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Refinable Quasi-Interpolatory Operators

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In this paper we are interested in the construction and analysis of convergence properties of some refinable quasi-interpolatory operators, based on a particular class of *totally positive refinable functions*.

1. Introduction

It is well-known that the refinable functions (r.f.), that is, the solutions of two scale refinement equations of the form

$$\varphi(x) = \sum_{i \in \mathbb{Z}} a_i \varphi(2x - i)$$

play a main role in several fields. In fact they are involved in the construction of wavelet bases, of convergent subdivision schemes, of shape preserving operators and so on.

A large class of refinable functions have been introduced in [1], [3]. These refinable functions will be denoted in the following by GP r.f.

They depend on one or more parameters, for particular values of which we get cardinal B-splines. However, supports being equal, GP r.f. are less smooth than the B-splines, and it is just this less smoothness that makes them more flexible and effective in several applications (see for instance [2], [6]).

In this paper, we are interested in the construction and the analysis of a class of refinable quasi-interpolatory operators, related to GP refinable functions. In Section 2 we give definitions and main properties of GP refinable functions; in Section 3 we introduce the quoted quasi-interpolatory refinable operators and give new results on their behaviour. In Section 4 we consider two particular cases of quasi-interpolatory refinable operators, introduced in Section 3, namely, operators of Bernstein-Schoenberg type and of generalized Bernstein-Schoenberg type, giving a new convergence result.

2. GP Refinable Functions and B-bases

The GP refinable functions introduced in [1] and [3] have compact support $[0, n+1]$, and are identified in terms of their *masks*, that is, of the vector $a = \{a_j\}_{j\in\mathbb{Z}}$. The entries of a depend explicitly on their index and on the length of the support:

$$a_k = \sum_{r=0}^{m} b_r^{(r)} \binom{n+1-2r}{k-r}, \qquad k = 0, 1, \ldots, n+1,$$

where m denotes the number of parameters and $b_r^{(r)}$ are defined recursively by the formula

$$b_\ell^{(r+1)} = b_\ell^{(r)} - \binom{m-2r}{\ell-r} b_r^{(r)}, \qquad r = 0, 1, \ldots, m-1, \quad \ell = r+1, \ldots, m.$$

The coefficients $b_\ell^{(0)}$, $\ell = 0, 1, \ldots, m$, are arbitrary positive numbers such that

$$\begin{cases} b_{m-r}^{(0)} = b_r^{(0)}, & r = 0, 1, \ldots, m \\ b_m^{(0)} = 2^{2m-n} - 2\sum_{\ell=0}^{m-1} b_\ell^{(0)}, & \\ \det\left(b_{2\ell-k}^{(0)}\right)_{k,\ell=1}^{p} > 0, & p = 1, 2, \ldots, 2m. \end{cases}$$

If $m = 1$, the masks reduce to

$$a_k = \frac{1}{2^h}\left[\binom{n+1}{k} + 4(2^{h-n}-1)\binom{n-1}{k-1}\right], \quad k = 0, 1, \ldots, n+1 \tag{1}$$

where $h \geq n$ is a real parameter and $n \geq 2$; if $h = n$ we get the mask of the B-spline of degree n.

The refinable functions GP, enjoy many properties useful in the applications:

i) compact support: $\operatorname{supp}\varphi = [0, n+1]$;

ii) central symmetry: $\varphi(x) = \varphi(n+1-x)$;

iii) total positivity, i.e.,

$$\varphi\begin{pmatrix} x_1 & x_2 & \ldots & x_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} := \det\left(\varphi(x_\ell - i_j)\right)_{\ell,j=1}^{p} \geq 0,$$

for any sequences $x_1 < x_2 < \ldots < x_p$ and $i_1 < i_2 < \ldots < i_p$, ($x_\ell \in \mathbb{R}$, $i_\ell \in \mathbb{Z}$);

iv) smoothness: $\varphi \in C^{n-m-1}(\mathbb{R})$;

v) order of polynomial reproducibility: $d = n - m \geq 0$, i.e.,

$$x^{\ell-1} = \sum_{k\in\mathbb{Z}} \beta_{jk}^{(\ell)} 2^{j/2} \varphi(2^j x - k), \qquad \ell = 1, 2, \ldots, d, \tag{2}$$

with explicit expressions of $\beta_j^{(\ell)} = \{\beta_{jk}^{(\ell)}\}_{k\in\mathbb{Z}}$, given in [5].

Moreover, any GP refinable function generates a Multiresolution Analysis (M.R.A.) on $\mathbb{R}$ and a M.R.A. on a finite interval I can be constructed using the so called refinable B-bases, obtained as follows.

Starting from the set

$$\Phi_j = \{2^{j/2}\varphi(2^j x - k)\}, \qquad j \in \mathbb{Z}^+,$$

the refinable B-basis W_j on I is related to Φ_j by the equation

$$\Phi_j = W_j A_j \tag{3}$$

where A_j is a suitable banded, totally positive (T.P.) and stochastic matrix [7]. The bases W_j preserve the properties of Φ_j, in the sense that they are T.P., normalized, centrally symmetric, generate a M.R.A. on $L^2(I)$, have the same order of reproducibility $d = n - m$ and there results

$$x^{\ell-1} = \sum_{k=0}^{N_j} \eta_{jk}^{(\ell)} w_{jk}(x), \qquad \ell = 1, 2, \ldots, d, \tag{4}$$

where the vector $\eta_j^{(\ell)} = \{\eta_{jk}^{(\ell)}\}_{k=0}^{N_j}$ is related to the vector $\beta_j^{(\ell)}$ in (2) by the equation $\eta_j^{(\ell)} = A_j \beta_j^{(\ell)}$, A_j being the matrix in (3).

Moreover, conditions of Dirichlet type are satisfied at the endpoint of I, which are of particular usefulness, for instance, in the solution of boundary differential problems by means of approximations based on refinable functions.

3. Quasi-Interpolatory Refinable Operators

Now we introduce some explicit refinable operators in terms of refinable B-bases, for functions defined on a finite interval I. For the sake of simple notations, we assume $I = [0, n+1]$, an assumption by no means restrictive. For the same reason, we restrict ourselves to consider GP r.f. depending of one parameter, whose masks are given by (1).

For any real-valued function f on I, we introduce the following refinable operator

$$Q_j\, f(x) = \sum_{i=0}^{N_j} (\lambda_{ji}\, f)\, w_{ji}(x), \qquad x \in I \tag{5}$$

where $N_j = 2^j(n+1)+n-1$, $j \in \mathbb{Z}^+$, $W_j = \{w_{ji}\}_{i=0}^{N_j}$ is any of the GP refinable B-bases, introduced in the previous section, with

$$\operatorname{supp} w_{ji} = \left[\max\{0, \frac{i-n}{2^j}\}, \min\{\frac{i+1}{2^j}, n+1\}\right],$$

and $\{\lambda_{ji}\}_{i=0}^{N_j}$ is a set of linear functionals.

In the case when W_j is a cardinal B-spline basis, $Q_j f$ reduces to the operator dealt with in [9].

If the linear space of functions f, we are considering, contains the system $\mathbb{P}_\ell$ of polynomials of degree $\ell - 1$ for some $1 \le \ell \le n-1$, it is reasonable to ask the question whether Q_j reproduces polynomials. In this regard, the following proposition holds.

Proposition 1. *The equality*

$$Q_j\,\pi = \pi \tag{6}$$

holds for every $\pi \in \mathbb{P}_\ell$, $1 \le \ell \le n-1$, if and only if the linear functionals λ_{ji} are such that

$$\lambda_{ji}x^{k-1} = \eta_{ji}^{(k)}, \qquad k = 1,2,\dots,\ell, \quad i = 0,1,\dots,N_j. \tag{7}$$

Proof. For any integer k, with $1 \le k \le \ell$, we have by (4):

$$x^{k-1} = \sum_{i=0}^{N_j} \eta_{ji}^{(k)} w_{ji}(x), \qquad x \in I.$$

Then, $Q_j\,x^{k-1} = x^{k-1} \Leftrightarrow \lambda_{ji}\,x^{k-1} = \eta_{ji}^{(k)}$.

The construction of suitable functionals λ_{ji} satisfying (7) can be realized starting from a set of ℓ linear functionals $\{\lambda_{jik}\}$, $k = 1,2,\dots,\ell$, satisfying the condition

$$\det\left(\lambda_{jik}\,x^{\nu-1}\right) \neq 0, \qquad k,\nu = 1,2,\dots,\ell. \tag{8}$$

In fact, the following corollary can be easily verified.

Corollary 1. *Let $\{\lambda_{jik}\}_{k=1}^{\ell}$ be a set of linear functionals satisfying* (8)*, and $\{\alpha_{jik}\}_{k=1}^{\ell}$ be the solution of the system*

$$\sum_{k=1}^{\ell} \alpha_{jik}(\lambda_{jik}\,x^{\nu-1}) = \eta_{ji}^{(\nu)}, \qquad \nu = 1,2,\dots,\ell.$$

Then, the operators $\lambda_{ji} := \sum_{k=1}^{\ell} \alpha_{jik}\lambda_{jik}$ satisfy condition (7) *and consequently,*

$$Q_j\,\pi = \pi, \qquad \pi \in \mathbb{P}_\ell, \quad 1 \le \ell \le n-1.$$

A possible choice for the functionals λ_{jik} is as follows. Assume

$$\lambda_{jik} f := \big[\tau_{ji1}, \tau_{ji2}, \ldots, \tau_{jik}\big] f, \qquad k = 1, 2, \ldots, \ell,$$

where $(\tau_{ji1}, \tau_{ji2}, \ldots, \tau_{jik})$ are distinct points in I, and consider the polynomials

$$p_{jik}(x) := \prod_{r=1}^{k-1} (x - \tau_{jir}).$$

Then,

$$\lambda_{jik}\, p_{jih} = \delta_{kh}, \qquad k, h = 1, 2, \ldots, \ell$$

and

$$\alpha_{jik} = \sum_{\nu=0}^{k-1} (-1)^{\nu} \eta_{ji}^{(k-\nu)} \mathrm{symm}_{\nu}\, (\tau_{ji1}, \tau_{ji2}, \ldots, \tau_{ji\nu}), \qquad k = 1, 2, \ldots, \ell.$$

In this case, the approximating refinable operator

$$Q_j\, f(x) = \sum_{i=0}^{N_j} \sum_{k=1}^{\ell} \alpha_{jik}\, [\tau_{ji1}, \tau_{ji2}, \ldots, \tau_{jik}] f\, w_{ji}(x)$$

satisfies (8), that is, it reproduces the polynomials from $\mathbb{P}_{\ell}$, $1 \le \ell \le n-1$.

4. Particular Cases

Quasi-interpolatory refinable operators have been considered and analyzed in [4], [7], [8]. The first case is provided by the Bernstein-Schoenberg type operators, which can be obtained from (5) setting $\lambda_{ji}\, f = f(\eta_{ji}^{(2)})$; that is, the operators defined for $f \in L^2(I)$ by

$$Q_j^{(1)} f(x) = \sum_{i=0}^{N_j} f(\eta_{ji}^{(2)})\, w_{ji}(x).$$

Several properties of $Q_j^{(1)} f$, which turns out to be a shape-preserving operator, are given in [4] and [5]. Here we state and prove the following theorem.

Theorem 1. *Let $f \in C^1(I)$. Then, for every $j \in \mathbb{Z}^+$,*

$$\|D^r(f - Q_j^{(1)} f)\|_\infty \le K\, 2^{-j(1-r)} \omega(f'; 2^{-j}), \qquad r = 0, 1 \tag{9}$$

where K is a constant independent of j and $\omega(f'; h)$ is the modulus of continuity of f'.

Proof. By (3), we can write

$$W_j = \Phi_j A^{-1}.$$

Then, taking into account the properties of matrix A, the Proposition 3.1 in [2] and the finite support of w_{ji}, we obtain the estimate

$$|w'_{ji}(x)| \le C\,2^j$$

where C is a constant independent of j.

For a fixed $t \in I,\ t \in [\frac{\ell}{2^j}, \frac{\ell+1}{2^j}]$, we denote by I_ℓ the smallest closed interval containing the support of $\{w_{ji}(x)\}_{i=\ell}^{\ell+n}$. Now, for $f \in C^1(I)$, we define

$$R(x) = f(x) - \sum_{k=0}^{1} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

Then $R'(x) = f'(x) - f'(t)$ and, from the Taylor expansion of R at the point t we have $R(x) = R'(\varsigma)(x-t)$, for some $\varsigma(x)$ between x and t. Since the operator $Q_j^{(1)} f$ reproduces any linear function, we can write

$$|D^r\big(f(x) - Q_j^{(1)} f(x)\big)| \le |D^r R(x)| + |D^r Q_j^{(1)} R(x)|, \qquad r = 0, 1.$$

For all $x \in I_\ell$, we obtain

$$|D^r R(x)| \le (n+1)^2\, 2^{-j(1-r)} \omega(f'; 2^{-j})$$

and recalling from [7] that $\eta_{ji}^{(2)} \in \operatorname{supp} w_{ji}$, the inequality

$$|D^r Q_j^{(1)} R(x)| = |\sum_{i=\ell}^{\ell+n} R(\eta_{ji}^{(2)}) D^r w_{ji}(x)| \le C(n+1)^3\, 2^{-j(1-r)} \omega(f'; 2^{-j}),$$

holds. Therefore, we get the error bound (9), with

$$K = \max\{(n+1)^2, C(n+1)^3\}.$$

The second example of quasi-interpolatory operators are the generalized Bernstein-Schoenberg type operators, obtained from (5) assuming $\lambda_{ji} f = \langle\, \chi_{ji}, f\,\rangle$:

$$Q_j^{(2)} f(x) = \sum_{i=0}^{N_j} \langle\, \chi_{ji}, f\,\rangle w_{ji}(x), \qquad x \in I, \quad f \in L^2(I),$$

where

$$\langle\, \chi_{ji}, f\,\rangle = \int_I \chi_{ji}(t) f(t)\, dt \qquad \text{and} \qquad \chi_{ji}(x) = \frac{w_{ji}(x)}{\int_I w_{ji}(t)\, dt}.$$

The main properties of these operators have been given in [7], [8].

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