Besov and Triebel spaces, Modulation spaces, Wiener Amalgams) can be constructed by decompositions in local subspaces, controlled by global norms. Moreover, the research of bases for these spaces based on decomposition techniques has given extremely fruitful applicative results. The paper presents an abstract version of the Gröbner-Feichtinger construction for a Hilbert space $\mathbb{H}$ as common skeleton of many interesting atomic decompositions (wavelets, Gabor frames, Local Fourier basis) and useful tool for defining new types of bases [5, 7]. The resulting bases are (structured) *frames* [2, 3] for $\mathbb{H}$.

Let $\mathbb{H}$ be a separable Hilbert space.

Definition 1. *A sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ is a frame for the Hilbert space $\mathbb{H}$ if there exist two positive constants $A, B > 0$ such that*

$$A\,\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\,\|f\|^2, \qquad \forall f \in \mathbb{H}. \tag{1}$$

The upper bound in condition (1) *is also known as the Bessel condition for the sequence $\{f_n\}_{n\in\mathbb{N}}$ and whenever it holds the sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be a Bessel sequence.*

*Doctorate in Computational Mathematics. The author wants to thank H. G. Feichtinger for the fruitful discussions, his valuable suggestions and hospitality of NuHAG (the Numerical Harmonic Analysis Group, Department of Mathematics, University of Vienna) during the preparation of this work.

Condition (1) ensures also that the *frame operator* $S : \mathbb{H} \to \mathbb{H}$ given by

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n,$$

is invertible. This implies that

$$f = SS^{-1}f = \sum_{n\in\mathbb{N}} \langle f, S^{-1}f_n\rangle f_n.$$

The sequence $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ is again a frame and it is called the *canonical dual* frame of $\{f_n\}_{n\in\mathbb{N}}$ with a frame operator S^{-1}. Since a frame is typically overcomplete in the sense that the coefficient maps $\{c_n\}_{n\in\mathbb{N}}$ such that $f = \sum_{n\in\mathbb{N}} c_n(f)f_n$ are in general not unique, there exist many possible duals $\{\tilde{f}_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ for which $f = \sum_{n\in\mathbb{N}}\langle f, \tilde{f}_n\rangle f_n$. The redundancy of a frame can play an important role in practical problems where robustness and error tolerance are fundamental as, for example, denoising, irregular sampling problems or pattern matching. How to calculate in efficient way (approximations of) corresponding canonical duals for general frames is still an open problem [1]. The following decomposition method provides also a novel construction of possible duals and a useful tool in order to check whether a system is a frame.

Definition 2. *A local construction or decomposition of the Hilbert space* $\mathbb{H}$ *is a sequence* $\{\mathbb{W}_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $\mathbb{H}$ *such that*

L1) $\mathbb{H} = \overline{\sum_{j\in\mathbb{Z}} \mathbb{W}_j} := \overline{\{\sum_{j\in F\subset\mathbb{Z}} c_j f_j : \#F < \infty, f_j \in \mathbb{W}_j\}}$;

L2) There exists $N \in \mathbb{N}$ *such that for all* $j \in \mathbb{Z}$ *there exists* $j^* \subset \mathbb{Z}$, $\#j^* \le N$ *with the property that*

$$\mathbb{W}_i \perp \mathbb{W}_j, \qquad \forall i \in \mathbb{Z}\backslash j^*, \quad \#\{i \in \mathbb{Z} : j \in i^*\} \le N.$$

L3) For all possible finite sequences $\{W_{i_1}, ..., W_{i_n}\}$ *one has*

$$\sum_{j=1}^{n} \mathbb{W}_{i_j} = \overline{\sum_{j=1}^{n} \mathbb{W}_{i_j}}.$$

Assume also the existence of a sequence of invertible bounded operators $\{D_j\}_{j\in\mathbb{Z}}$, $D_j : \mathbb{H} \to \mathbb{H}$, such that $D_j(\mathbb{W}_0) = \mathbb{W}_j$. In such a case, in order to define a decomposition, it is sufficient to give the couple $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{Z}})$. In particular, if $D_j = D^j$, where $D : \mathbb{H} \to \mathbb{H}$ is a unitary operator, then we will say that the decomposition is *coherent.* One typical example is given by orthonormal wavelet spaces [3]. Let us introduce now a generalization of frames.

Definition 3. *Let* $\mathbb{W}_0 \subset \mathbb{H}$ *be a closed subspace. A sequence* $\{\psi_k^0\}_{k\in\mathbb{Z}} \subset \mathbb{H}$ *is a local family of atoms for* $\mathbb{W}_0$ *in* $\mathbb{H}$ *if*

Ψ1) There exists a sequence $\{c_k\}_{k\in\mathbb{Z}} \subset (\mathbb{W}_0)'$ *and* $\tilde{B} > 0$ *such that*

$$f = \sum_{k\in\mathbb{Z}} c_k(f)\psi_k^0, \quad \sum_{k\in\mathbb{Z}} |c_k(f)|^2 \le \tilde{B}\,\|f\|^2, \qquad \forall f \in \mathbb{W}_0. \tag{2}$$

Ψ2) There exists $B > 0$ *such that*

$$\sum_{k\in\mathbb{Z}} |\langle f, \psi_k^0\rangle|^2 \le B\,\|f\|^2, \qquad \forall f \in \mathbb{W}_0. \tag{3}$$

Remark. Since $c_k \in (\mathbb{W}_0)'$, for all $k \in \mathbb{Z}$, there exists $\tilde{\psi}_k^0 \in \mathbb{W}_0$ such that $c_k(f) = \langle f, \tilde{\psi}_k^0\rangle$. We call $\{\tilde{\psi}_k^0\}_{k\in\mathbb{Z}}$ a dual of the local family $\{\psi_k^0\}_{k\in\mathbb{Z}}$. Under the assumptions (2) and (3), there exist $A, \tilde{A} > 0$, such that for all $f \in \mathbb{W}_0$

$$A\,\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, \psi_k^0\rangle|^2 \le B\,\|f\|^2,$$
$$\tilde{A}\,\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, \tilde{\psi}_k^0\rangle|^2 \le \tilde{B}\,\|f\|^2.$$

In particular, $\{\tilde{\psi}_k^0\}_{k\in\mathbb{Z}}$ is a frame for $\mathbb{W}_0$.

Definition 4. *Given a decomposition* $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{Z}})$ *of* $\mathbb{H}$ *we will call a system of bounded quasi-projectors or a bounded resolution of the identity any set* $\mathbb{P} = \{P_j\}_{j\in\mathbb{Z}}$ *of operators such that* $P_j : \mathbb{H} \to \mathbb{W}_j$*, and*

P1) $\sum_{j\in\mathbb{Z}} P_j = I_{\mathbb{H}}$;

P2) $\sum_{j\in\mathbb{Z}} \|P_j f\|^2 \le C_2\,\|f\|^2$;

The system is said to be self-adjoint and compatible with the canonical projections if

P3) $P_j = P_j^* \quad \forall j$;

P4) $P_j \circ \pi_{W_j} = P_j \quad \forall j$*, where* π_{W_j} *is the canonical projection on* $\mathbb{W}_j$.

Proposition 1. *Given a decomposition* $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{N}})$ *of* $\mathbb{H}$ *such that* $\pi_{W_j}(\mathbb{W}_i) \subset \mathbb{W}_i \bigcap \mathbb{W}_j$ *and* $[\pi_{W_i}, \pi_{W_j}] = \pi_{W_i}\pi_{W_j} - \pi_{W_j}\pi_{W_i} \equiv 0\ \forall i, j$*, one can always construct a system* $\mathbb{P} = \{P_j\}_{j\in\mathbb{Z}}$ *with properties (P1-P4). In fact, if* τ_n *is the sequence of bounded operators given by*

$$I_{\mathbb{H}} - \tau_n = \prod_{j=0}^{n} (I - \pi_{W_j}),$$

then

$$\mathcal{P}_n = \tau_n - \tau_{n-1} = \pi_{W_n} \prod_{j=0}^{n-1} (I_{\mathbb{H}} - \pi_{W_j}), \qquad n > 0, \quad \mathcal{P}_0 = \tau_0,$$

defines a system of bounded quasi-projectors.

Lemma 1. *Let $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{Z}})$ be a decomposition of $\mathbb{H}$ and assume that a system of bounded quasi-projectors $\mathbb{P} = \{P_j\}_{j\in\mathbb{Z}}$ with the property of being self-adjoint and compatible with the canonical projections is given. Then, for all $f \in \mathbb{H}$,*

$$C_1 \|f\|^2 \leq \sum_{j\in\mathbb{Z}} \|P_j f\|^2 \leq C_2 \|f\|^2.$$

A more abstract class of resolutions of the identity can be derived as follows:

Proposition 2. *Given a decomposition $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{Z}})$ of the Hilbert space $\mathbb{H}$, the operator $S_\pi(f) = \sum_{k\in\mathbb{Z}} \pi_{W_k}(f)$ is positive, self-adjoint and invertible. Hence, the operators given by $\mathcal{P}_j = \pi_{W_j} S_\pi^{-1}$, $j \in \mathbb{Z}$, are a system of quasi-projectors with $\mathcal{P}_j^* = S_\pi^{-1} \pi_{W_j}$ $\forall j$, and for all $f \in \mathbb{H}$,*

$$C_1 \|f\|^2 \leq \sum_{j\in\mathbb{Z}} \|P_j f\|^2 \leq C_2 \|f\|^2. \tag{4}$$

Moreover, if $[\pi_{W_j}, \pi_{W_i}] \equiv 0$ $\forall i, j$, then the system has also properties (P3-P4).

Proof. Cotlar's lemma and L2) ensure that S_π is a well defined continuous operator and that the convergence of the series is in the strong operator topology. Let us consider $\mathbb{V}_n = \sum_{j=0}^n \mathbb{W}_j$ and let $\mathbb{P}_n$ be the set of the permutations on $\{0, ..., n\}$. For all $k \in \{0, ..., n\}$ one can define the equivalence relation on $\mathbb{P}_n$ given by

$$\sigma \sim_k \sigma' \leftrightarrow \{\sigma_1, ..., \sigma_k\} = \{\sigma'_1, ..., \sigma'_k\}.$$

Let $[\mathbb{P}_n]_k = \mathbb{P}_n / \sim_k$ and define the sets

$$\mathbb{Q}_{k,\sigma} := (\mathbb{W}_{\sigma_1} + ... + \mathbb{W}_{\sigma_k})^\perp \bigcap \mathbb{W}_{\sigma_{k+1}} \bigcap ... \bigcap \mathbb{W}_{\sigma_n},$$

for all $k \in \{0, ..., n\}$ and $\sigma \in [\mathbb{P}_n]_k$, where "$\perp$" is taken in $\mathbb{V}_n$. The spaces $\mathbb{Q}_{k,\sigma}$ are closed and such that $\mathbb{V}_n = \bigoplus_{k,\sigma\in[\mathbb{P}_n]_k} \mathbb{Q}_{k,\sigma}$. Hence, for all $f \in \mathbb{V}_n$ one has

$$\|f\|^2 = \sum_{k,\sigma\in[\mathbb{P}_n]_k} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2.$$

Moreover, $\mathbb{Q}_{k,\sigma} \bigcap \mathbb{W}_j \neq 0$ if and only if $\mathbb{Q}_{k,\sigma} \subset \mathbb{W}_j$. Let us define the set

$$Q_j := \{(k, \sigma) : \sigma \in [\mathbb{P}_n]_k, \mathbb{Q}_{k,\sigma} \subset \mathbb{W}_j\}.$$

One has

$$\|\pi_{W_j} f\|^2 = \sum_{(k,\sigma)\in Q_j} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2$$

and

$$\sum_{k,\sigma\in[\mathbb{P}_n]_k} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2 \leq \sum_{j=0}^n \sum_{(k,\sigma)\in Q_j} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2 = \sum_{j=0}^n \|\pi_{W_j} f\|^2.$$

By property $L2)$ and by definition, one has that $\#\{j : \mathbb{Q}_{k,\sigma} \subset \mathbb{W}_j\} \leq N$ and then

$$\sum_{j=0}^{n} \sum_{(k,\sigma)\in Q_j} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2 \leq N \sum_{k,\sigma\in[\mathbb{P}_n]_k} \|\pi_{\mathbb{Q}_{k,\sigma}} f\|^2.$$

Hence, for all $n \in \mathbb{N}$ and all $f \in \mathbb{V}_n$,

$$\|f\|^2 \leq \sum_{j=0}^{n} \|\pi_{W_j}(f)\|^2 \leq N\|f\|^2. \tag{5}$$

Clearly, $\mathbb{V}_n \subset \mathbb{V}_{n+1}$ and $\mathbb{H} = \overline{\bigcup_{n\in\mathbb{N}} \mathbb{V}_n}$ (L1). For all $n \in \mathbb{N}$ and all $f \in \mathbb{H}$ one has by (5)

$$\|\pi_{V_n} f\|^2 \leq \sum_{j=0}^{n} \|\pi_{W_j}(\pi_{V_n} f)\|^2 \leq N\|\pi_{V_n} f\|^2. \tag{6}$$

Since $\mathbb{W}_j \subset \mathbb{V}_n$ for all $j \in \{0, ..., n\}$ then $\pi_{W_j}(\pi_{V_n} f) = \pi_{W_j}(f)$. By taking the limit for $n \to \infty$, (6) extends to

$$\|f\|^2 \leq \sum_{j\in\mathbb{N}} \|\pi_{W_j}(f)\|^2 \leq N\|f\|^2,$$

for all $f \in \mathbb{H}$. Hence the operator S_π is positive, self-adjoint and invertible and

$$f = S_\pi S_\pi^{-1} f = \sum_{j\in\mathbb{Z}} \pi_{W_j}(S_\pi^{-1} f)$$

(P1). Moreover, formula (4) holds (P2). Observe now that S_π^{-1} commutes with π_{W_j} if and only if S_π commutes with π_{W_j}. In particular, if $[\pi_{W_j}, \pi_{W_i}] \equiv 0$, then S_π commutes with π_{W_j} for all j and hence $\mathcal{P}_j^* = \mathcal{P}_j$ (P3), and $\mathcal{P}_j(\pi_{W_j}) = \mathcal{P}_j$ (P4).

Theorem 1. *Consider a decomposition* $(\mathbb{W}_0, \{D_j\}_{j\in\mathbb{Z}})$ *of* $\mathbb{H}$ *such that there exist two positive sequences* $(\alpha_j)_{j\in\mathbb{Z}}$, $(\beta_j)_{j\in\mathbb{Z}}$ *with* $\alpha_j \asymp \beta_j$, *for* $j \to \infty$, *and such that*

$$\alpha_j \|f\|^2 \leq \|D_j f\|^2 \leq \beta_j \|f\|^2, \quad \forall f \in \mathbb{W}_0, \forall j \in \mathbb{Z}.$$

Assume also that a system of bounded quasi-projectors $\mathbb{P} = \{P_j\}_{j\in\mathbb{Z}}$ *with property (4) is given. Let* $\{\psi_k^0\}_{k\in\mathbb{Z}} \subset \mathbb{H}$ *be a local family of atoms for* $\mathbb{W}_0$ *in* $\mathbb{H}$. *Then, for all* $f \in \mathbb{H}$,

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, P_j^*(D_j^{-1})^* \tilde{\psi}_k^0 \rangle D_j \psi_k^0. \tag{7}$$

Moreover, if $c_{k,j}^{\mathbb{P}}(f) = \langle f, P_j^*(D_j^{-1})^* \tilde{\psi}_k^0 \rangle$, *then the sequence* $c^{\mathbb{P}}(f) = (c_{k,j}^{\mathbb{P}}(f))_{j,k\in\mathbb{Z}}$ *has the property that for all* $\xi_j \asymp \alpha_j \asymp \beta_j$

$$\|c^{\mathbb{P}}(f)\|_{l^2_{\xi_j^{1/2}}(\mathbb{Z}\times\mathbb{Z})} \asymp \|f\|.$$

If $\{\alpha_j^{-\frac{1}{2}} D_j \psi_k^0\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence for $\mathbb{H}$ (B), then $\{\xi_j^{-\frac{1}{2}} D_j \psi_k^0\}_{j,k\in\mathbb{Z}}$ is a frame for $\mathbb{H}$ and the convergence of the sums in (7) *is unconditional. Moreover, if $\{\psi_k^0\}_{k\in\mathbb{Z}}$ is a frame for $\mathbb{W}_0$ and $\alpha_j \asymp \beta_j \asymp \beta$, then condition (B) is automatically verified.*

Proof. First of all, observe that if $\{\psi_k^0\}_{k\in\mathbb{Z}} \subset \mathbb{H}$ is a local family of atoms for $\mathbb{W}_0$ in $\mathbb{H}$, then $\{D_j\psi_k^0\}_{k\in\mathbb{Z}} \subset \mathbb{H}$ is a local family of atoms for $\mathbb{W}_j$ in $\mathbb{H}$. By property P1), for all $f \in \mathbb{H}$, $f = \sum_{j\in\mathbb{Z}} P_j f$. Since $P_j f \in \mathbb{W}_j$ one has

$$P_j f = \sum_{k\in\mathbb{Z}} \langle P_j f, (D_j^{-1})^* \tilde{\psi}_k^0 \rangle D_j \psi_k^0$$

which implies

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, P_j^* (D_j^{-1})^* \tilde{\psi}_k^0 \rangle D_j \psi_k^0.$$

Moreover,

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |c_{k,j}^{\mathbb{P}}(f)|^2 \xi_j &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle P_j f, (D_j^{-1})^* \tilde{\psi}_k^0 \rangle|^2 \xi_j \\ &\le \tilde{B} \sum_{j\in\mathbb{Z}} \|P_j f\|^2 \xi_j \, \alpha_j^{-1} \le C \, \tilde{B} \, C_2 \, \|f\|^2. \end{aligned}$$

The lower bound estimate can be derived analogously. If $\{\alpha_j^{-1/2} D_j \psi_k^0\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence for $\mathbb{H}$, one has immediately that $\{\xi_j^{-\frac{1}{2}} D_j \psi_k^0\}_{j,k\in\mathbb{Z}}$ is also a frame for $\mathbb{H}$: in fact,

$$\|f\|^4 = \left(\sum_{k,j\in\mathbb{Z}} c_{k,j}^{\mathbb{P}}(f) \xi_j^{\frac{1}{2}} \, \xi_j^{-\frac{1}{2}} \langle f, D_j \psi_k^0 \rangle \right)^2 \le C \, \tilde{B} \, C_2 \, \|f\|^2 \sum_{k,j\in\mathbb{Z}} |\langle f, \xi_j^{-\frac{1}{2}} D_j \psi_k^0 \rangle|^2.$$

Assume, now, that $\{\psi_k^0\}_{k\in\mathbb{Z}}$ is a frame for $\mathbb{W}_0$. As a consequence

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, D_j \psi_k^0 \rangle|^2 &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle \pi_{W_j}(f), D_j \psi_k^0 \rangle|^2 \\ &\le \beta \, B \sum_{j\in\mathbb{Z}} \|\pi_{W_j}(f)\|^2 \le \beta \, B \, N \, \|f\|^2. \end{aligned}$$

Hence, $\{D_j \psi_k^0\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence for $\mathbb{H}$.

The abstract decomposition of Hilbert spaces technique makes easier to check whether a system is a frame: under conditions which ensure the existence of a local family of atoms, it is sufficient to verify that the (global) system is a Bessel sequence. The construction improves flexibility in the possible choice of duals, depending only on the choice of suitable systems of quasi-projectors and local duals. Applications are given in order to derive flexible intermediate Gabor-wavelet frames in α-Modulation spaces [5, 9] and in [7] where local Circular Harmonic frames are used for efficient 2D pattern matching of digital images.

References

[1] O. Christensen, Finite-dimensional approximation of the inverse frame operator, *J. Four. Anal. Appl.* **6**, 1 (2000), 79–91.

[2] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341–366.

[3] I. Daubechies, "Ten Lectures on Wavelets", SIAM, 1992.

[4] H. G. Feichtinger, Banach spaces of distributions defined by decomposition methods II, *Math. Nachr.* **132** (1987), 207–237.

[5] H. G. Feichtinger and M. Fornasier, Flexible Gabor-wavelet continuous and discrete frames in α-Modulation spaces, talk *in* "2^{nd} Int. Gabor Workshop 2001", Vienna, Dec. 2001.

[6] H. G. Feichtinger and P. Gröbner, Banach spaces of distributions defined by decomposition methods I, *Math. Nachr.* **123** (1985), 97–120.

[7] M. Fornasier and D. Toniolo, Compactly supported circular harmonics: fast, robust and efficient 2D pattern recognition, preprint, submitted.

[8] M. Frazier and B. Jawerth, Decomposition of Besov spaces, *Indiana Univ. Math. J.* **34** (1985), 777–799.

[9] P. Gröbner, "Banachräume glatter Funktionen and Zarlegungmethoden", PhD Thesis, University of Vienna, 1992.

[10] H. Triebel, "Theory of Function Spaces", Geest and Portig, Leipzig, 1983; Birkhäuser, Basel, 1983.

Massimo Fornasier
Department of Pure and Applied Mathematics
University of Padova
Via Belzoni, 7
35131 Padova
ITALY
E-mail: `mfornasi@math.unipd.it`