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A Quadratic Spline of Sendov Type

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For a compact interval $[a, b]$, $a < b$, on the real axis we denote by $[a, b]$ the space of all real-valued continuous functions on $[a, b]$, equipped with the sup norm given by

$$\|f\|_{C[a,b]} = \|f\|_\infty = \max\{|f(x)| : \ x \in [a, b]\}.$$

For a natural number r we write

$$C^r[a, b] = \{f \in C[a, b] : \ f^{(r)} \in C[a, b]\},$$

and

$$W_{r,\infty}[a, b] = \{f \in C[a, b] : \ f^{(r-1)} \text{ abs. cont., } \|f^{(r)}\|_{L_\infty[a,b]} < \infty\}$$

where $\|f\|_{L_\infty[a,b]} = \|f\|_{L_\infty} = \sup\{|f(x)| : \ x \in [a, b]\}$.

The following theorem due to Brudnyi [1] is very important in approximation theory.

Theorem 1. *Let $f \in C[0, 1]$ and s be a prescribed natural number. Then there exists a family of functions $\{f_{s,h} : \ 0 < h < s^{-1}\}$ from $W_{s,\infty}[0, 1]$ such that*

$$\|f - f_{s,h}\|_\infty \le A_s \omega_s(f; h),$$
$$\|f^{(s)}_{s,h}\|_{L_\infty} \le B_s h^{-s} \omega_s(f; h),$$

where the constants A_s and B_s depend only on s and ω_s is the s-th order modulus of continuity.

It is of interest to have information about the magnitude of the constants A_s and B_s. Zhuk [7] gave lower bounds for the constants A_s and B_s for the case $s = 1$ and $s = 2$ using an extension of the function f to a larger interval. In [4] a pointwise refinement of Zhuk results was obtained.

A genuinely different approach to constructing smoothing functions f_h is to define appropriate spline functions whose definition does not require an extension of f. This was done by Sendov [6]. Sendov proved the following

Theorem 2. *Let $f \in C[0,1]$. Then there exists a family of functions*

$$\Big\{f_h:\ h=\frac{1}{m},\ m\geq 2,\ m\in\mathbb{N}\Big\}\subset W_{2,\infty}[0,1]$$

such that

$$\|f-f_{1/m}\|_\infty \leq \frac{9}{8}\,\omega_2\Big(f;\frac{1}{m}\Big),$$
$$\|f''_{1/m}\|_\infty \leq m^2\,\omega_2\Big(f;\frac{1}{m}\Big).$$

Sendov's functions $f_{1,m}$ are quadratic splines $S_2(f;\cdot)\in W_{2,\infty}[0,1]$. Gonska and Kovacheva [4] proved that the constant 9/8 figuring in Theorem 2 can be replaced by 1.

Our aim is to construct smoothing functions f_h such that

$$\|f-f_h\|_\infty \leq A\,\omega_2^\varphi(f;h),$$
$$\|f''_h\|_\infty \leq B\,\omega_2^\varphi(f;h),$$

where the constants A and B are independent on the function f and $\omega_2^\varphi(f;\cdot)$ is Ditzian-Totik modulus defined by

$$\omega_2^\varphi(f;t)=\sup_{0\leq h\leq t}\|\Delta^2_{h\varphi}f\|_\infty$$

where

$$\Delta^2_{h\varphi}f(x)=\begin{cases} f(x-h\varphi(x))-2f(x)+f(x+h\varphi(x)), & \\ & \text{if } [x-h\varphi(x),x+h\varphi(x)]\subset[0,1]\\ 0, & \text{otherwise,}\end{cases}$$

$\varphi(x)=\sqrt{x(1-x)}$. The functions f_h will be quadratic splines of Sendov-type. Such functions were considered for the first time by Gonska and Tachev [3].

In [2] Gavrea considered a new quadratic C^1-spline g and proved the following:

Theorem 3. *Let $f\in C[0,1]$. Then*

$$\|f-g\|_\infty \leq \omega_2^\varphi\Big(f;\sin\frac{\pi}{2(m+1)}\Big),$$
$$\|\varphi^2 g''\|_\infty \leq \frac{2}{\sin^2\frac{\pi}{4(m+1)}}\,\omega_2^\varphi\Big(f;\sin\frac{\pi}{2(m+1)}\Big),$$

m being a natural number.

The following result (see [2]) will be used in this paper.

Let θ_1, θ_2 *be two distinct points such that* $0 \le \theta_1 < \theta_2 \le \frac{\pi}{2}$. *If* $a = \sin^2\theta_1$, $b = \sin^2\theta_2$, *then for every* $f \in C[0,1]$ *the following estimate holds*

$$|L_1(f;a,b)(x) - f(x)| \le \omega_2^{\varphi}(f;\sin(\theta_2 - \theta_1)) \tag{1}$$

where $L_1(f;a,b)$ *is the Lagrange polynomial which interpolates the function* f *at the points* a *and* b.

Let Δ_{2n+3} : $0 = x_0 < x_1 < \cdots < x_n < x_{n+2} < \cdots < x_{2n+2} = 1$ be a partition of the interval $[0,1]$ such that

$$x_1 - x_0 \le x_2 - x_1 \le \cdots \le x_n - x_{n-1},$$
$$x_n - x_{n-1} \ge x_{n+1} - x_n \ge \cdots \ge x_{2n+2} - x_{2n+1}.$$

We denote by $\widetilde{S}_n(f)$ the continuous polygonal line having as knots the points x_k, $k = 0,1,\dots,2n+2$. With each such knot x_k, $k = 1,2,\dots,2n+2$, we associate the numbers a_k and b_k, defined in the following way:

$$a_1 = 0, \qquad b_1 = 2x_1$$

and

$$a_k = \frac{x_k + x_{k-1}}{2}, \quad b_k - x_k = x_k - a_k, \qquad k = 2,\dots,n+1.$$

For $k = n+2,\dots,2n+1$ we define the numbers a_k and b_k by symmetry with respect to $1/2$. It is supposed that $2x_1 \le a_2$.

We construct the function g in the following way. For $x \in [a_k, b_k]$, $k = 1,2,\dots,2n+1$, g is the second degree Bernstein polynomial over the interval $[a_k,b_k]$, determined by the ordinates $\widetilde{S}_n(f;a_k)$, $f(x_k)$, $\widetilde{S}_n(f;b_k)$. For $x \in [b_k, a_{k+1}]$, $k = 1,2,\dots,2n$, $g(x) = \widetilde{S}_n(f;x)$.

It is easy to show that for $x \in [a_k,b_k]$, $k = 1,2,\dots,2n+1$, the function g is given by

$$\begin{aligned} g(x) = {} & \frac{1}{2}\frac{x_{k+1} - x_{k-1}}{x_k - x_{k-1}}[x_{k-1}, x_k, x_{k+1}; f]\Big(x - \frac{x_{k-1} + x_k}{2}\Big)^2 \\ & + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\Big(x - \frac{x_{k-1} + x_k}{2}\Big) + \frac{f(x_k) + f(x_{k-1})}{2}. \end{aligned} \tag{2}$$

Using the estimate (1) we obtain the following theorem.

Theorem 4. *Let* $f \in C[0,1]$. *Then*

$$\|f - g\|_\infty \le \omega_2^{\varphi}(f;d_n), \tag{3}$$

$$\|\varphi^2 g''\|_\infty \le C_n \omega_2^{\varphi}(f;d_n) \tag{4}$$

where $d_n := \max\limits_{k\in\{1,2,\dots,2n+1\}} \sin(\theta_{k+1} - \theta_{k-1})$, $\theta_i = \arcsin\sqrt{x_i}$, $i = 0,1,\dots,2n+2$, *and*

$$c_n = \max_{k\in\{1,2,\dots,2n+1\}} \frac{x_{k+1} - x_{k-1}}{(x_{k+1} - x_k)(x_k - x_{k-1})^2}\,\|\varphi^2\|_{[a_k,b_k]}.$$

Let m be a fixed natural number, $m \geq 1$, and

$$\Delta_m : \; 0 = x_0 < x_1 < \cdots < x_{2m+2} = 1$$

where $x_k = \sin^2 \frac{k\pi}{4(m+1)}$, $k = 0, 1, \ldots, 2m+2$.

Theorem 5. *Let $f \in C[0,1]$ and g be the function constructed above for the partition Δ_m. Then*

$$\|f - g\|_\infty \leq \omega_2^\varphi\Big(f; \sin \frac{\pi}{2(m+1)}\Big), \tag{5}$$

$$\|\varphi^2 g''\|_\infty \leq \frac{4}{3\sin^2 \frac{\pi}{4(m+1)}}\, \omega_2^\varphi\Big(f; \sin \frac{\pi}{2(m+1)}\Big). \tag{6}$$

Proof. From (3) we obtain (5). For the proof of inequality (6) we distinguish three cases: (I) $x \in [0, 2x_1]$, (II) $x \in [a_k, b_k]$, $k = 2, \ldots, m$, (III) $k = m+1$.

Case I. From (2) we get

$$g''(x) = \frac{x_2}{2x_1}[0, x_1, x_2; f] = \frac{x_2}{2x_1^2(x_2 - x_1)}\big(L_1(f; 0, x_2)(x_1) - f(x_1)\big). \tag{7}$$

From (7) and (3), after a tedious manipulation, we get

$$|x(1-x)g''(x)| \leq \frac{4\cos\frac{2\pi}{4(m+1)} \cdot \cos\frac{\pi}{4(m+1)}}{\sin^2\frac{\pi}{4(m+1)}\big(4\cos^2\frac{\pi}{4(m+1)} - 1\big)}\, \omega_2^\varphi\Big(f; \sin\frac{\pi}{2(m+1)}\Big). \tag{8}$$

It is easy to show that

$$\frac{\cos\frac{2\pi}{4(m+1)} \cdot \cos\frac{\pi}{4(m+1)}}{4\cos^2\frac{\pi}{4(m+1)} - 1} \leq \frac{1}{3}. \tag{9}$$

From (8) and (9) we obtain (6).

Case II. $x \in [a_k, b_k]$, $k = 2, \ldots, m$. From (4) we get

$$\|\varphi^2 g''\|_\infty \leq \frac{b_k(1-b_k)(x_{k+1} - x_{k-1})}{(x_{k+1} - x_k)(x_k - x_{k-1})^2}\, \omega_2^\varphi\Big(f; \sin\frac{\pi}{2(m+1)}\Big).$$

It is straightforward to show that

$$\frac{b_k(1-b_k)(x_{k+1} - x_{k-1})}{(x_{k+1} - x_k)(x_k - x_{k-1})^2} \leq \frac{1}{4\sin^2\frac{\pi}{4(m+1)}}\, g\Big(\Big(\tan\frac{(2k-1)\pi}{4(m+1)}\Big)^{-1}\Big) \tag{10}$$

where

$$g(t) = \Big(1 + \frac{1}{t\sin 2\alpha + \cos 2\alpha}\Big)(t^2\sin^2\alpha - 1 + 2t\sin 2\alpha + 2\cos 2\alpha)$$

for $t \in \big[\tan\alpha, \frac{1}{\tan 3\alpha}\big]$, $\alpha = \frac{\pi}{4(m+1)}$.

The function g is increasing and thus

$$g(t) \le \frac{176}{45} < \frac{16}{3}. \tag{11}$$

The inequalities (10) and (11) solve the Case II.

Case III. $k = m+1$.

Because $x(1-x) \le \dfrac{1}{4}$ we have

$$\begin{aligned}\|\varphi^2 g''\|_\infty &\le \frac{1}{4} \cdot \frac{1}{\sin^2 \frac{\pi}{4(m+1)} \cdot \cos^3 \frac{\pi}{4(m+1)}}\, \omega_2^\varphi\Big(f; \sin \frac{\pi}{2(m+1)}\Big) \\ &\le \frac{4}{3\sin^2 \frac{\pi}{4(m+1)}}\, \omega_2^\varphi\Big(f; \sin\frac{\pi}{2(m+1)}\Big).\end{aligned}$$

The last inequality proves our theorem.

From Theorem 4 we obtain the following theorems (see [2], [3]).

Theorem 6. *Let m be a natural number, $m \ge 1$, and $L : C[0,1] \to C[0,1]$ a linear positive operator which preserves linear functions. For any h, $h \in \big[\sin\frac{\pi}{2(m+1)}, 1\big]$, the following inequality holds*

$$|(Lf)(x) - f(x)| \le \Big[2 + \frac{4}{3\sin^2\frac{\pi}{4(m+1)}}\big((Lu)(x) - u(x)\big)\Big]\omega_2^\varphi(f;h)$$

where $u(x) = x\ln x + (1-x)\ln(1-x)$ for $x \in (0,1)$, $u(0) = u(1) = 0$.

The K_2^φ-functional is defined by

$$K_2^\varphi(f;t) = \inf_{g\in W_{2,\infty}^\varphi} \big\{\|f-g\|_\infty + t^2\|\varphi^2 g''\|_\infty\big\}$$

where

$$W_{2,\infty}^\varphi = \big\{g :\ g' \in AC_{loc}[0,1],\ \|\varphi^2 g''\|_\infty < \infty\big\}.$$

Using quadratic C^1-spline g, we get

Theorem 7. *Let $t \in \big[\sin\frac{\pi}{2(m+1)}, 1\big]$ and $f \in C[0,1]$. Then*

$$K_2^\varphi(f;t) \le \frac{19+8\sqrt{2}}{3}\, \omega_2^\varphi(f;t).$$

Remark. Tachev [5] obtained

$$K_2^\varphi(f;t) \le 15\, \omega_2^\varphi(f;t).$$

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