A Quadratic Spline of Sendov Type

Ioan Gavrea

For a compact interval \([a, b]\), \(a < b\), on the real axis we denote by \([a, b]\) the space of all real-valued continuous functions on \([a, b]\), equipped with the sup norm given by

\[
\|f\|_{C[a,b]} = \|f\|_\infty = \max\{|f(x)| : x \in [a, b]\}.
\]

For a natural number \(r\) we write

\[
C^r[a,b] = \{ f \in C[a,b] : f^{(r)} \in C[a,b] \},
\]

and

\[
W_r,\infty[a,b] = \{ f \in C[a,b] : f^{(r-1)} \text{ abs. cont.}, \|f^{(r)}\|_{L_\infty[a,b]} < \infty \}
\]

where \(\|f\|_{L_\infty[a,b]} = \|f\|_{L_\infty} = \sup \{|f(x)| : x \in [a, b]\}\).

The following theorem due to Brudnyi [1] is very important in approximation theory.

**Theorem 1.** Let \(f \in C[0,1]\) and \(s\) be a prescribed natural number. Then there exists a family of functions \(\{f_{s,h} : 0 < h < s^{-1}\}\) from \(W_s,\infty[0,1]\) such that

\[
\|f - f_{s,h}\|_\infty \leq A_s \omega_s(f,h),
\]

\[
\|f_{s,h}^{(s)}\|_{L_\infty} \leq B_s h^{-s} \omega_s(f,h),
\]

where the constants \(A_s\) and \(B_s\) depend only on \(s\) and \(\omega_s\) is the \(s\)-th order modulus of continuity.

It is of interest to have information about the magnitude of the constants \(A_s\) and \(B_s\). Zhuk [7] gave lower bounds for the constants \(A_s\) and \(B_s\) for the case \(s = 1\) and \(s = 2\) using an extension of the function \(f\) to a larger interval. In [4] a pointwise refinement of Zhuk results was obtained.

A genuinely different approach to constructing smoothing functions \(f_h\) is to define appropriate spline functions whose definition does not require an extension of \(f\). This was done by Sendov [6]. Sendov proved the following
Theorem 2. Let $f \in C[0,1]$. Then there exists a family of functions $$\{ f_h : h = \frac{1}{m}, \ m \geq 2, \ m \in \mathbb{N} \} \subset W_{2,\infty}[0,1]$$ such that $$\| f - f_{1/m} \|_{\infty} \leq \frac{9}{8} \omega_2(f; \frac{1}{m}),$$ $$\| f_h'' \|_{\infty} \leq m^2 \omega_2(f; \frac{1}{m}).$$

Sendov's functions $f_{1,m}$ are quadratic splines $S_2(f; \cdot) \in W_{2,\infty}[0,1]$. Gonska and Kovacheva [4] proved that the constant $9/8$ figuring in Theorem 2 can be replaced by $1$.

Our aim is to construct smoothing functions $f_h$ such that $$\| f - f_h \|_{\infty} \leq A \omega_2(f; h),$$ $$\| f_h'' \|_{\infty} \leq B \omega_2(f; h),$$ where the constants $A$ and $B$ are independent on the function $f$ and $\omega_2(f; \cdot)$ is Ditzian-Totik modulus defined by $$\omega_2(f; t) = \sup_{0 \leq h \leq t} \| \Delta_{h,\varphi}^2 f \|_{\infty}$$ where

$$\Delta_{h,\varphi}^2 f(x) = \begin{cases} f(x - h\varphi(x)) - 2f(x) + f(x + h\varphi(x)), & \text{if } [x - h\varphi(x), x + h\varphi(x)] \subset [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

$\varphi(x) = \sqrt{x(1-x)}$. The functions $f_h$ will be quadratic splines of Sendov-type. Such functions were considered for the first time by Gonska and Tachev [3].

In [2] Gavrea considered a new quadratic $C^1$-spline $g$ and proved the following:

Theorem 3. Let $f \in C[0,1]$. Then $$\| f - g \|_{\infty} \leq \omega_2(f; \sin \frac{\pi}{2(m+1)}),$$ $$\| \varphi^2 g'' \|_{\infty} \leq \frac{2}{\sin^2 \frac{\pi}{2(m+1)}} \omega_2(f; \sin \frac{\pi}{2(m+1)}),$$ $m$ being a natural number.

The following result (see [2]) will be used in this paper.
Let $\theta_1, \theta_2$ be two distinct points such that $0 \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$. If $a = \sin^2 \theta_1$, $b = \sin^2 \theta_2$, then for every $f \in C[0, 1]$ the following estimate holds

$$|L_1(f; a, b)(x) - f(x)| \leq \omega_2^f(f; \sin(\theta_2 - \theta_1))$$

where $L_1(f; a, b)$ is the Lagrange polynomial which interpolates the function $f$ at the points $a$ and $b$.

Let $\Delta_{2n+3} : 0 = x_0 < x_1 < \cdots < x_n < x_{n+2} < \cdots < x_{2n+2} = 1$ be a partition of the interval $[0, 1]$ such that

$$x_1 - x_0 \leq x_2 - x_1 \leq \cdots \leq x_n - x_{n-1},$$

$$x_n - x_{n-1} \geq x_{n+1} - x_n \geq \cdots \geq x_{2n+2} - x_{2n+1}.$$

We denote by $\tilde{S}_n(f)$ the continuous polygonal line having as knots the points $x_k$, $k = 0, 1, \ldots, 2n + 2$. With each such knot $x_k$, $k = 1, 2, \ldots, 2n + 2$, we associate the numbers $a_k$ and $b_k$, defined in the following way:

$$a_1 = 0, \quad b_1 = 2x_1$$

and

$$a_k = \frac{x_k + x_{k-1}}{2}, \quad b_k - x_k = x_k - a_k, \quad k = 2, \ldots, n + 1.$$  

For $k = n + 2, \ldots, 2n + 1$ we define the numbers $a_k$ and $b_k$ by symmetry with respect to $1/2$. It is supposed that $2x_1 \leq a_2$.

We construct the function $g$ in the following way. For $x \in [a_k, b_k]$, $k = 1, 2, \ldots, 2n + 1$, $g$ is the second degree Bernstein polynomial over the interval $[a_k, b_k]$, determined by the ordinates $\tilde{S}_n(f; a_k)$, $\tilde{S}_n(f; b_k)$. For $x \in [b_k, a_{k+1}]$, $k = 1, 2, \ldots, 2n$, $g(x) = \tilde{S}_n(f; x)$.

It is easy to show that for $x \in [a_k, b_k]$, $k = 1, 2, \ldots, 2n + 1$, the function $g$ is given by

$$g(x) = \frac{1}{2} \frac{x_{k+1} - x_{k-1}}{x_k - x_{k-1}} \left[ x_{k-1}, x_k, x_{k+1}; f \right] \left( x - \frac{x_{k-1} + x_k}{2} \right)^2 + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \left( x - \frac{x_{k-1} + x_k}{2} \right) + f(x_k) + f(x_{k-1}).$$

Using the estimate (1) we obtain the following theorem.

**Theorem 4.** Let $f \in C[0, 1]$. Then

$$\|f - g\|_\infty \leq \omega_2^f(f; d_n),$$

$$\|\varphi^2 g''\|_\infty \leq C_n \omega_2^f(f; d_n)$$

where $d_n := \max_{k \in \{1, 2, \ldots, 2n+1\}} \sin(\theta_{k+1} - \theta_{k-1})$, \(\theta_i = \arcsin \sqrt{x_i}, i = 0, 1, \ldots, 2n+2\), and

$$c_n = \max_{k \in \{1, 2, \ldots, 2n+1\}} \frac{x_{k+1} - x_{k-1}}{(x_{k+1} - x_k)(x_k - x_{k-1})^2} \|\varphi^2\|_{[a_k, b_k]}.$$
Let $m$ be a fixed natural number, $m \geq 1$, and

$$\Delta_m : \ 0 = x_0 < x_1 < \cdots < x_{2m+2} = 1$$

where $x_k = \sin^2 \frac{k\pi}{4(m+1)}$, $k = 0, 1, \ldots, 2m + 2$.

**Theorem 5.** Let $f \in C[0, 1]$ and $g$ be the function constructed above for the partition $\Delta_m$. Then

$$\|f - g\|_\infty \leq \omega_2^\varphi \left( f; \sin \frac{\pi}{2(m+1)} \right),$$

$$\|\varphi^2 g''\|_\infty \leq \frac{4}{3\sin^2 \frac{\pi}{4(m+1)}} \omega_2^\varphi \left( f; \sin \frac{\pi}{2(m+1)} \right).$$

**Proof.** From (3) we obtain (5). For the proof of inequality (6) we distinguish three cases: (I) $x \in [0, 2x_1]$, (II) $x \in [a_k, b_k]$, $k = 2, \ldots, m$, (III) $k = m + 1$.

**Case I.** From (2) we get

$$g''(x) = \frac{x_2}{2x_1} (0, x_1, x_2; f) = \frac{x_2}{2x_1} (x_2 - x_1) \left( L_1(f; 0, x_2)(x_1) - f(x_1) \right).$$

From (7) and (3), after a tedious manipulation, we get

$$|x(1-x)g''(x)| \leq \frac{4 \cos \frac{2\pi}{4(m+1)} \cdot \cos \frac{\pi}{4(m+1)}}{\sin \frac{\pi}{4(m+1)} (4\cos^2 \frac{\pi}{4(m+1)} - 1)} \omega_2^\varphi \left( f; \sin \frac{\pi}{2(m+1)} \right).$$

It is easy to show that

$$\frac{\cos \frac{2\pi}{4(m+1)} \cdot \cos \frac{\pi}{4(m+1)}}{4\cos^2 \frac{\pi}{4(m+1)} - 1} \leq \frac{1}{3}. \quad (9)$$

From (8) and (9) we obtain (6).

**Case II.** $x \in [a_k, b_k]$, $k = 2, \ldots, m$. From (4) we get

$$\|\varphi^2 g''\|_\infty \leq \frac{b_k (1-b_k)(x_{k+1} - x_k-1)}{(x_{k+1} - x_k)(x_k - x_{k-1})^2} \omega_2^\varphi \left( f; \sin \frac{\pi}{2(m+1)} \right).$$

It is straightforward to show that

$$\frac{b_k (1-b_k)(x_{k+1} - x_k-1)}{(x_{k+1} - x_k)(x_k - x_{k-1})^2} \leq \frac{1}{4\sin^2 \frac{\pi}{4(m+1)}} g \left( \tan \frac{(2k-1)\pi}{4(m+1)} \right)^{-1} \quad (10)$$

where

$$g(t) = \left( 1 + \frac{1}{t \sin 2\alpha + \cos 2\alpha} \right) (t^2 \sin^2 \alpha - 1 + 2t \sin 2\alpha + 2 \cos 2\alpha)$$

for $t \in [\tan \alpha, \frac{1}{\tan 3\alpha}]$, $\alpha = \frac{\pi}{4(m+1)}$. 


The function $g$ is increasing and thus
\[ g(t) \leq \frac{176}{45} < \frac{16}{3}. \] (11)
The inequalities (10) and (11) solve the Case II.

**Case III.** $k = m + 1$.

Because $x(1 - x) \leq \frac{1}{4}$ we have
\[ \|\varphi^2 g''\|_{\infty} \leq \frac{1}{4} \cdot \sin^2 \frac{\pi}{4(m+1)} \cdot \cos^3 \frac{\pi}{4(m+1)} \cdot \omega^2_{\varphi} \left( f; \sin \frac{\pi}{2(m+1)} \right) \]
\[ \leq \frac{4}{3\sin^2 \frac{\pi}{4(m+1)}} \omega^2_{\varphi} \left( f; \sin \frac{\pi}{2(m+1)} \right). \]
The last inequality proves our theorem.

From Theorem 4 we obtain the following theorems (see [2], [3]).

**Theorem 6.** Let $m$ be a natural number, $m \geq 1$, and $L : C[0, 1] \to C[0, 1]$ a linear positive operator which preserves linear functions. For any $h$, $h \in \left[ \sin \frac{\pi}{2(m+1)}, 1 \right]$, the following inequality holds
\[ |(Lf)(x) - f(x)| \leq \left[ 2 + \frac{4}{3\sin^2 \frac{\pi}{4(m+1)}} \left( (Lu)(x) - u(x) \right) \right] \omega^2_{\varphi} (f; h) \]
where $u(x) = x \ln x + (1 - x) \ln(1 - x)$ for $x \in (0, 1)$, $u(0) = u(1) = 0$.

The $K^2_{\varphi}$-functional is defined by
\[ K^2_{\varphi} (f; t) = \inf_{g \in W_{\varphi}^{2, \infty}} \{ \|f - g\|_{\infty} + t^2 \|\varphi^2 g''\|_{\infty} \} \]
where
\[ W_{\varphi}^{2, \infty} = \{ g : g' \in AC_{\text{loc}}[0, 1], \|\varphi^2 g''\|_{\infty} < \infty \}. \]
Using quadratic $C^1$-spline $g$, we get

**Theorem 7.** Let $t \in \left[ \sin \frac{\pi}{2(m+1)}, 1 \right]$ and $f \in C[0, 1]$. Then
\[ K^2_{\varphi} (f; t) \leq \frac{19 + 8\sqrt{2}}{3} \omega^2_{\varphi} (f; t). \]

**Remark.** Tachev [5] obtained
\[ K^2_{\varphi} (f; t) \leq 15 \omega^2_{\varphi} (f; t). \]
References


